Eventually Strong Wrpp Semigroups
Whose Idempotents Satisfy Permutation Identities

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Abstract
The aim of this paper is to study eventually strong wrpp semigroups whose idempotents satisfy permutation identities, that is, so-called PI-strong wrpp semigroups. After some properties are obtained, the structure of such semigroups are investigated. In particular, the structure of special cases are established.

Keywords: eventually strong wrpp, normal band, eventually PI-strong wrpp, spined product

1. Introduction and preliminaries
Let \( S \) be a semigroup, \( A \) a subset of \( S \) and let
\[
\sigma = \left( \begin{array}{cccc}
1 & 2 & \ldots & n \\
\sigma(1) & \sigma(2) & \ldots & \sigma(n)
\end{array} \right)
\]
A non-identity permutation on \( n \) objects. Then \( A \) is said to satisfy the permutation identity determined by \( \sigma \) (in short, to satisfy a permutation identity if there is no ambiguity) if
\[
(x_1 x_2 \ldots x_n)_{A} \quad (x_1 x_2 \ldots x_n)_{A} = x_{\sigma(1)} x_{\sigma(2)} \ldots x_{\sigma(n)},
\]
Where \( x_1, x_2, \ldots, x_n \) is the product of \( x_1, x_2, \ldots, x_n \) in \( S \) if \( A = S \), then \( S \) is called a PI-semigroup.

In connection with this, regular semigroup whose idempotents satisfy permutation identities were investigated by Yamada (1967, P.371). Strong wrpp semigroup whose idempotents satisfy permutation identities were investigated by Guo(1996, P.1947) Eventually strong wrpp semigroup whose idempotents satisfy permutation identities were studied by Du et al.(2001,P.424).

Du(2001, P.5) introduced the relation \( L^{(n)} \) which generalizes the relation \( L^{(1)} \), and the concept of eventually wrpp semigroup was introduced.

Let \( a, b \in S \). Then \( aL^{(n)} b \iff axRby \) for all \( x, y \in S \). we write as \( L_{a}^{(n)} \) with respect to \( L^{(n)} \) - class containing \( a \) for any \( a \in S \). Clearly, \( L^{(0)} \subseteq L^{(n)} \). In particular, we have \( L^{(1)} = L^{(n)} \) when \( S = S^{1} \).

We denote
\[
I_{a} = \{ e \in E(S) \} \quad \forall x \in S \) eax = ax, \quad \text{and} \quad xae = xa
\]
for all \( a \in E(S) \).

A semigroup is called eventually strong wrpp semigroup if each \( L_{a}^{(n)} \)-class of \( S \) contains an idempotent, and \( |L_{a}^{(n)} \cap I_{a}| = 1 \) for all \( a \in E(S) \). Here, denotes unique idempotent by \( a^{*} \).

A eventually strong wrpp semigroup \( S \) is called eventually PI-strong wrpp semigroup if idempotents of \( S \) satisfy a permutation identity.

Throughout this paper, the terminologies and notations are not defined can be found in Howie(1976).

Lemma 1.1 (Yamada, 1967, P.371-392) let \( B \) be a band. Then the following conditions are equivalent:
(1) \( B \) satisfy a permutation identity;
(2) \( B \) is a normal band;
(3) \( B \) is a strong semilattice of rectangular bands.
Lemma 1.2 (Tang, 1997, PP.1499-1504) Let \( Y \) be a semilattice, and \( S = [Y; S, \Phi_{*, \alpha}] \) a strong semilattice of semigroup \( S_0 \). If for any \( a \in S \), \( b \in S_{\alpha}(a, b) \in R \), then \( \alpha \rightarrow \beta \).

Definition 1.3 A semigroup \( S \) is called \( R \)-cancellative monoids, if for any \( a, b \in S \), \( (ab)c = (ca)b \) and \( a(c, b) \in R \Rightarrow (a, b) \in R \).

