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Geometric Condition of Singularity of $S_3^2(\Delta_{MS}^2)$

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Abstract

The aim of this paper is to investigate the geometric condition of singularity of $S_3^2(\Delta_{MS}^2)$. The algebraic of singularityof $S_3^2(\Delta_{MS}^2)$ is obtained in (Luo and Chen, 2005). The result of this paper will be useful to further study the geometric condition of singularity of $S_{\mu+1}^{\mu}(\Delta_{MS}^{\mu})(\mu > 3)$.

Keywords: Singularity, Spline space, Geometric condition

1. Introduction

The definition of multivariate spline is stated as follows(Wang, 1994): for a given partition Δ of a region Ω, the linearspace

$$S_k^{\mu}(\Delta) := \{s \mid s|_{T_i} \in P_k, s \in C^{\mu}(\Omega), \forall T_i \in \Delta\}$$

is called spline space of degree k with smoothness μ, where T_i is a cell of the Δ and P_k is the polynomial space of total degree $\leq k$.

Luo & Chen(Luo and Chen, 2005) investigated the singularity of the space $S_{\mu+1}^{\mu}(\Delta_{MS}^{\mu})(\mu \geq 2)$ and gave out an algebraic necessary and sufficient condition to the singularity. Take $\mu = 1$ for instance, i.e. Morgan-Scott triangulation. Shi(shi,1991) and Diener(Diener,1990) obtained the geometric significance of the necessary and sufficient condition of $\dim(S_2^1(\Delta_{MS})) = 7$, respectively. Du(Du, 2003) gave another type of the necessary and sufficient condition of the singularity of $S_2^1(\Delta_{MS})$ from the viewpoint of the projective geometry, that is, if the six quasi-inner edges are regarded as six points in the projective plane then they lie on a conic.

Now, we research the condition of $\mu = 2$.

2. Algebraic of Singularity of $S_3^2(\Delta_{MS}^2)$

The singularity of the spline space $S_3^2(\Delta_{MS}^2)$ is investigated by Luo and Chen(Luo and Chen, 2005) using the Generation Basis method. They obtained a necessary and sufficient condition in algebraic form. Δ_{MS}^2 is seen in Figure 1.

Denoted by

$$\begin{cases} l_1 = a_1 u + b_1 w \\ l_2 = a_2 u + b_2 w, \\ l_3 = a_3 u + b_3 w \end{cases} \begin{cases} l_4 = a_4 w + b_4 v \\ l_5 = a_5 w + b_5 v \\ l_6 = a_6 w + b_6 v \end{cases} \text{and} \begin{cases} l_7 = a_7 v + b_7 u \\ l_8 = a_8 v + b_8 u \\ l_9 = a_9 v + b_9 u \end{cases} \quad (1)$$

Then, the following conclusion in algebraic form is true

Theorem 1. (LuoandChen,2005) The spline space $S_3^2(\Delta_{MS}^2)$ is singular ($\dim(S_3^2(\Delta_{MS}^2)) = 11$) if and only if

$$\frac{a_1 a_2 a_3}{b_1 b_2 b_3} \cdot \frac{a_4 a_5 a_6}{b_4 b_5 b_6} \cdot \frac{a_7 a_8 a_9}{b_7 b_8 b_9} = -1 \quad (2)$$

Let a, b, c be three distinct non-infinity lines in P_2. Denoted by the intersection points between lines a, b, c and $l_i (i = 1,2,3), l_j (i = 4,5,6), l_i (i = 7,8,9)$ respectively. $u =< b, c >.,v =< c, a >, w =< a, b >$

Let l_2', l_5' and l_8' be

$$l_2' = b_2 u + a_2 w, l_5' = b_5 w + a_5 v, l_8' = b_8 v + a_8 u .$$

Without loss of generality, we assume that the six points determined by intersections of Aa, Bb, Cc and intersections of l_2', l_5', l_8' are distinct from each other in the triangulation . Under this assumption, we shall prove the following important conclusion.

Theorem 2. The spline space $s_3^2(\Delta_{MS}^2)$ is singular if and only if the six points determined by intersections of Aa, Bb, Cc and intersections of l_2', l_5', l_8' lie on a conic.

Proof: Without loss of generality, we regard the lines u, v, w as basic lines, and let u = (1, 0, 0), w = (0, 1, 0), v = (0, 0, 1).

