Some Properties of Relative Topological Space

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Abstract
In this paper, some topological properties were studied, especially including the Cartesian Product of relative $T_i$, Hausdorff and superregular and the property of 2-paracompact under the perfect mapping was also discussed.

Keywords: Y is $T_i$ in X, Y is Hausdorff in X, Y is superregular in X, Y is 2-paracompact in X.

1. Introduction
Relative topological properties are extension of classic topological invariants. In 1989, the relative topological properties were discussed by A.V. Arhangel’skii and V.M. Geneli in (Arhangel’skii, 1996, p. 1-13), and A.V. Arhangel’skii gave the first systematic text on relative topological properties in 1996. In recent years, some further new results of the relative topology were obtained respective by A.V. Arhangel’skii, J. Tartir and W. Just, O. Pavlov and M. Matveer, I. Yaschenko, V.V. Tkachuk, M. G. Tkachenko and R. G. Wilson, etc.

In my paper, some relative topological properties were studied and four results were given.

2. The Cartesian Product of Relative Topological Space
X is a space, $Y \subset X$, the concept of Y is $T_i$ in X, Hausdorff and superregular in X were introduced in (Arhangel’skii, 1996, p. 1-13).

In this part, the Cartesian Product of them were discussed, and I gave there results.

Definition 2.1 Y is $T_i$ in X: If for each $y \in Y$, the set $\{y\}$ is closed in X.

Definition 2.2 Y is Hausdorff in X: If for every two distinct points $x$ and $y$ of Y, there are disjoint open subsets $u$ and $v$ of X, such that: $x \in u$ and $y \in v$.

Definition 2.3 Y is superregular in X: If for each $y$ of Y and each closed subset $p$ of X, such that $y \notin p$, there are disjoint open subsets $u$ and $v$ of X, such that: $y \in u$ and $p \subset v$.

Lemma 2.4 Let $\{X_s : s \in A\}$ is a family of topological space. Then for each $a \in A$, the projection $P_a : \{X_a : a \in A\} \rightarrow X_a$ is an onto and open continuous mapping.

Lemma 2.5 Y is superregular in X if and only if for each $y$ and arbitrary open subset $u$ of X which contains $y$, there exists an open subset $v$ of X, such that: $y \in v \subset \overline{v} \subset u$.

Proof. “$\Rightarrow$” Let $y$ is an arbitrary point of Y, and $u$ is an open subset of X which contains $y$. Then, $u'$ is closed in X and such that: $y \notin u'$. Since Y is superregular in X, so there exist two disjoint open subsets $u'$ and $v$, such that: $y \in v$ and $u' \subset u$. Obviously, $v \subset u'$. So, $\overline{v} \subset u' = u' \subset u$. That is: $v \subset u$.

“$\Leftarrow$” Let $y$ is an arbitrary point of Y and $p$ is a closed subset of X which doesn’t contains $y$. Obviously, $X \setminus p$ is an open subset of X and such that: $x \in X \setminus p$. So, there exists an open subset $v$ of X, such that: $y \in v \subset \overline{v} \subset X \setminus p$. Let $u = X \setminus v$, it is obvious that $u$ is a neighborhood of $p$ and such that: $u \cap v = \emptyset$. So, Y is superregular in X.

Theory 2.6 If $Y_s$ is $T_i$ in $X_s$ for each $s \in S$. Then, the Cartesian Product $\times\{Y_s : s \in S\}$ is $T_i$ in $\times\{X_s : s \in S\}$.

Proof. Let $y$ is an arbitrary point of $\times\{Y_s : s \in S\}$. For each $y_s \in Y_s$, since $Y_s$ is $T_i$ in $X_s$, so $\{y_s\}$ is $T_i$ in $X_s$, so $\{y_s\}$ is closed in $X_s$. By the TH2.3.4 (Arhangel’skii & Nogura, 1998, p. 49-58), then, $y = \times\{y_s : s \in S\}$ is closed in $\times\{X_s : s \in S\}$. That
is Hausdorff in $Y$. For each $s \in S$, the Cartesian Product $\times \{Y : s \in S\}$ is Hausdorff.

**Theorem 2.7** If $Y$ is Hausdorff in $X$, then the Cartesian Product $\times \{Y : s \in S\}$ is Hausdorff in $\times \{X : s \in S\}$.

