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Optimal Portfolio Selection Models with Uncertain Returns

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Abstract

This paper provides two new models for portfolio selection in which the securities are assumed to be uncertain variables that are neither random nor fuzzy. Since there is no efficient method to solve the proposed models, the original problems are transformed into their crisp equivalents programming when the returns are chosen some special uncertain variables such as rectangular uncertain variable, triangular uncertain variable, trapezoidal uncertain variable and normal uncertain variable. Finally, its feasibility and effectiveness of the method is illustrated by numerical example.

Keywords: Uncertain variable, Portfolio selection, Mean-variance, Crisp equivalent programming

1. Introduction

Portfolio selection is concerned with an individual who is trying to allocate one's wealth among alternative securities such that the investment goal can be achieved. The problem was initialized by Markowitz (1952, p.77), and the solution of his mean-variance methodology has been serving as a basis of the development of modern financial theory. The pioneer work Markowitz combined probability and optimization theory to model the investment behavior under uncertainty. Quantifying investment return as the mean of returns of the securities, and investment risk as the variance from the mean, Markowitz formulated his models mathematically in two ways: minimizing variance for a given expected value or maximizing expected value for a given variance.

Portfolio theory has been greatly improved since Markowitz. The researches mainly focused on two directions. One direction is how to define the investment risk. In 1952, Markowitz stated that variance could be regarded as risk. Since then, mathematical analysis on portfolio management has developed greatly, and variance has become the most popular mathematical definition of risk for portfolio selection. Scholars developed a variety of models using variance to quantify risk in various situations, for example, variance models proposed by Best (2000, p.195), Chopra (1998,p.53), Gram (2003, p. 546), Deng (2005, p.278) and Huang (2007, p.396). Since in case when return distributions securities are asymmetric, the selected portfolio based on variance may have a potential danger to sacrifice too much expected return in eliminating both high return extremes and low return extremes, semi-variance was proposed as an alternative definition of risk by Markowitz (1952, p.77) and lots of models were built to minimize semi-variance in different cases, for example, Chow (1994, p.231), Grootveld (1999, p.304), Homaifar (1990, p.677), Markowitz (1993, p.307) and Rom (1994, p.431). Another alternative definition of risk is the probability of an adverse outcome Roy (1952, p.431). There are also many research works that minimize the probability of an adverse outcome such as Mao (1970, p.657) and Williams (1997, p.77). However, in reality, some people may only be sensitive to one preset disastrous loss level and regard the chance of occurring this bad case as risk, risk curse was proposed as the fourth definition of risk such as Huang (2008, p.351) and Huang (2008, p.1102). Another direction is how to choose return rate. Portfolio selection was initially handled in stochastic environments. After then, the problem was dealt with in fuzzy, random fuzzy and birandom environment. There are a variety of models in this line. Let us mention some of the representatives in recent years. For example, we have fuzzy chance-constrained model by Huang (2006, p. 500), fuzzy mean semi-variance model by Huang (2008, p.1) and random fuzzy mean-variance model by Grootveld (1999, p.304).

These studies solved the problem in different stochastic, fuzzy or random and fuzzy simultaneously environments with different risk. In practice, the real life decisions are usually made in the state of uncertainty, when the uncertainty of the return rate behaves neither randomness nor fuzziness, we need a new tool to deal with it. In such situations, the use of uncertain theory to represent unknown parameters provides an interesting help. In other words, we may employ the uncertain theory which was initialized by Liu (2009) to deal with this new type of uncertainty.

The paper is organized as follows. After recalling some definitions and results about uncertain measure and uncertain variable in section 2, the mean-variance models for portfolio selection is introduced in section 3. Then section 4 discusses its crisp equivalents when the return rates are chosen as some special uncertain variables such as rectangular uncertain variable, triangular uncertain variable, trapezoidal uncertain variable and normal uncertain variable. In section 5, we provide a numerical example to illustrate the potential application and the effectiveness of the new models. Finally, we conclude the paper in section 6.

