P-completely Regular Semigroup

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Abstract
In order to prove a completely regular semigroup with the strong semilattice structure is P-completely regular semigroup. Using the strong semilattice structure and the property of congruence. The sufficient condition of which a completely regular semigroup is P-completely regular semigroup is have the strong semilattice structure, and The subclass NBG of the completely regular semigroup is P-completely regular semigroup.

Keywords: Completely regular semigroup, The strong semilattice, Homomorphisms

1. Preliminaries

Definition 1.1. A semigroup $S$ is a semilattice $Y$ of semigroup $S_a (a \in Y)$ if there exists an epimorphism $\varphi$ of $S$ onto the semilattice $Y$ with $a\varphi^{-1} = S_a (a \in Y)$. We write $S = [Y; S_a]$.

Definition 1.2. Let $S = [Y; S_a]$ be a semilattice of semigroups. If for every $a \in Y$ every congruence on $S_a$ can be extended to a congruence on $S$, then $S$ is said to be a $P$-semigroup.

Definition 1.3. A semigroup $S$ is called completely regular, if for every $a \in S$, there exists an element $x \in S$ such that $a = axa$ and $ax = xa$.

In [1], we have known that completely regular semigroup $S = [Y; S_a]$ is a semilattice of completely simple semigroups $S_a$. In fact, every $S_a$ is a $D$-class of $S$. If every congruence on $D$-class of $S$ can be extended to a congruence on $S$, then $S$ is said to be a $P$-completely regular semigroup.

Definition 1.4. Let $S = [Y; S_a]$ be a semilattice of semigroup. For each pair $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, let $\varphi_{\alpha, \beta} : S_a \to S_{\beta}$ be a homomorphism such that

(i) $\varphi_{\alpha, \alpha} = 1_{S_a}$,

(ii) $\varphi_{\alpha, \beta} \varphi_{\beta, \gamma} = \varphi_{\alpha, \gamma}$ if $\alpha \geq \beta \geq \gamma$

on $S = \bigcup_{a \in Y} S_a$ define a multiplication by

$ab = (a\varphi_{a, a}) (b\varphi_{b, b})$ (a $\in \alpha$, b $\in \beta$)

with this multiplication $S$ is a strong semilattice $Y$ of semigroup $S_a$ to be denoted by $S = [Y; S_a, \varphi_{a, \beta}]$.

In this paper, we mainly give some views on a open problem" characterize all the $P -$ completely regular semigroups ", and find the sufficient condition that a completely regular semigroup is the $P$-completely regular semigroup.

We shall use the same notations and terminology to [1]. In this paper, we are interested in the symbols:

$CR$ of completely regular semigroups,

$NB$ of normal bands,

$Con (S)$ of all congruences on $S$,

$NBG$ of normal cryptogroups,

$ONBG$ of normal orthogroups,

$Clifford$ of clifford semigroups.
2. The Main Result

Let $S = [Y, S_\beta, \varphi_{a,\beta}]$ be a strong semilattice of semigroups. If $\rho_a \in Con(S_\beta)$, then for every $\beta \in Y$ we can define a new relation $\rho_{a}^* = \omega_{\beta a} \rho_a \mid_{S_\beta}$ as follows:

\[
\rho_{a_{\mid S}}^* = \{((a, b) \in S_\beta \times S_\beta | (a, b) \in \rho_a) \cup 1_{S_\beta} \mid \beta = \alpha \}
\]

\[
\{((a, b) \in S_\beta \times S_\beta | (a, b) \in \rho_a) \cup 1_{S_\beta} \mid \beta < \alpha \quad (1') \}
\]

\[
\{((a, b) \in S_\beta \times S_\beta | (a, b) \in \rho_a) \cup 1_{S_\beta} \mid \beta > \alpha \quad (1'') \}
\]

**Theorem 2.1.** Let $S = [Y, S_\beta, \varphi_{a,\beta}]$ be a strong semilattice of semigroups, if $\rho_a \in Con(S_\beta)$, then $\rho_{a_{\mid S}}^* \in Con(S_\beta)$.