2. Some lemma

In what follows \( S \) is always a eventually strong semigroup whose idempotents satisfy permutation identity (*) . Let \( \sigma \in E(S) \), we let

\[
\sigma = \{ a \in S \mid a^* = e \}.
\]

Lemma 2.1 The following conditions hold:
1. A subsemigroups of \( S \) satisfy formula (*);
2. \( E(S) \) is a normal band;
3. \( L(S) \) or \( L(T) \) for any a subsemigroup \( T \) of \( S \);
4. \( ab = aa^*b = ab^*b = a^*b^*ab \) for any \( a, b \in S \).

Proof. Proof of (1) and (2) refer to Yamada (1967, PP. 371-392). (3) is trivial. We only show that (4). According to \( xax = xa, ax = xa \) and \( a = a^*a \), we have \( ab = ab^*a = ab^*b = ab^*b \).

Dually, we have \( ab = a^*b^*ab \).

Lemma 2.2 \( (ab)^* = a^*b^* \) holds for any \( a, b \in S \).

Proof. Let all \( x, y \in S \). Then

\[
abxRaby \Rightarrow a^*b^*b^*R^*a^*b^*b^* (aL^*a^*, a^*b = a^*b^*b^*)
\]

\[
\Rightarrow b^*a^*b^*y \quad (R \text{ is a left congruence})
\]

\[
\Rightarrow (b^*)^*a^*b^*R^*a^*b^*y \quad (S \text{ satisfies (*)})
\]

\[
\Rightarrow b^*a^*b^*y = b^*a^*b^*xR^*b^*a^*b^*y = ba^*b^*y \quad (S \text{ satisfies (**)})
\]

\[
\Rightarrow a^*b^*b^*R^*a^*b^*y
\]

\[
\Rightarrow \text{Consequently, } \alpha^*b^* \in L_0 \cap E(S) \text{. By lemma 2.1(4), we know that } (ab)^* = a^*b^*.
\]

Lemma 2.3 Let \( e \in E(S) \). Then \( S_e \) is an inflation of commutative \( R \)-cancellative monoids.

Proof. According to lemma 2.2, we know that \( S_e \) is a subsemigroup of \( S \). Noticed that

\[
eS_e = S \text{, so } S_e^2 = \{ ab \mid a, b \in S \} = \{ ab \mid a, b \in S \} \subseteq S_e
\]

Thus \( eS_e = S_e \iff e \in S_e \), so \( S_e^2 \) contains a identity element \( e \), and satisfies (*). For any \( a, b \in S_e \), we have

\[
ab = e^*b^*c^*e = e^*c = e^*a = a
\]

That is, \( S_e^2 \) is commutative. For any \( c \in S_e^2 \) such that \( eaeRcb \), by \( cL_e \), we have \( a = eaeRcb = b \). Thus \( S_e^2 \) is a \( R \)-left cancellative, so \( S_e^2 \) is a \( R \)-cancellative monoid. The mapping \( \phi_e \) defined by the following rule:

\[
\phi_e : S_e \rightarrow S_e, \quad x \mapsto ex
\]

Then for any \( x, y \in S_e \), we have \( xy = exy = exy = xy \phi_e \). So \( S_e \) is an inflation of commutative \( R \)-cancellative monoids \( S_e^2 \).

Lemma 2.4 Let \( B \) be a normal band, and \( \eta \) the least semilattice congruence on \( B \). Then \( B \) is a eventually PI- strong wrpp semigroup and \( L(B) = \eta \).

Proof. Let \( [Y; E_{a,b}] \) be a strong semilattice decomposition of \( B \) with structure homomorphism \( \xi \), where \( Y \) is a semilattice, and \( E_{a,b}(a \in Y) \) is a rectangular band. For any \( a \in B \), we have \( a \in L_0 \cap E_{a,b} \). If \( e \in E_{a,b} \), then we have \( eax = axe = xa \) and \( eae = ea = ea = ea = ea = ea = a \). Therefore, \( e = ae = a \), so \( a = a^* \). According to lemma 1.1, we know that \( B \) is a eventually PI- strong wrpp semigroup.
Let \( a \in E, b \in E (a, b \in Y) \), and \( aL' b \). By \( a = a \xi_{\alpha, \alpha} b \), we have \( b = b \xi_{\beta, \beta} b \). According to lemma 1.2, \( \beta = a \xi_{\alpha, \alpha} \). Similarly, \( \alpha = a \xi_{\alpha, \alpha} \). Therefore \( L''(B) \subseteq \eta \).

