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Strongly Convergence Theorem of m-accretive Operators

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Abstract

Let X be a real Banach space, we will introduce a modifieations of the Mann iterations in a uniformly smooth Banach space, $x_{n+1} = \alpha_n(\lambda u + (1-\lambda)x_n) + (1-\alpha_n)J_r x_n$, Where $\{\alpha_n\}, \lambda$ satisfied some conditions, then we will prove the strongly convergence of the sequence $\{x_n\}$ to a zero of accretive operators. This theorem extend (Kim and Xu, Nonlinear Analysis) results. (Kim, 2005, pp.51-60)

Keywords: Strong convergence, Non-expansive mapping, Accretive operator

1. Preliminaries

Recall that a (possibly multivalued) operator A with domain *D(A)* and *R(A)* in X is accretive if for each $x_i \in D(A)$, and $y_i \in Ax_i$ (i=0,1,2), there exits a $j \in J(x_2 - x_1)$ such that $(y_2 - y_1, j) \geq 0$. An accretive operator A is m-accretive if $R(I+rA)=X$, for each $r>0$. The set of zero of A is denoted by F. Hence $F = \{z \in D(A) : 0 \in A(z)\} = A^{-1}(0)$. We denote by J_r the resolvent of A. Note that if A is m-accretive then $J_r : X \to X$ is nonexpansive and F(J_r)=F.

In order to prove our results, we need the following lemmas:

Lemma 1: Assume $\{\alpha_n\}$ is a sequence of nonnegative numbers such that (Xu, 2002, pp.240-256)

$\alpha_{n+1} \leq (1-\gamma_n)\alpha_n + \gamma_n \delta_n, \quad n \geq 0$, where $\{\gamma_n\}$ is a sequence in (0,1) and $\{\delta_n\}$ is a sequence in □ such that

(i) $\sum_{i=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n\to\infty} \delta_n \leq 0$ or $\sum_{i=1}^{\infty} \gamma_n |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Lemma 2: Let X be a smooth real Banach space and let J be the duality map of X. Then there holds the inequality

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle x, J(x+y)\rangle, \quad x, y \in X.$$

Lemma 3: Let $\{x_n\}$ and $\{y_n\}$ be bounded sequence in a Banach space X such that (Suzuki, 2005, pp.103-123)

$x_{n+1} = \gamma_n x_n + (1-\gamma_n) y_n$, $n \geq 0$. Where $\{\gamma_n\}$ is a sequence in [0,1] such that $0 < \liminf_{n\to\infty} \gamma_n \leq \limsup_{n\to\infty} \gamma_n < 1$, assume

$$\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} + x_n\|) \leq 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

2. Main results

Theorem: Let X be a uniformly smooth Banach space, C a closed convex subset of X and $T : C \to C$ a nonexpansive mapping such that $Fix(T) \neq \phi$. For any given sequence $\{\alpha_n\}$ in [0,1], $\lambda \in (0,1)$ and $u, x_0 \in C$, defined $\{x_n\}$ by the iterative algorithm $x_{n+1} = \alpha_n(\lambda u + (1-\lambda)x_n) + (1-\alpha_n)J_{r_n}x_n$, (2.1)

(i) $\lim_{n\to\infty} \alpha_n = 0$ (ii) $\sum_{i=0}^{\infty} \alpha_n = \infty$. Then $\{x_n\}$ convergens strongly to a zero of A.

Proof: We first show that $\{x_n\}$ is bounded. Indeed, take a point $p \in Fix(T)$ to get, using the nonexpansive of J_r,

$$\begin{aligned}\|x_{n+1} - p\| &\le \alpha_n\lambda\|u-p\| + \alpha_n(1-\lambda)\|x_n - p\| + (1-\alpha_n)\|J_r x_n - p\| \\ &\le \alpha_n\lambda\|u-p\| + \alpha_n(1-\lambda)\|x_n - p\| + (1-\alpha_n)\|x_n - p\| \\ &\le \alpha_n\lambda\|u-p\| + (1-\alpha_n\lambda)\|x_n - p\|\end{aligned}$$

So we have for all $n \ge 0$, $\|x_n - p\| \le \max\{\|u-p\|, \|x_0 - p\|\}$, (2.2)

So $\{x_n\}$ is bounded, and also $\{J_{r_n}x_n\}$. We have

$$\|x_{n+1} - J_{r_n}x_n\| = \alpha_n\|\lambda u + (1-\lambda)x_n - J_{r_n}x_n\| \to 0 \ (n\to\infty) \tag{2.3}$$

From (2.1) we have $x_{n+1} = \alpha_n(1-\lambda)x_n + \alpha_n\lambda u + (1-\alpha_n)J_r x_n$

$$= \alpha_n(1-\lambda)x_n + [1-\alpha_n(1-\lambda)]\frac{\alpha_n\lambda u + (1-\alpha_n)J_r x_n}{1-\alpha_n(1-\lambda)}$$

We set $\gamma_n = 1-\alpha_n(1-\lambda)$ and $y_n = \frac{\alpha_n\lambda}{\gamma_n}u + \frac{1-\alpha_n}{\gamma_n}J_r x_n$

