



Some Properties of Relative Topological Space

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Abstract

In this paper, some topological properties were studied, especially including the Cartesian Product of relative T_1 , Hausdorff and superregular and the property of 2-paracompact under the perfect mapping was also discussed.

Keywords: T_1 in X , Y is Hausdorff in X , Y is superregular in X , Y is 2-paracompact in X .

1. Introduction

Relative topological properties are extension of classic topological invariants. In 1989, the relative topological properties were discussed by A.V. Arhangel'skii and H.M.M. Genecli in (Arhangel'skii, 1996, p.1-13), and A.V. Arhangel'skii gave the first systematic text on relative topological properties in 1996. In recent years, some further new results of the relative topology were obtained respectively by A.V. Arhangel'skii, J. Tartir and W. Just, O. Pavlov and M. Matveer, I. Yaschenko, V.V. Tkachuk, M.G. Tkachenko and R.G. Wilson, etc.

In my paper, some relative topological properties were studied and four results were given.

2. The Cartesian Product of Relative Topological Space

X is a space, $Y \subset X$, the concept of Y is T_1 , Hausdorff and superregular in X were introduced in (Arhangel'skii, 1996, p.1-13). In this part, the Cartesian Product of them were discussed, and I gave there results.

Definition 2.1 Y is T_1 in X : If for each $y \in Y$, the set $\{y\}$ is closed in X .

Definition 2.2 Y is Hausdorff in X : If for every two distinct points x and y of Y , there are disjoint open subsets u and v of X , such that: $x \in u$ and $y \in v$.

Definition 2.3 Y is superregular in X : If for each y of Y and each closed subset p of X , such that $y \notin p$, there are disjoint open subsets u and v of X , such that: $y \in u$ and $p \subset v$.

Lemma 2.4 Let $\{X_a\}_{a \in A}$ is a family of topological space. Then for each $a \in A$, the projection $P_a : \times\{X_a : a \in A\} \rightarrow X_a$ is an onto and open continuous mapping.

Lemma 2.5 Y is superregular in X if and only if for each y and arbitrary open subset u of X which contains y , there exists an open subset v of X , such that: $y \in v \subset \bar{v} \subset u$

Proof. " \Rightarrow " Let y is an arbitrary point of Y , and u is an open subset of X which contains y . Then, u' is closed in X and such that: $y \notin u'$. Since Y is superregular in X , so there exist two disjoint open subsets u_1 and v , such that: $y \in v$ and $u' \subset u_1$. Obviously, $v \subset u_1'$. So, $\bar{v} \subset u_1' = u_1' \subset u$. That is: $v \subset u$.

" \Leftarrow " Let y is an arbitrary point of Y and p is a closed subset of X which doesn't contains y . Obviously, $X \setminus p$ is an open subset of X and such that: $x \in X \setminus p$. So, there exists an open subset v of X , such that: $y \in v \subset \bar{v} \subset X \setminus p$. Let $u = X \setminus \bar{v}$, it is obvious that u is a neighborhood of p and such that: $u \cap v = \emptyset$. So, Y is superregular in X .

Theory 2.6 If Y_s is T_1 in X_s for each $s \in S$. Then, the Cartesian Product $\times\{Y_s : s \in S\}$ is T_1 in $\times\{X_s : s \in S\}$.

Proof. Let y is an arbitrary point of $\times\{Y_s : s \in S\}$. For each $y_s \in Y_s$, since Y_s is T_1 in X_s , so $\{y_s\}$ is T_1 in X_s , so $\{y_s\}$

is closed in X_s . By the TH2.3.4 (Arhangel'skii & Nogura, 1998, p.49-58), then, $y = \times\{y_s : s \in S\}$ is closed in $\times\{X_s : s \in S\}$. That

is the Cartesian Product $\times\{Y_s : s \in S\}$ is T_1 in $\times\{X_s : s \in S\}$.

Theory 2.7 If Y_s is Hausdorff in X_s For each $s \in S$. Then, the Cartesian Product $\times\{Y_s : s \in S\}$ is Hausdorff in $\times\{X_s : s \in S\}$.

Proof. Let x and y are two distinct points of $\times\{Y_s : s \in S\}$. Then, there exists a $s \in S$, such that $x_s \neq y_s$. Obviously $x_s \in X_s, y_s \in X_s$. Since Y_s is Hausdorff in X_s , so there exist two open subsets u and v of X_s , Such that: $x_s \in u_s$, and $y_s \in v_s$ and $u \cap v = \emptyset$. Let $u = p_s^{-1}(u_s)$ $v = p_s^{-1}(v_s)$, by the lemma 2.4, u and v are open in $\times\{X_s : s \in S\}$. It is obvious that: $x \in u$ and $y \in v$ and $u \cap v = \emptyset$. That is, $\times\{Y_s : s \in S\}$ is Hausdorff in $\times\{X_s : s \in S\}$.