2. Preliminaries

Let Γ be a nonempty set, and let A be a σ-algebra over Γ. Each element $\Lambda \in$ A is called an event. In order to provide an axiomatic definition of uncertain measure, it is necessary to assign to each event Λ a number $M\{\Lambda\}$ which indicates the level that Λ will occur. In order to ensure that the number $M\{\Lambda\}$ has certain mathematical properties, Liu (2009) proposed the following five axioms:

Axiom 1 (Normality) $M(\Gamma)=1$;

Axiom 2 (Monotonicity) $M(\Lambda_1) \le M(\Lambda_2)$ whenever $\Lambda_1 \subseteq \Lambda_2$;

Axiom 3 (Self-duality) $M(\Lambda)+M(\Lambda^c)=1$ for every event Λ;

Axiom 4 (Countable subadditivity) For every countable sequence of events $\{\Lambda_i\}$, we have

$$M(\bigcup_{i=1}^{\infty}\Lambda_i) \le \sum_{i=1}^{\infty} M(\Lambda_i).$$

The following is the definition of uncertain measure.

Definition 1 (Liu (2009)). The set function is called an uncertain measure if it satisfies the normality, monotonicity, self-duality and countable subadditivity axioms.

Example 1 Let $\Gamma=\{\gamma_1,\gamma_2\}$. For this case, there are only 4events. Define

$$M\{\gamma_1\}=0.4,\ \ M\{\gamma_2\}=0.6,\quad M(\phi)=0,\ \ M(\Gamma)=1,$$

then M is an uncertain measure because it satisfies the four axioms.

Definition 2 (Liu (2009)). Let Γ be a nonempty set, A a σ-algebra over Γ, and M an uncertain measure. Then the triplet (Γ, A, M) is called an uncertain space.

The product uncertain measure is defined as follows.

Axiom 5 (Liu (2009)). Product Measure Axiom) Let Γ_k be nonempty sets on which M_k are uncertain measures, $k=1,\ 2,\cdots,n$, respectively. Then the product uncertain measure on Γ is

$$M\{\Lambda\}=\begin{cases}\sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\ \min\limits_{1\le k\le n} M_k\{\Lambda_k\}, & \text{if}\ \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda}\ \min\limits_{1\le k\le n} M_k\{\Lambda_k\}>0.5,\\ 1-\sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c}\ \min\limits_{1\le k\le n} M_k\{\Lambda_k\}, & \text{if}\ \sup\limits_{\Lambda_1\times\Lambda_2\times\cdots\times\Lambda_n\subset\Lambda^c}\ \min\limits_{1\le k\le n} M_k\{\Lambda_k\}>0.5.\end{cases}$$

For each event $\Lambda \in$ A, denoted by $M = M_1 \wedge M_2 \wedge \cdots \wedge M_n$.

Definition 3 (Liu (2009)). An uncertain variable is a measurable function ξ from an uncertainty space $(\Gamma, \mathrm{A}, \mathrm{M})$ to the set of real numbers, i.e., for any Borel set B of real numbers , the set

$$\{\xi \in \mathrm{B}\}=\{\gamma \in \Gamma \mid \xi(\gamma) \in \mathrm{B}\}$$

is an event.

A random variable can be characterized by a probability density function and a fuzzy variable may be described by a membership function, uncertain variable can be characterized by identification function.

Definition 4 (Liu (2009)). An uncertain variable ξ is said to have a first identification function λ if

(1) $\lambda(x)$ is a nonnegative function on R such that

$$\sup_{x\ne y}\lambda(x)+\lambda(y)=1;$$

(2) For any set B of real numbers, we have

$$M\{\xi\in\mathrm{B}\}=\begin{cases}\sup\limits_{x\in\mathrm{B}}\lambda(x), & if\ \sup\limits_{x\in\mathrm{B}}\lambda(x)<0.5,\\ 1-\sup\limits_{x\in\mathrm{B}^C}\lambda(x), & \text{if}\ \sup\limits_{x\in\mathrm{B}^C}\lambda(x)\ge 0.5.\end{cases}$$

Definition 5 (Liu (2009)). The uncertainty distribution $\Phi:\ R\to[0,1]$ of an uncertain variable ξ is defined by

$\Phi(x) = M\{\xi \le x\}$.