From (1), we have

$$l_1 = (a_1, b_1, 0) \quad l_4 = (0, a_4, b_4) \quad l_7 = (b_7, 0, a_7)$$

$$l_3 = (a_3, b_3, 0), \quad l_6 = (0, a_6, b_6), \text{ and } l_9 = (b_9, 0, a_9)$$

$$l_2' = (b_2, a_2, 0), \quad l_5' = (0, b_5, a_5), \quad l_8' = (a_8, 0, b_8)$$

and

$$A = l_1 \times l_9 = (b_1 a_9, -a_1 a_9, -b_1 b_9), B = l_6 \times l_7 = (a_6 a_7, b_6 b_7 - a_6 b_7), C = l_3 \times l_4 = (b_3 b_4, -a_3 a_4, a_3 a_4)$$

$$a = w \times v = (1,0,0), b = u \times w = (0,0,1), c = u \times v = (0,-1,0).$$

So the lines Aa, Bb and Cc can be expressed as follows:

$$Aa = A \times a = (0, -b_1 b_9, a_1 a_9), Bb = B \times b = (b_6 b_7, -a_6 a_7, 0), Cc = C \times c = (a_3 a_4, 0, -b_3 b_4).$$

By direct calculations, the intersections of Aa, Bb, Cc and the intersections of l_2', l_5', l_8' are formed to be

$$v_1 = Aa \times Bb = (a_1 a_9 a_6 a_7, a_1 a_9 b_6 b_7, b_1 b_9 b_6 b_7), \quad v_2 = Bb \times Cc = (a_6 a_7 b_3 b_4, b_6 b_7 b_3 b_4, a_6 a_7 a_3 a_4),$$

$$v_3 = Cc \times Aa = (-b_3 b_4 b_1 b_9, -a_3 a_4 a_1 a_9, -a_3 a_4 b_1 b_9), \quad v_4 = l_2' \times l_5' = (a_2 a_5, -b_2 a_5, b_2 a_5)$$

$$v_5 = l_5' \times l_8' = (b_5 b_8, a_5 a_8, -b_5 a_8), v_6 = l_8' \times l_2' = (-b_8 a_2, b_8 b_2, a_8 a_2)$$

We now give the equivalent condition that $v_1, v_2, \cdots, v_6$ lie on a conic by Pascal's Theorem. To do this, the three intersection points of three subtense of the hexagon with vertices $v_1, v_2, \cdots, v_6$ are

$$B_1 = (v_1 \times v_5) \times (v_2 \times v_6) = (b_1 b_5 b_6 b_7 b_8 b_9 + a_1 b_5 a_6 a_7 a_8 a_9)(b_2 b_3 b_4 a_6 a_7 b_8 + a_2 b_3 b_4 b_6 b_7 b_8)$$
$$- (a_1 a_5 a_6 a_7 a_8 a_9 - a_1 b_5 b_6 b_7 b_8 a_9)(-a_2 a_3 a_4 a_6 a_7 b_8 - a_2 b_3 b_4 a_6 a_7 a_8)(a_1 a_5 a_6 a_7 a_8 a_9 -$$

$$a_1b_5b_6b_7b_8a_9)(a_2b_3b_4b_6b_7a_8 - b_2a_3a_4a_6a_7b_8) - (-a_1b_5b_6b_7a_8a_9 - b_1a_5b_6b_7a_8b_9)$$

$$(a_6a_7b_3b_4b_8b_2 + b_6b_7b_3b_4b_8a_2)(-a_1a_9b_6b_7b_5a_8 - b_1b_9b_6b_7a_5a_8)(-a_6a_7a_3a_4a_2b_8$$
$$-a_6a_7b_3b_4a_2a_8) - (b_1b_9b_6b_7b_5b_8 + a_1a_9a_6a_7b_5a_8)(b_6b_7b_3b_4a_8a_2 - a_6a_7a_3a_4b_2b_8),$$

$$B_2 = (v_1 \times v_4) \times (v_3 \times v_6) = (b_1a_2a_5b_6b_7b_9 - a_1b_2b_5a_6a_7a_9)(-b_1b_2b_3b_4b_8b_9 - a_1a_2a_3a_4b_8a_9)$$
$$-(-a_1b_2a_5a_6a_7a_9 - a_1a_2a_5b_6b_7a_9)(b_1a_2a_3a_4b_8b_9 + b_1a_2b_3b_4a_8b_9)(-a_1b_2a_5a_6a_7a_9 -$$

$$a_1a_2a_5b_6b_7a_9)(-a_1a_2a_3a_4a_8a_9 + b_1b_2a_3a_4b_8b_9) - (a_1b_2b_5b_6b_7a_9 + b_1b_2a_5b_6b_7a_8b_9)$$