Conversely, let \( aLb \). Then \( \alpha = \beta \), and for any \( x \in E, y \in E \), we have

\[
axRy \iff a \xi_{\alpha, \alpha} x \xi_{\alpha, \alpha} R a \xi_{\alpha, \alpha} y \xi_{\alpha, \alpha} \iff a \cdot y = a \cdot \alpha
\]

\[
\iff b \xi_{\beta, \beta} x \xi_{\beta, \beta} R b \xi_{\beta, \beta} y \xi_{\beta, \beta}
\]

\[
\iff b \cdot x R b \cdot y
\]

Thus \( aL' b \), so \( \eta \subseteq L''(B) \).

Lemma 2.5 the following conditions are equivalent:

1. \( E(S) \) is a rectangular band;
2. \( S \) is \( L''(B) \)-single;
3. \( S \) is an inflation of the semidirect product of a commutative \( R \)-cancellative monoids and a rectangular band.

Proof. (1) \( \Rightarrow \) (2). Let \( E(S) \) be a rectangular band. For any \( a, b \in S \), we have

\[
axRy \iff a \cdot xRa \cdot y \quad (aL''(a))
\]

\[
\iff b \cdot a \cdot xRb \cdot a \cdot y \quad (R \text{ is a left congruence})
\]

\[
\iff b \cdot a \cdot x \cdot Rb \cdot a \cdot y \iff b \cdot x \cdot Rb \cdot y \quad \text{(by lemma 2.1(4))}
\]

\[
\iff b \cdot x \cdot Rb \cdot y \quad \text{(by lemma 2.1(4))}
\]

\[
\iff b \cdot x \cdot Rb \cdot y
\]

Therefore, \( aL' b \).

(2) \( \Rightarrow \) (3). Let \( S \) be \( L''(B) \)-single. By lemma 2.1 (2) and (3), we know that \( E(S) \) is a normal band, and \( L''(B) \lvert_{E(S)} = \omega \lvert_{E(S)} \). So \( E(S) \) is a rectangular band since \( L''(B) \lvert_{E(S)} \) is the least semilattice congruence on \( E(S) \). According to lemma 2.1(4) and lemma 2.2, we have \( S^i = \bigcup_{x \in E(S)} S^e \). The mapping defined by the following rule:

\[
\varphi : S \rightarrow S^i, \quad x \mapsto xx^*
\]

By lemma 2.3 and its proof, we know that definition of \( \varphi \) is good. For any \( a, b \in S \), we have \( ab = a \cdot \varphi b \cdot \varphi \). Thus \( S \) is an inflation of \( S^i \). We select a fixed \( e \in E(S) \). The mapping defined by the following rule:

\[
\Psi : S^i \rightarrow S^i \times E(S), \quad x \mapsto (exe, x^*)
\]

Let \( x, y \in S^i \) such that \( x \Psi = y \Psi \). Then \( x = y^* \) and \( x = x^* ex^* xx^* ex^* = x^* ey^* y^* = y \). Therefore \( \Psi \) is injective. For any \( (x, f) \in S^i \times E(S) \), we have \( (fx) \Psi = (x, f) \Psi \). Thus \( \Psi \) is surjective. On the other hand, for any \( x, y \in S^i \), we have

\[
(xy) \Psi = (exye, (xy)^*) = (exxe y e x^* y^*)
\]

\[
= (exxe y e x^* y^*)
\]

\[
= (ex y e x^* y^*)
\]

\[
= x^* y y \Psi
\]

Therefore, \( \Psi \) is an isomorphic mapping.

