Geometric Condition of Singularity of $S^2_3(Δ^2_{MS})$

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Abstract
The aim of this paper is to investigate the geometric condition of singularity of $S^2_3(Δ^2_{MS})$. The algebraic of singularity of $S^2_3(Δ^2_{MS})$ is obtained in (Luo and Chen, 2005). The result of this paper will be useful to further study the geometric condition of singularity of $S^\mu_{\mu+1}(Δ^\mu_{MS})(\mu > 3)$.

Keywords: Singularity, Spline space, Geometric condition

1. Introduction
The definition of multivariate spline is stated as follows (Wang, 1994): for a given partition $Δ$ of a region $Ω$, the linearspace

$$S^\mu_k(Δ) := \{s|s|_{T_i} \in P_k, s \in C^\mu(Ω), \forall T_i \in Δ\}$$

is called spline space of degree $k$ with smoothness $\mu$, where $T_i$ is a cell of the $Δ$ and $P_k$ is the polynomial space of total degree $\leq k$.

Luo & Chen (Luo and Chen, 2005) investigated the singularity of the space $S^\mu_{\mu+1}(Δ^\mu_{MS})(\mu \geq 2)$ and gave out an algebraic necessary and sufficient condition to the singularity. Take $\mu = 1$ for instance, i.e. Morgan-Scott triangulation. Shi (Shi, 1991) and Diener (Diener, 1990) obtained the geometric significance of the necessary and sufficient condition of $\dim(S^1_2(Δ^1_{MS})) = 7$, respectively. Du (Du, 2003) gave another type of the necessary and sufficient condition of the singularity of $S^2_3(Δ^2_{MS})$ from the viewpoint of the projective geometry, that is, if the six quasi-inner edges are regarded as six points in the projective plane then they lie on a conic.

Now, we research the condition of $\mu = 2$.

2. Algebraic of Singularity of $S^2_3(Δ^2_{MS})$

The singularity of the spline space $S^2_3(Δ^2_{MS})$ is investigated by Luo and Chen (Luo and Chen, 2005) using the Generation Basis method. They obtained a necessary and sufficient condition in algebraic form. $Δ^2_{MS}$ is seen in Figure 1.

Denoted by
Then, the following conclusion in algebraic form is true

**Theorem 1.** (Luo and Chen, 2005) The spline space \( S_3^2(\Delta_{MS}^2) \) is singular if and only if

\[
\frac{a_1a_2a_3}{b_1b_2b_3} + \frac{a_4a_5a_6}{b_4b_5b_6} + \frac{a_7a_8a_9}{b_7b_8b_9} = -1
\]

(2)

Let \( a, b, c \) be three distinct non-infinity lines in \( P_2 \). Denoted by the intersection points between lines \( a, b, c \) and \( l_i (i = 1, 2, 3), l_j (i = 4, 5, 6), l_k (i = 7, 8, 9) \) respectively. \( u = b > v = c, w = a > \)

Let \( l'_1, l'_2, l'_3 \) and \( l'_6 \) be

\[
l'_1 = b w + a_3 v, l'_2 = b w + a_4 v, l'_3 = b v + a_5 u.
\]

Without loss of generality, we assume that the six points determined by intersections of \( Aa, Bb, Cc \) and intersections of \( l'_1, l'_2, l'_3, l'_6 \) are distinct from each other in the triangulation. Under this assumption, we shall prove the following important conclusion.

**Theorem 2.** The spline space \( S_3^2(\Delta_{MS}^2) \) is singular if and only if the six points determined by intersections of \( Aa, Bb, Cc \) and intersections of \( l'_1, l'_2, l'_3, l'_6 \) lie on a conic.

**Proof:** Without loss of generality, we regard the lines \( u, v, w \) as basic lines, and let \( u = (1, 0, 0), v = (0, 1, 0), w = (0, 0, 1) \).

From (1), we have

\[
l_1 = (a_1, b_1, 0), \quad l_4 = (0, a_4, b_4), \quad l_7 = (b_7, 0, a_7),
\]

\[
l_3 = (a_3, b_3, 0), \quad l_6 = (0, a_6, b_6), \quad l_9 = (b_9, 0, a_9),
\]

\[
l'_2 = (b_2, a_2, 0), \quad l'_5 = (0, b_5, a_5), \quad l'_8 = (a_8, 0, b_8),
\]

and

\[
A = l_1 \times l_9 = (b_1a_9 - a_1b_9, b_1b_9 - a_1b_9, b_1a_9 - b_1a_9),
\]

\[
B = l_6 \times l_7 = (a_6a_7, b_6b_7 - a_6b_7, b_6a_7 - a_6b_7),
\]

\[
C = l_5 \times l_4 = (b_5a_4 - a_5a_4, a_5a_4 - a_5a_4, b_5a_4 - b_5a_4)
\]

\[
a = w \times v = (1, 0, 0), \quad b = u \times w = (0, 0, 1), \quad c = u \times v = (0, -1, 0).
\]

So the lines \( Aa, Bb, Cc \) can be expressed as follows:

\[
Aa = A \times a = (0, -b_1b_9, a_9a_1), \quad Bb = B \times b = (b_1b_9, -a_9a_1, 0), \quad Cc = C \times c = (a_1a_4, 0, -b_3b_4).
\]