Definition 6 (Liu (2009)). Let ξ be an uncertain variable. Then the expected value of ξ is defined by

$$E[\xi] = \int_0^{+\infty} M\{\xi \ge x\}\, dr - \int_{-\infty}^{0} M\{\xi \le x\} dr,$$

provided that at least one of the two integrals is finite.

Definition 7 (Liu (2009)). Let ξ be an uncertain variable with finite expected value e, then the variance of ξ is defined by $V[\xi] = E[(\xi - e)^2]$.

The detailed exposition on the uncertain theory have been recorded in the literature, the interested readers may consult it.

3. Mean-variance Model

In Markowitz models, security returns were regarded as random variables. As discussed in introduction, there does exist situations that security returns may be uncertain variable parameters. In this situation, we can use uncertain variables to describe the security returns.

Let x_i denote the investment proportion in the ith security, ξ_i represents uncertain return of the ith security, $i = 1, 2, \cdots, n$, respectively, and a the maximum risk level that the investor can tolerate. Following Markowitz's idea, we quantify investment return by the expected value of a portfolio, and risk by the variance. Then an optimal portfolio should be the one with maximal expected return for the given variance level. To express it in mathematical formula, the mean-variance model is as follows:

$$\begin{aligned}
&\max\ E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\
&\text{Subject to}: \\
&V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \le a, \\
&x_1 + x_2 + \cdots + x_n = 1, \\
&x_i \ge 0,\ i = 1, 2, \cdots, n.
\end{aligned} \tag{1}$$

where E denotes the expected value operator , and V the variance operator of the uncertain total return rate, a the maximum risk level the investor can tolerate.

When the investors preset an expected return level that they feel satisfactory, and want to minimize the risk for this given level of return, the optimization model becomes

$$\begin{aligned}
&\min\ V[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \\
&\text{Subject to}: \\
&E[x_1\xi_1 + x_2\xi_2 + \cdots + x_n\xi_n] \ge b, \\
&x_1 + x_2 + \cdots + x_n = 1, \\
&x_i \ge 0,\ i = 1, 2, \cdots, n.
\end{aligned} \tag{2}$$

where b denotes the minimum expected investment return that the investors can accept.

4. Deterministic Equivalents

The traditional solution methods require conversion of the objective function and the constraints to their respective deterministic equivalents. As we know, this process is usually hard to perform and only successful for some special cases. Let us consider the following forms of the uncertain return rates.

Case 1. Suppose that the return rate ξ_i of the ith security is rectangular uncertain variable $\xi_i = (a_i, b_i), i = 1, 2, \cdots, n$, then

$\sum_{i=1}^n x_i\xi_i$ is rectangular uncertain variable $(\sum_{i=1}^n x_i a_i, \sum_{i=1}^n x_i b_i)$. According to the properties of rectangular uncertain variable, we have

$$E[\textstyle\sum_{i=1}^n x_i\xi_i] = (\sum_{i=1}^n x_i a_i + \sum_{i=1}^n x_i b_i)/2 = \sum_{i=1}^n x_i(a_i + b_i)/2,$$

$$V[\textstyle\sum_{i=1}^n x_i\xi_i] = (\sum_{i=1}^n x_i b_i - \sum_{i=1}^n x_i a_i)^2/8 = [\sum_{i=1}^n x_i(b_i - a_i)]^2/8.$$

Since the term $[\sum_{i=1}^n x_i(b_i - a_i)]^2$ is nonnegative, $V[\sum_{i=1}^n x_i\xi_i] \le a$ is equivalent to $\sum_{i=1}^n x_i(b_i - a_i) \le 2\sqrt{2a}$. In this case, models (1) and (2) can be converted into its deterministic equivalents as follows.