**Proof.** We need prove from two parts as follows:

(1) Assume $\beta > \alpha, u, v \in S_\beta$ and $(u, v) \in \rho_a^* \mid_{S_\beta}$, then by the definition of $\rho_a^*$, we have $(u\varphi_{a,\beta}, v\varphi_{a,\beta}) = \rho_a^* \mid_{S_\beta} = \rho_a^*$. For any $c \in S_\beta, c\varphi_{a,\beta} \in S_\beta$, since $\rho_a \in Con(S_\beta)$, this imply that

\[
((ac)\varphi_{a,\beta}, (bd)\varphi_{a,\beta}) \in \rho_a^* \mid_{S_\beta}
\]

\[
((a\varphi_{a,\beta})(c\varphi_{a,\beta}), (d\varphi_{a,\beta})) \in \rho_a^* \mid_{S_\beta}
\]

\[
(u, v) \in \rho_a^* \mid_{S_\beta}
\]

Similarly, we can show that $(cu, cv) \in \rho_a^* \mid_{S_\beta}$. Thus $\rho_a^* \mid_{S_\beta} \in Con(S_\beta)$.

(2) Assume $\beta < \alpha, m, n, u, v \in S_\beta$ and $(m, n), (u, v) \in \rho_a^* \mid_{S_\beta}$. Form the definition of $\rho_a^*$, there exist $a, b, c, d \in S_a$, so that $b\varphi_{a,\beta} = v, d\varphi_{a,\beta} = n$ and $(a, b) \in \rho_a^*, (c, d) \in \rho_a^*$. Since $\rho_a \in Con(S_\beta)$, we have $(ac, bd) \in \rho_a$.

\[
\Rightarrow ((ac)\varphi_{a,\beta} = (bd)\varphi_{a,\beta}) \in \rho_a^* \mid_{S_\beta}
\]

\[
((a\varphi_{a,\beta})\varphi_{a,\beta} = (d\varphi_{a,\beta})) \in \rho_a^* \mid_{S_\beta}
\]

\[
(u, v) \in \rho_a^* \mid_{S_\beta}
\]

Thus $\rho_a^* \mid_{S_\beta} \in Con(S_\beta)$.

From (1) and (2), we conclude that $\rho_a^* \mid_{S_\beta} \in Con(S_\beta)$ for any $\beta \in Y$.

We have immediately the following corollary and it’s proofs are omitted.

**Corollary 2.2.** Let $S = [Y, S_\beta, \varphi_{a,\beta}]$ be a strong semilattice of semigroups, and $a,b \in S_\beta$. If $(a, b) \in \rho_a^*$, then $(a\varphi_{a,\gamma}b, b\varphi_{a,\gamma}) \in \rho_a$ for any $\gamma \leq \beta$.

**Theorem 2.3.** If $S = [Y, S_\beta, \varphi_{a,\beta}]$ be a strong semilattice of semigroups, then $S$ is a $P$-semigroup.

**Proof.** We need prove for every $a \in Y$ every congruence on $S_a$ can be extended to a congruence on $S$, that is to say, we only need prove $\rho_a^* \in Con(S)$. Let $(a, b) \in \rho_a^*$. By the definition of $\rho_a^*$ as (1), we know $a$ is in the same subsemigroup of $S$ with $b$. Assume that $a, b \in S_\beta$, then for any $c \in S, c \in S_\gamma$, by Definition 1.4., we have

\[
ac = (a\varphi_{a,\gamma})(c\varphi_{a,\gamma}) \quad \text{and} \quad bc = (b\varphi_{a,\gamma})(c\varphi_{a,\gamma})
\]

Since $(a, b) \in \rho_a^* \mid_{S_\beta}$, thus

\[
\Rightarrow (a\varphi_{a,\gamma}b, b\varphi_{a,\gamma}) \in \rho_a^* \mid_{S_\beta} \in Con(S_{\beta})
\]

\[
((a\varphi_{a,\gamma})\varphi_{a,\gamma} = (d\varphi_{a,\gamma})) \in \rho_a^* \mid_{S_\beta}
\]

\[
(u, v) \in \rho_a^* \mid_{S_\beta}
\]

Similarly, we may show that $(ac, bc) \in \rho_a^*$. This show that $\rho_a^* \in Con(S)$. By arbitrariness of $\alpha$, we get $S$ is $P$-semigroup.

From Theorem 2.3, we know the sufficient condition of which a semigroup is the $P$-semigroup. If $S \in CR$, then we have immediately the following corollary.

**Corollary 2.4.** Let $S \in CR$. If $S = [Y, S_\beta, \varphi_{a,\beta}] \in NBG$, then $S$ is $P$-completely regular semigroup.

From Corollary 2.4, it is obvious that all the subclass of $NBG$, i.e. $NB, ONBG, Clifford$, is $P$-
completely regular semigroup.

References