By direct calculations, the intersections of \( Aa, Bb, Cc \) and the intersections of \( l'_1, l'_2, l'_3, l'_6 \) are formed to be

\[
v_1 = Aa \times Bb = (a_1a_9a_4a_5a_6, a_1b_9b_4b_5b_6, b_1b_9b_4b_5b_6),
\]

\[
v_2 = Bb \times Cc = (a_1a_9a_4a_5a_6, a_1b_9b_4b_5b_6, b_1b_9b_4b_5b_6),
\]

\[
v_3 = Cc \times Aa = (b_1b_9b_4b_5b_6, -a_9a_1a_4a_5a_6, -a_9b_9b_4b_5b_6),
\]

\[
v_4 = l'_1 \times l'_5 = (a_2a_5, b_2a_5, -b_2a_5),
\]

\[
v_5 = l'_2 \times l'_3 = (b_2b_5, a_2a_5, -b_2a_5),
\]

\[
v_6 = l'_8 \times l'_2 = (b_5a_4, b_5b_4, a_5a_4),
\]

We now give the equivalent condition that \( v_1, v_2, \ldots, v_6 \) lie on a conic by Pascal’s Theorem. To do this, the three intersection points of three subtense of the hexagon with vertices \( v_1, v_2, \ldots, v_6 \) are

\[
B_1 = (v_1 \times v_3) \times (v_2 \times v_6) = (b_1b_2b_3b_4b_5b_6 + a_1b_2a_3a_4a_5a_6, b_1b_2b_3b_4a_6a_7b_8 + a_2b_2b_3b_4b_5b_6),
\]

\[
-(a_1a_3a_5a_6a_7a_8a_9 - a_1b_5b_7b_9a_9)(-a_2a_5a_4a_6a_7b_8 - a_2b_5b_7b_9a_9)(a_1a_3a_5a_6a_7a_8a_9 -
\]

\[
(a_1a_3a_5a_6a_7a_8a_9 - a_1b_5b_7b_9a_9)(-a_2a_5a_4a_6a_7b_8 - a_2b_5b_7b_9a_9)(a_1a_3a_5a_6a_7a_8a_9 -
\]
\[ a, b, c, d, e, f \] \begin{eqnarray*}
\frac{a_1 a_2 a_3 \cdot a_4 a_5 a_6 \cdot a_7 a_8 a_9}{b_1 b_2 b_3 \cdot b_4 b_5 b_6 \cdot b_7 b_8 b_9} &=& -1
\end{eqnarray*}

4. Example

In this section, we shall give two examples to illustrate our main results distinctly. One of the conic is elliptic conic, the other is hyperbolic conic.

Example 1. Consider a given triangulation shown in Fig. 2, where

\[ A = (1/2, 2, 1), B = (-4, -2, 1), C = (4, -2, 1), a = (0, -1, 1), b = (1, 0, 1), \]
\[ c = (-1, 0, 1), u : -y = 0, v : -x - y - z = 0, w : x - y - z = 0, \]
\[ l_1 : -4x - y + 4/3z = 0, l_2 : -2/3x - y + 2/3z = 0, l_3 : -1/4x - y - z = 0, \]
\[ l_4 : 1/4x - y - z = 0, l_5 : 2/3x - y + 2/3z = 0, l_6 : 4/3x - y + 4/3z = 0, \]
\[ l_7 : 2x - y - 2z = 0, l_8 : -1/2x - y - z = 0, l_9 : -42/47x - y - 42/47z = 0. \]

It can be proved that the spline space \( \mathcal{S}_3^3(\Delta_{MS}) \) of piecewise polynomial of degree 3 with smoothness 2 is singular.

The corresponding \( l'_1, l'_2, l'_3 \) are
In this example, the conic corresponding to Theorem 2 is

$$302x^2 - 2861xy + 15800y^2 - 1421x + 12880y + 2624 = 0$$

which forms elliptic conic and is shown in Fig. 2.

**Example 2.** The following example shows that a conic corresponding to Theorem 2 forms hyperbolic conic. Given a singular triangulation $\Delta_{MS}^2$ for spline space of piecewise polynomial of degree 3 with smoothness shown in Fig. 3, where

$$A = (1/2,2,1), \quad B = (-4,-3,1), \quad C = (3,-4,1), \quad a = (0,-2,1), \quad b = (1,-1,1),$$

$$c = (-1,0,1), \quad u : -1/2x - y - 1/2z = 0, \quad v : -2x - y - 2z = 0, \quad w : x - y - 2z = 0,$$

$$l_1 : -6x - y + 5z = 0, \quad l_2 : -3/2x - y + 1/2z = 0, \quad l_3 : -2/3x - y - 2z = 0,$$

$$l_4 : 1/2x - y - 2z = 0, \quad l_5 : x - y + z = 0, \quad l_6 : 4/3x - y + 4/3z = 0,$$

$$l_7 : 1/7x - y - 8/7z = 0, \quad l_8 : 2x - y - 2z = 0, \quad l_9 : -38/61x - y - 38/61z = 0$$

The corresponding $l'_7, l'_8, l'_9$ are

$$l'_7 = 5/14x - 19/14 - y = 0, \quad l'_8 = -3x - 2 - y = 0, \quad l'_9 = -229/122x - 229/122 - y = 0,$$

and the corresponding duality figure of the triangulation is shown in Fig. 3. A hyperbolic curve passing through the six points as mentioned Theorem 2 is shown in 3 and

$$\frac{1143}{2803}x^2 + \frac{1028}{1963}y^2 - \frac{1022}{715}xy - \frac{503}{256}xz + \frac{1161}{1112}yz + z = 0$$

**References**


Figure 1. $\Delta^2_{MS}$

Figure 2. Example 1