Then we get $x_{n+1} = (1-\gamma_n)x_n + \gamma_n y_n$ (2.4)

It easily to get $\{y_n\}$ (since $\alpha_n \to 0, \gamma_n \to 1$ and $\{x_n\}$ is bounded).

$$y_{n+1} - y_n = \left(\frac{\alpha_{n+1}}{\gamma_{n+1}} - \frac{\alpha_n}{\gamma_n}\right)\lambda u + \frac{1-\alpha_{n+1}}{\gamma_{n+1}}(J_r x_{n+1} - J_r x_n) + \left(\frac{1-\alpha_{n+1}}{\gamma_{n+1}} - \frac{1-\alpha_n}{\gamma_n}\right)J_r x_n$$

Since J_r is nonexpansive and $\frac{1-\alpha_{n+1}}{\gamma_{n+1}}$ <1, we get

$$\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \le \left(\frac{\alpha_{n+1}}{\gamma_{n+1}} - \frac{\alpha_n}{\gamma_n}\right)\lambda\|u\| + \left(\frac{1-\alpha_{n+1}}{\gamma_{n+1}} - \frac{1-\alpha_n}{\gamma_n}\right)\|J_r x_n\|$$

Since $\alpha_n \to 0, \gamma_n \to 1$, so $\limsup_{n\to\infty}\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\| \le 0$

Applying Lemma 3, we have $\lim_{n\to\infty}\|y_n - x_n\| = 0$, (2.5)

since $\alpha_n \to 0$, we get $\lim_{n\to\infty}\|y_n - J_r x_n\| = \lim_{n\to\infty}\frac{\alpha_n\lambda}{\gamma_n}\|u - J_r x_n\| = 0$ (2.6)

from (2.5) and (2.6) we get $\lim_{n\to\infty}\|x_n - J_r x_n\| = 0$ (2.7)

since z_t is the unique solution to the equation (1.1), we can write

$$z_t - x_n = (1-t)(J_r z_t - x_n) + t(u - x_n)$$

Apply lemma 2 to get

$$\|z_t - x_n\|^2 \le (1-t)^2 \|J_r z_t - x_n\|^2 + 2t\langle u - z_n, J(z_t - x_n)\rangle$$

$$\le (1-t)^2 \|J_r z_t - x_n\|^2 + \|J_r x_n - x_n\|(2\|z_t - x_n\| + \|J_r x_n - x_n\|)$$

$$+2\langle u - z_t, J(z_t - x_n)\rangle + 2t\|z_t - x_n\|^2 \tag{2.8}$$

Since $\{z_t\},\{x_n\}$ are bounded, there exists a constant M>0,

$$\langle u - z_t, J(x_n - z_t)\rangle \le (t + \frac{\|J_r x_n - x_n\|}{t})M \tag{2.9}$$

So we get $\limsup\limits_{n\to\infty}\langle u - z_t, J(x_n - z_t)\rangle \le tM$ (2.10)

Put $p = Q(u) = s - \lim\limits_{t\to 0} z_t$

Since J is uniformly continuous on bounded sets of X, we find that

$\|J(x_n - z_t) - J(x_n - p)\| \le \varepsilon_t$, uniformly on n. (2.11)

Where $\varepsilon_t > 0$ and $\lim\limits_{t\to 0}\varepsilon_t = 0$, Let $\beta > 0$ satisfy $\|u - z_t\| \le \beta$ and $\|z_t - p\| \le \beta$

For all $t \in (0,1)$ and n. We have

$$\langle u - p, J(x_n - p)\rangle = \langle u - z_t, J(x_n - z_t)\rangle + \langle u - z_t, J(x_n - p) - J(x_n - z_t)\rangle + \langle z_t - p, J(x_n - p)\rangle$$

$$\le \langle u - z_t, J(x_n - p)\rangle + \beta(\varepsilon_t + \|z_t - p\|)$$

It follows (2.10) that, for all $t \in (0,1)$,

$$\limsup_{n\to\infty}\langle u - p, J(x_n - p)\rangle \le tM + \beta(\varepsilon_t + \|z_t - p\|)$$

Let $t \to 0$, we can get $\limsup\limits_{n\to\infty}\langle u - p, J(x_n - p)\rangle \le 0$ (2.12)

Finally, we claim that $x_n \to p$ in norm. Apply Lemma 2 to get

$$\|x_{n+1} - p\|^2 = \|\alpha_n(1-\lambda)(x_n - p) + (1-\alpha_n)(J_r x_n - p) + \alpha_n\lambda(u - p)\|^2$$

$$\le \|\alpha_n(1-\lambda)(x_n - p) + (1-\alpha_n)(J_r x_n - p)\|^2 + 2\alpha_n\lambda\langle u - p, J(x_{n+1} - p)\rangle$$

$$\le (1-\alpha_n)\lambda\|x_n - p\|^2 + 2\alpha_n\lambda\langle u - p, J(x_{n+1} - p)\rangle \tag{2.13}$$

Applying lemma 1, and from (2.12) and (2.13) we conclude that $\|x_n - p\| \to 0$.

So the proof is completed.

References

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