(3) \( \Rightarrow \) (1). It is trivial.

Lemma 2.6 Let \( \rho = \{(a, b) \in S \times S \mid a \cdot \eta b \cdot \eta \} \). Then \( \rho \) is a semilattice congruence on \( S \), and each \( \rho \)-class of \( S \) is a
Proof. Let \( \{(a, b) \mid \gamma a = \gamma b\} \). Then by lemma 2.2, we know that \( \gamma \) is a congruence of \( S \), and such that \( \gamma S \in J \subset S \). Let \( Q \) is the least semilattice congruence on \( /S \). We easily verify that left side graph is commutative. Therefore \( U \) is a semilattice congruence. Let \( T \) is a class of \( S \). Then \( (\cdot)/T \) is a rectangular band, and \( (\cdot)/T \in /S \). For any \( a \in T \), by lemma 3(3), we have (**). If for any \( (\cdot)/T \in /S \) such that (**), and for all \( x \in T \), then \( \cdot \in T \), and for any \( f \in T \), we have \( \cdot + f = f = ++(\cdot) = \cdot = f \). Thus, \( + = a \), so \( T \) is an eventually PI-strong wrpp semigroup.

By lemma 2.5, \( T \) is \( L \)-single.

We easily show that the following:

Lemma 2.7 the following conditions are equivalent:

1. \( T \) is a strong semilattice of the direct product of commutative \( R \) cancellative monoids and rectangular bands
2. \( T \) is a spined product of strong semilattice of commutative \( R \) cancellative monoids and normal band with respect to its greatest homomorphism image.

3. Main results and its proof

Next we give main results in this paper.

Theorem 3.1 the following conditions are equivalent:

1. \( S \) is an eventually PI-strong wrpp semigroup;
2. \( S \) is a strong semilattice of inflation on the direct product commutative \( R \) cancellative monoids and rectangular band;
3. \( S \) is an inflation of strong semilattice on the direct product commutative \( R \) cancellative monoids and rectangular band;
4. \( S \) is a inflation of spined product of strong semilattice of commutative \( R \) cancellative monoids and normal band with respect to its the common
Thus $\Phi_{\alpha, \beta}$ is a structure homomorphism, where $\alpha, \beta \in Y$, and $\alpha \geq \beta$.

(2) $\Rightarrow$ (3). Let $S = [Y ; S_a, \Phi_{a, b}]$ is a strong semilattice of $S$, where $\alpha, \beta \in Y$, $S_a$ is an inflation of the direct product of a commutative $R$-cancellative monoid $M_\alpha$ and a rectangular band $E_\alpha$. Then $S'_\alpha = M_\alpha \times E_\alpha$, and for $\beta \leq \alpha$, we have

$$ (M_\alpha \times E_\alpha) \Phi_{\alpha, \beta} = S'_\beta \Phi_{\alpha, \beta} \subseteq M_\beta \times E_\beta. $$

Therefore, $M = \bigcup_{\alpha \in Y} M_\alpha \times E_\alpha$ forms a subsemigroup of $S$, and

$$ M = [Y ; M_\alpha, \Phi_{\alpha, \beta} \mid M_\alpha \times E_\alpha \text{ and the mapping } E_\alpha \text{ onto } E_\beta], $$

respectively, identity element of $M_\alpha$ write as $e_\alpha$, and $M = [Y, M_\alpha, \Phi_{\alpha, \beta}, B = [Y, E_\alpha, \Psi_{\alpha, \beta}].$ According to lemma 9, $S = M \times B$. For any $x, y \in S_\alpha (\alpha \in Y)$

Thus $\Phi_{\alpha, \beta}$ is a structure homomorphism, where $\alpha, \beta \in Y$, and $\alpha \geq \beta$.