Theory 2.8 If Y_s is superregular in X_s for each $s \in S$. Then, the Cartesian Product $\times\{Y_s : s \in S\}$ is superregular in $\times\{X_s : s \in S\}$.

Proof. For each $y \in \times\{Y_s : s \in S\}$ and arbitrary open subset u of $\times\{X_s : s \in S\}$, such that: $y \in u$. There exists a member $v = \{v_{\alpha_s} : s \in S\}$ of the some fixed base of $\times\{X_s : s \in S\}$, such that: $y \in v \subset u$. For each y_s and v_{α_s} , since Y_s is superregular in X_s , so there exists an open subset v_{α_s}' , such that: $y \in v_{\alpha_s}' \subset v_{\alpha_s} \subset u$. Let $v' = \{v_{\alpha_s}' : s \in S\}$. Obviously, $y \in v' \subset v \subset u$. By the lemma 2.5, $\times\{Y_s : s \in S\}$ is superregular in $\times\{X_s : s \in S\}$.

3. The Property of Relative Compactness under the Perfect Mapping.

The definition of 2-paracompact was introduced in (Arhangrl'skii, 2002, p.153-201). Some properties of topological spaces under the perfect mapping were given in (Engelking, 1997). In this part, I studied the property of 2-paracompact under the perfect mapping, and gave a result about it.

Definition 3.1 Y is 2-paracompact in X : If for each open covering A of X , there exists an open family covering \mathfrak{R} of X , such that: \mathfrak{R} refines A , $Y \subset \cup \mathfrak{R}$ and \mathfrak{R} is locally finite at each y of Y .

Theorem 3.2 Let $f: X \rightarrow Y$ is a perfect mapping. If Y_1 is 2-paracompact in Y . Then, $f^{-1}(Y_1)$ is 2-paracompact in X .

Proof. Let $A = \{u_s : s \in S\}$ is an open covering of X . Since f is a perfect mapping, so for each $y \in Y$, the fiber $f^{-1}(y)$ is a compact subset of X . Thus, there exists a finite subset $S(y)$ of S , such that: $f^{-1}(y) \subset \cup_{s \in S(y)} u_s = u_{y(s)}$. Since f is a perfect mapping, by the TH1.4.13 in (Engelking, 1997), there exists an open neighborhood $w_{y(s)}$ of y , such that: $f^{-1}(y) \subset f^{-1}(w_{y(s)}) \subset u_{y(s)}$. We may also assume that: $f^{-1}(w_{y(s)})$ is $v_{y(s)}$. That is $v_{y(s)} = f^{-1}(w_{y(s)})$. Then, it is obvious that: $v_{y(s)}$ is open in X and such that: $f^{-1}(y) \subset v_{y(s)} = f^{-1}(f(v_{y(s)})) \subset u_{y(s)}$ and $f(v_{y(s)})$ is an open subset of Y . Obviously, $\mathfrak{R}_1 = \{f(v_{y(s)}) : y \in Y\}$ is an open covering of Y . Since Y_1 is 2-paracompact in Y , so there exists an open family $\mathfrak{R}_2 = \{v_a : a \in A\}$ of Y by open subsets of Y_1 , such that: \mathfrak{R}_2 refines $\mathfrak{R}_1, Y_1 \subset \cup \mathfrak{R}_2$ and \mathfrak{R}_2 is locally finite at each $y \in Y$. We may also assume that $f(v_{y(s)})$ which contains v_a is $f(y_a(s))$. Since f is perfect mapping, thus, $\mathfrak{R}_3 = \{f^{-1}(v_a) : a \in A\}$ is an open family of X and locally Finite each $x \in f^{-1}(Y_1)$ and $f^{-1}(Y_1) \subset \cup \mathfrak{R}_3$. Obviously $f^{-1}(v_a) \subset f^{-1}(f(v_{y(s)})) = v_{y_a(s)} = u_{y_a(s)}$. Let $\mathfrak{R} = \{f^{-1}(v_a) \cap u_s : a \in A, s \in S_a(y)\}$. Then, \mathfrak{R} is an open family of X and such that \mathfrak{R} refines A , $f^{-1}(Y_1) \subset \cup \mathfrak{R}$ and \mathfrak{R} is locally finite at each $x \in f^{-1}(Y_1)$. That is $f^{-1}(Y_1)$ is 2-paracompact in X .

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