$$
\begin{aligned}
&\max \ \textstyle\sum_{i=1}^{n} x_i(a_i+b_i) \\
&\text{Subject to:} \\
&\textstyle\sum_{i=1}^{n} x_i(b_i-a_i) \le 2\sqrt{2a}, \\
&x_1+x_2+\cdots+x_n=1, \\
&x_i \ge 0,\ i=1,2,\cdots,n.
\end{aligned}
\tag{3}
$$

and

$$
\begin{aligned}
&\min \ \textstyle\sum_{i=1}^{n} x_i(b_i-a_i) \\
&\text{Subject to:} \\
&\textstyle\sum_{i=1}^{n} x_i(b_i+a_i) \ge 2b, \\
&x_1+x_2+\cdots+x_n=1, \\
&x_i \ge 0,\ i=1,2,\cdots,n.
\end{aligned}
\tag{4}
$$

Case 2. If the return rates are all trapezoidal uncertain variables, Let ξ_i be (a_i, b_i, c_i, d_i), where $a_i < b_i \le c_i < d_i, i=1,2,\cdots,n$. Then $\sum_{i=1}^{n} x_i\xi_i$ is $(\sum_{i=1}^{n} x_ia_i, \sum_{i=1}^{n} x_ib_i, \sum_{i=1}^{n} x_ic_i, \sum_{i=1}^{n} x_id_i)$. In accordance with the properties of trapezoidal uncertain variable, we have

$$E[\textstyle\sum_{i=1}^{n} x_i\xi_i] = (\sum_{i=1}^{n} x_ia_i + \sum_{i=1}^{n} x_ib_i + \sum_{i=1}^{n} x_ic_i + \sum_{i=1}^{n} x_id_i)/4 = \sum_{i=1}^{n} x_i(a_i+b_i+c_i+d_i)/4,$$

$$V[\textstyle\sum_{i=1}^{n} x_i\xi_i] = \dfrac{4\alpha^2+3\alpha\beta+\beta^2+9\alpha\gamma+3\beta\gamma+6\gamma^2}{48} + \dfrac{[(\alpha-\beta-2\gamma)^+]^3}{384\alpha}$$

where

$$\alpha = (\textstyle\sum_{i=1}^{n} x_ib_i - \sum_{i=1}^{n} x_ia_i) \vee (\sum_{i=1}^{n} x_id_i - \sum_{i=1}^{n} x_ic_i),\ \beta = (\sum_{i=1}^{n} x_ib_i - \sum_{i=1}^{n} x_ia_i) \wedge (\sum_{i=1}^{n} x_id_i - \sum_{i=1}^{n} x_ic_i),$$

and $\gamma = \sum_{i=1}^{n} x_ic_i - \sum_{i=1}^{n} x_ib_i$, so the model (1) and (2) can be changed into the following formulas:

$$
\begin{aligned}
&\max \ \textstyle\sum_{i=1}^{n} x_i(a_i+b_i+c_i+d_i) \\
&\text{Subject to:} \\
&V[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n] \le a, \\
&x_1+x_2+\cdots+x_n=1, \\
&x_i \ge 0,\ i=1,2,\cdots,n.
\end{aligned}
\tag{5}
$$

and

$$
\begin{aligned}
&\min \ V[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n] \\
&\text{Subject to:} \\
&\textstyle\sum_{i=1}^{n} x_i(a_i+b_i+c_i+d_i) \ge 4b, \\
&x_1+x_2+\cdots+x_n=1, \\
&x_i \ge 0,\ i=1,2,\cdots,n.
\end{aligned}
\tag{6}
$$

In models (5) and (6), the variance $V[x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n]$ is formulated as

$$V[\textstyle\sum_{i=1}^{n} x_i\xi_i] = \dfrac{4\alpha^2+3\alpha\beta+\beta^2+9\alpha\gamma+3\beta\gamma+6\gamma^2}{48} + \dfrac{[(\alpha-\beta-2\gamma)^+]^3}{384\alpha}.$$

Case 3. An uncertain variable ξ is called normal if it has a normal uncertainty distribution

$$\Phi(x) = (1+\exp(\frac{\pi(e-x)}{\sqrt{3}\sigma}))^{-1}, x \in R,$$

denoted by $N(e,\sigma)$ where e and σ are real number with $\sigma > 0$. Suppose that the return rate of ith security is normally distributed with parameters e_i and $\sigma_i > 0, i=1,2,\cdots,n$. Then we have

$$E[\textstyle\sum_{i=1}^{n} x_i\xi_i] = \sum_{i=1}^{n} x_ie_i, \quad V[\sum_{i=1}^{n} x_i\xi_i] = (\sum_{i=1}^{n} x_i\sigma_i)^2.$$