(4) $\Rightarrow$ (5). Let $S$ is an inflation of a strong semilattice of a commutative $R$-cancellative monoid $M$ and a normal band $\beta$ with respect to the mapping $\beta : S \rightarrow M, \beta x \rightarrow x^\beta$.

$$ xy = x \Phi_{\alpha, \beta} \Phi_{\beta, \alpha} = x \Phi_{\alpha, \beta} (x \Phi_{\beta, \alpha}) = (x \Phi_{\beta, \alpha}) y \Phi_{\beta, \alpha} $$

Therefore, $S$ is an inflation of $M$ with respect to the mapping $\beta : S \rightarrow M, \beta x \rightarrow x^\beta$.

$$ \Phi_{\alpha, \beta} = (k, e) \in M_\alpha \times E_\alpha, x \beta = (l, f) \in M_\beta \times E_\beta, b \beta = (m, g) \in M_\beta \times E_\beta, y \beta = (n, h) \in M_\beta \times E_\beta, e_\alpha \text{ is an identity element of } M_\alpha. $$

Then $axR\beta \Leftrightarrow (kl, ef) R(kn, eh) \Leftrightarrow (ke, ef) R(ke, n, eh) \Leftrightarrow (e, k, ef) R(e, n, eh)$.

Thus $\beta ax = \beta a \lambda \Leftrightarrow \alpha \lambda \gamma \Leftrightarrow \alpha \gamma \Leftrightarrow \alpha \lambda \gamma$ is a cancellative monoid.

$$ \Psi_{\alpha, \beta}(a, b, x, y) \in S, $$

such that $a \beta x = \beta a \lambda \Leftrightarrow \alpha \lambda \gamma \Leftrightarrow \alpha \gamma \Leftrightarrow \alpha \lambda \gamma$ is a cancellative monoid.

Let $\psi_{\alpha} \in \Phi_{\alpha, \beta}$, and $x \beta = (l, f) \in M_\beta \times E_\beta, b \beta = (m, g) \in M_\beta \times E_\beta, y \beta = (n, h) \in M_\beta \times E_\beta, e_\alpha \text{ is an identity element of } M_\alpha. $
$R$ – cancellative monoid and $B$ is a normal band, we obtain

$$axyb = (klmn, egh) = (klmn, egh) = ayxb.$$

Therefore, (5) holds.

Definition 3.2 A wrpp semigroup $S$ is called to satisfy conditions (L) if there exist a unique $e \in L^+ \cap E(S)$ such that $eae = a$ for all $a \in S$.

Corollary 3.3 An eventually PI-strong semigroup $S$ is a semigroup with satisfying the conditions (L) and satisfying the permutation identity (*).

If and only if $S_1 = S$.

Proof. Let $S$ be an eventually PI-strong wrpp semigroup. By theorem 3.1, we know that $S$ is an inflation of the spined product of a commutative $R$-cancellative monoid $M$ and a normal band $B$ with respect to its common greatest semilattice homomorphism image $Y$. If $S_1 = S$, then $S = M \times B$. According to the proof of theorem 3.1, for any $(k,e) \in M \times E \subseteq S, (k,e) = (e,k) = (k,e)$ and $(k,e)(k,e) = (k,e)$, we can imply that $S$ is a semigroup with satisfying the conditions (L) and satisfying the permutation identity (*). Conversely, if $S$ is a semigroup with satisfying the conditions (L) and satisfying the permutation identity (*), then we have $a^2 a = a$ for any $a \in S$. Therefore $S_1 = S$.

Corollary 3.4 The following conditions are equivalent:

1. $S$ is a semigroup with satisfying the conditions (L) and satisfying the permutation identity (*);
2. $S$ is an eventually PI-strong wrpp semigroup, and $S_1 = S$;
3. $S$ is a strong semilattice of the direct product of a commutative $R$-cancellative monoid and a rectangular band;
4. $S$ is a spined product of a strong semilattice of commutative $R$-cancellative monoid and a normal band;
5. $S$ is a semigroup with satisfying the conditions (L) and satisfying a permutation identity $x_1 x_2 x_3 x_4 = x_1 x_2 x_3 x_4$.

References