$$(-b_3b_4b_1b_9b_8b_2 - a_3a_4a_1a_9b_8a_2)(a_1a_9b_6b_7b_2b_5 + b_1b_9b_6b_7b_2a_5)(a_3a_4b_1b_9b_8a_2$$
$$+ b_3b_4b_1b_9a_8a_2) - (b_1b_9b_6b_7a_2a_5 - a_1a_9a_6a_7b_2b_5)(-a_3a_4a_1a_9a_8a_2 + a_3a_4b_1b_9b_8b_2),$$

$$B_3 = (v_2 \times v_4) \times (v_3 \times v_5) = (a_6a_7a_3a_4a_2a_5 - a_6a_7b_3b_4b_2b_5)(-b_3b_4b_1b_9a_5a_8 + a_3a_4a_1a_9b_5b_8)$$
$$-(-a_6a_7b_3b_4b_2a_5 - b_6b_7b_3b_4a_2a_5)(-a_3a_4b_1b_9b_5b_8 - b_3b_4b_1b_9b_5b_8)(-a_6a_7b_3b_4b_2a_5$$

$$-b_6b_7b_3b_4a_2a_5)(a_3a_4a_1a_9b_5a_8 + a_3a_4b_1b_9a_5a_7) - (b_6b_7b_3b_4b_2b_5 + a_6a_7a_3a_4b_2a_5)$$

$$(-b_3b_4b_1b_9a_5a_8 + a_3a_4a_1a_9b_5b_8)(b_6b_7b_3b_4b_2b_5 + a_6a_7a_3a_4b_2a_5)(-a_3a_4b_1b_9b_5b_8$$
$$-b_3b_4b_1b_9b_5a_8) - (a_6a_7a_3a_4a_2a_5 - a_6a_7b_4b_3b_2b_5)(a_3a_4a_1a_9b_5b_8 + a_3a_4b_1b_9a_5a_8)$$

The directed area of triangle determined by B_1, B_2 and B_3 is

$$(B_1, B_2, B_3) = -(b_5b_8b_2 + a_2a_5a_8)^2(b_7b_6b_2 + b_2a_6a_7)(b_5b_1b_9 + a_1a_5a_9)(b_3a_8b_4 + a_3a_4b_8)$$

$$(b_1b_3b_4b_6b_7b_9 - a_1a_3a_4a_6a_7a_9)^2(b_1b_2b_3b_4b_5b_6b_7b_8b_9 + a_1a_2a_3a_4a_5a_6a_7a_8a_9)$$

Since the six points $v_1, v_2, \cdots, v_6$ are all distinct, we have

$$-(b_5b_8b_2 + a_2a_5a_8)^2(b_7b_6b_2 + b_2a_6a_7)(b_5b_1b_9 + a_1a_5a_9)(b_3ba_8b_4 + a_3a_4b_8)$$

$$(b_1b_3b_4b_6b_7b_9 - a_1a_3a_4a_6a_7a_9)^2 \neq 0.$$

Hence, it follows from Pascal's Theorem (says that $v_1, v_2, \cdots, v_6$ lie on a conic if an only if $(B_1, B_2, B_3) = 0$ that the necessary and sufficient condition that $v_1, v_2, \cdots, v_6$ lie on a conic is

$$\frac{a_1a_2a_3}{b_1b_2b_3} \cdot \frac{a_4a_5a_6}{b_4b_5b_6} \cdot \frac{a_7a_8a_9}{b_7b_8b_9} = -1$$

4. Example

In this section, we shall give two examples to illustrate our main results distinctly. One of the conic is elliptic conic, the other is hyperbolic conic.

Example 1. Consider a given triangulation shown in Fig. 2, where

$$A = (1/2,2,1), B = (-4,-2,1), C = (4,-2,1), a = (0,-1,1), b = (1,0,1),$$

$$c = (-1,0,1), u: -y = 0, v: -x - y - z = 0, w: x - y - z = 0,$$

$$l_1: -4x - y + 4/3z = 0, l_2: -2/3x - y + 2/3z = 0, l_3: -1/4x - y - z = 0,$$

$$l_4: 1/4x - y - z = 0, l_5: 2/3x - y + 2/3z = 0, l_6: 4/3x - y + 4/3z = 0$$

$$l_7: 2x - y - 2z = 0, l_8: -1/2x - y - z = 0, l_9: -42/47x - y - 42/47z = 0.$$

It can be proved that the spline space $S_3^2(\Delta_{MS}^2)$ of piecewise polynomial of degree 3 with smoothness 2 is singular. The corresponding l'_7, l'_8, l'_9 are

$l_7' = -x - y + z = 0, l_8' = 1/2x - y - z = 0, l_9' = -5/47x - y - 5/47z = 0$.