So the model (1) and (2) can be converted into the following linear equivalents:

$$\begin{aligned} &\max \ \textstyle\sum_{i=1}^{n} x_i e_i \\ &\text{Subject to:} \\ &\textstyle\sum_{i=1}^{n} x_i \sigma_i \le \sqrt{a}, \\ &x_1 + x_2 + \cdots + x_n = 1, x_i \ge 0,\ i = 1, 2, \cdots, n. \end{aligned} \tag{7}$$

and

$$\begin{aligned} &\min \ \textstyle\sum_{i=1}^{n} x_i \sigma_i \\ &\text{Subject to:} \\ &\textstyle\sum_{i=1}^{n} x_i e_i \ge b, \\ &x_1 + x_2 + \cdots + x_n = 1, \\ &x_i \ge 0,\ i = 1, 2, \cdots, n. \end{aligned} \tag{8}$$

Thus we can solve the models (3)-(8) by traditional method.

5. Numerical Example

Example 2 Assume that there are 5 securities. Among them, returns of five are all normal uncertain variables $\xi_i = N(e_i, \sigma_i), i = 1, 2, 3, 4, 5$. Let the return rates be

$$\xi_1 = N(0,1),\ \xi_2 = N(1,2),\ \xi_3 = N(2,3), \xi_4 = N(3,4), \xi_5 = N(4,5),$$

Then

$$\textstyle\sum_{i=1}^{5} x_i \xi_i = N(x_2 + 2x_3 + 3x_4 + 4x_5, x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5)$$

Thus, we have

$$E[x_1\xi_1 + x_2\xi_2 + x_3\xi_3 + x_4\xi_5] = x_2 + 2x_3 + 3x_4 + 4x_5,$$
$$V[x_1\xi_1 + x_2\xi_2 + x_3\xi_3 + x_4\xi_5] = x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5.$$

Suppose that the risk is not allowed to exceed 1.5, and the minimum excepted return the investor can accept is 2, then the models (7) and (8) are as follows:

$$\begin{aligned} &\max \ x_2 + 2x_3 + 3x_4 + 4x_5 \\ &\text{Subject to:} \\ &x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \le 1.5, \\ &x_1 + x_2 + \cdots + x_5 = 1, x_i \ge 0,\ i = 1, 2, \cdots, 5 \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\min \ x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \\ &\text{Subject to:} \\ &x_2 + 2x_3 + 3x_4 + 4x_5 \ge 2, \\ &x_1 + x_2 + \cdots + x_5 = 1, x_i \ge 0,\ i = 1, 2, \cdots, 5. \end{aligned} \tag{10}$$

By use of Matlab 7.0 on PC we obtain the optimal solutions of model (9) and (10). The optimal solution of model (9) is

$$(0.6962, 0.1478, 0.1186, 0.0344, 0.0029),$$

and the value of objective function is 0.5000. This means that in order to gain maximum expected return with the risk not greater than 1.5, the investor should assign his money according to the optimal. The corresponding maximum expected return is 0.5.

The optimal solution of model (10) is

$$(0.3316, 0.0002, 0.2035, 0.2663, 0.1985),$$

and the value of objective function is 3.0000. This means that in order to minimize the risk with the expected value not less than 2, the investor should assign his money according to the optimal. The corresponding minimum risk is 3.

6. Conclusions

In this paper, uncertain variable is applied to portfolio selection problems, and two types of uncertain programming models for portfolio selection with uncertain returns are provided. In order to solve the proposed models by traditional methods we discuss the crisp equivalents when the uncertain returns are chosen to be some special uncertain variables and give one example to explain the efficiency of the method. The paper does not include the conditions when the return rates are general uncertain variables, this can be interesting areas for future researches.

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