**Proof.** Let $x$ and $y$ be two distinct points of $\times \{Y : s \in S\}$. Then, there exists a $s \in S$, such that $x_s \neq y_s$. Obviously $x_s \in X_s$ and $y_s \in Y_s$. Since $Y$ is Hausdorff in $X$, there exist two open subsets $u$ and $v$ of $X_s$ such that $x_s \in u$, and $y_s \in v$, and $u \cap v = \emptyset$. Let $u = p_s^{-1}(u_s)$ and $v = p_s^{-1}(v_s)$ by the Lemma 2.4. Then, $u$ and $v$ are open in $\times \{X : s \in S\}$. It is obvious that $x \in u$ and $y \in v$ and $u \cap v = \emptyset$. That is, $\{Y : s \in S\}$ is Hausdorff in $\times \{X : s \in S\}$.

**Theorem 2.8** If $Y$ is superregular in $X$, then the Cartesian Product $\times \{Y : s \in S\}$ is superregular in $\times \{X : s \in S\}$.

**Proof.** For each $y \in \times \{Y : s \in S\}$ and arbitrary open subset $u$ of $\times \{X : s \in S\}$, such that $y \in u$. There exists a member $v = \{v_s : s \in S\}$ of the some fixed base of $\times \{X : s \in S\}$, such that $y \in v \subseteq u$. For each $y_s$ and $v_s$, since $Y$ is superregular in $X_s$, there exists an open subset $v_s$, such that $y_s \in v_s \subseteq v_s$. Let $v_s = \{v_s : s \in S\}$. Obviously, $y \in v \subseteq v \subseteq u$. By the lemma 2.5, $\{Y : s \in S\}$ is superregular in $\times \{X : s \in S\}$.

### 3. The Property of Relative Compactness under the Perfect Mapping.

The definition of 2-paracompact was introduced in (Arhangel’skii, 2002, p.153-201). Some properties of topological spaces under the perfect mapping were given in (Engelking, 1997). In this part, I studied the property of 2-paracompact under the perfect mapping, and gave a result about it.

**Definition 3.1** $Y$ is 2-paracompact in $X$ if for each open covering $\mathcal{A}$ of $X$, there exists an open family covering $\mathcal{B}$ of $Y$ such that $\mathcal{A}$ refines $\mathcal{B}$, and $\mathcal{B}$ is locally finite at each $y \in Y$.

**Theorem 3.2** Let $f: X \to Y$. Then, $f^{-1}(Y)$ is 2-paracompact in $X$.

**Proof.** Let $\mathcal{A} = \{u_s : s \in S\}$ be an open covering of $X$. Since $f$ is a perfect mapping, so for each $y \in Y$, the fiber $f^{-1}(y)$ is a compact subset of $X$. Thus, there exists a finite subset $s(y)$ of $S$, such that $f^{-1}(y) = \bigcup_{s \in s(y)} u_s = u_{s(y)}$. Since $f$ is a perfect mapping, by the TH 1.4.13 in (Engelking, 1997), there exists an open neighborhood $u_{s(y)}$ of $y$, such that $f^{-1}(y) \subseteq f^{-1}(u_{s(y)}) \subseteq u_{s(y)}$. We may also assume that $f^{-1}(u_{s(y)}) = v_{s(y)}$. Then, $v_{s(y)} = f^{-1}(u_{s(y)})$. It is obvious that $v_{s(y)}$ is open in $X$ and such that $f^{-1}(y) \subseteq v_{s(y)} = f^{-1}(f(v_{s(y)})) = f^{-1}(u_{s(y)})$ and $f(v_{s(y)})$ is an open subset of $Y$. Obviously, $\{f(v_{s(y)}) : y \in Y\}$ is an open covering of $Y$. Since $Y$ is 2-paracompact in $X$, there exists an open family $\mathcal{B}_1 = \{v_a : a \in A\}$ of $Y$ by open subsets of $Y$, such that $\mathcal{B}_1$ refines $\mathcal{B}_2$, $Y \subseteq \bigcup \mathcal{B}_2$ and $\mathcal{B}_2$ is locally finite at each $y \in Y$. We may also assume that $f(v_{s(y)})$ which contains $v_a$ is $f(y_a)$. Since $f$ is perfect mapping, thus, $f^{-1}(v_a) = \{f^{-1}(v_a) : a \in A\}$ is an open family of $X$ and locally Finite each $x \in f^{-1}(Y)$ and $f^{-1}(Y) \subseteq \bigcup \mathcal{B}_2$. Obviously $f^{-1}(v_a) \subseteq f^{-1}(f(v_{s(y)})) = v_{s(y)} = u_{s(y)}$. Let $\mathcal{B} = \{f^{-1}(v_a) : a \in A, s \in S(y)\}$. Then, $\mathcal{B}$ is an open family of $X$ and such that $\mathcal{B}$ refines $\mathcal{A}$, $f^{-1}(Y) \subseteq \bigcup \mathcal{B}$ and $\mathcal{B}$ is locally finite at each $x \in f^{-1}(Y)$. That is $f^{-1}(Y)$ is 2-paracompact in $X$.

**References**