In this example, the conic corresponding to Theorem 2 is

$$302x^2 - 2861xy + 15800y^2 - 1421x + 12880y + 2624 = 0,$$

which forms elliptic conic and is shown in Fig. 2

Example 2. The following example shows that a conic corresponding to Theorem 2 forms hyperbolic conic. Given a singular triangulation Δ_{MS}^2 for spline space of piecewise polynomial of degree 3 with smoothness shown in Fig. 3, where

$$A = (1/2,2,1), B = (-4,-3,1), C = (3,-4,1), a = (0,-2,1), b = (1,-1,1),$$

$$c = (-1,0,1), u : -1/2x - y - 1/2z = 0, v : -2x - y - 2z = 0, w : x - y - 2z = 0,$$

$$l_1 : -6x - y + 5z = 0, l_2 : -3/2x - y + 1/2z = 0, l_3 : -2/3x - y - 2z = 0,$$

$$l_4 : 1/4x - y - 2z = 0, l_5 : x - y + z = 0, l_6 : 4/3x - y + 4/3z = 0,$$

$$l_7 : 1/7x - y - 8/7z = 0, l_8 : 2x - y - 2z = 0, l_9 : -38/61x - y - 38/61z = 0$$

The corresponding l_7', l_8', l_9' are

$$l_7' = 5/14x - 19/14 - y = 0, l_8' = -3x - 2 - y = 0, l_9' = -229/122x - 229/122 - y = 0,$$

and the corresponding duality figure of the triangulation is shown in Fig. 3. A hyperbolic curve passing through the six points as mentioned Theorem 2 is shown in 3 and

$$\frac{1143}{2803}x^2 + \frac{1028}{1963}y^2 - \frac{1022}{715}xy - \frac{503}{256}xz + \frac{1161}{1112}yz + z = 0$$

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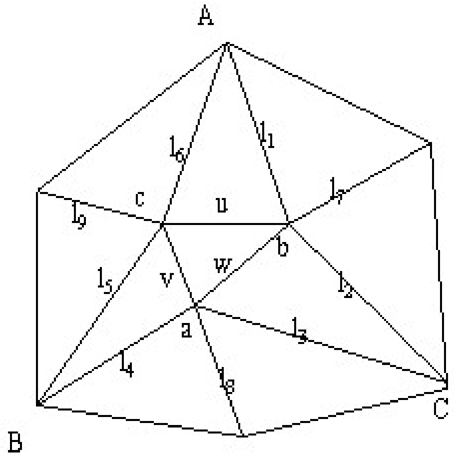

Figure 1. Δ^2_{MS}

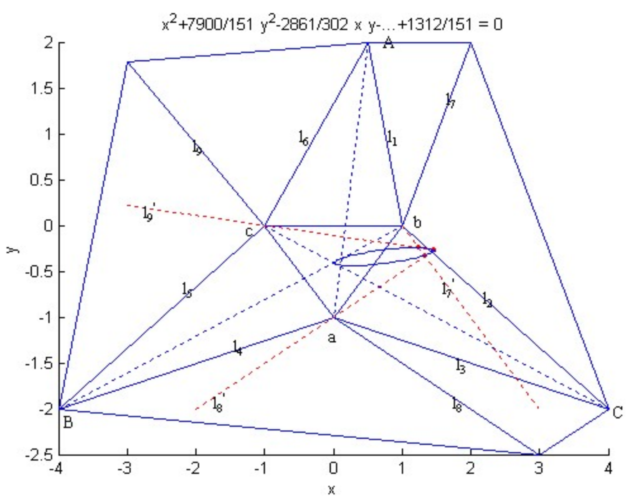

Figure 2. Example 1

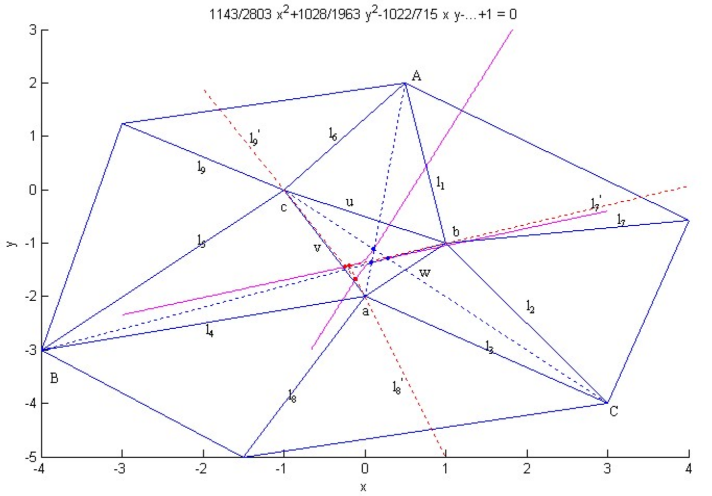

Figure 3. Example 2