



P-completely Regular Semigroup

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Abstract

In order to prove a completely regular semigroup with the strong semilattice structure is P -completely regular semigroup. Using the strong semilattice structure and the property of congruence. The sufficient condition of which a completely regular semigroup is P -completely regular semigroup is have the strong semilattice structure, and The subclass NBG of the completely regular semigroup is P -completely regular semigroup.

Keywords: Completely regular semigroup, The strong semilattice, Homomorphisms

1. Preliminaries

Definition 1.1. A semigroup S is a semilattice Y of semigroup S_α ($\alpha \in Y$) if there exists an epimorphism φ of S onto the semilattice Y with $\alpha\varphi^{-1} = S_\alpha$ ($\alpha \in Y$). We write $S = [Y; S_\alpha]$.

Definition 1.2. Let $S = [Y; S_\alpha]$ be a semilattice of semigroups. If for every $\alpha \in Y$ every congruence on S_α can be extended to a congruence on S , then S is said to be a P -semigroup.

Definition 1.3. A semigroup S is called completely regular, if for every $a \in S$, there exists an element $x \in S$ such that $a = axa$ and $ax = xa$.

In [1], we have known that completely regular semigroup $S = [Y; S_\alpha]$ is a semilattice of completely simple semigroups S_α . In fact, every S_α is a D -class of S . If every congruence on D -class of S can be extended to a congruence on S , then S is said to be a P -completely regular semigroup.

Definition 1.4. Let $S = [Y; S_\alpha]$ be a semilattice of semigroup. For each pair $\alpha, \beta \in Y$ such that $\alpha \geq \beta$, let $\varphi_{\alpha, \beta} : S_\alpha \rightarrow S_\beta$ be a homomorphism such that

- (i) $\varphi_{\alpha, \alpha} = 1_{S_\alpha}$,
- (ii) $\varphi_{\alpha, \beta} \varphi_{\beta, \gamma} = \varphi_{\alpha, \gamma}$ if $\alpha \geq \beta \geq \gamma$

on $S = \cup_{\alpha \in Y} S_\alpha$ define a multiplication by

$$ab = (a\varphi_{\alpha, \alpha\beta})(b\varphi_{\beta, \alpha\beta}) \quad (a \in \alpha, b \in \beta)$$

with this multiplication S is a strong semilattice Y of semigroup S_α to be denoted by $S = [Y; S_\alpha, \varphi_{\alpha, \beta}]$.

In this paper, we mainly give some views on an open problem “characterize all the P -completely regular semigroups”, and find the sufficient condition that a completely regular semigroup is the P -completely regular semigroup.

We shall use the same notations and terminology to [1]. In this paper, we are interested in the symbols:

CR	of completely regular semigroups,
NB	of normal bands,
$Con(S)$	of all congruences on S ,
NBG	of normal cryptogroups,
$ONBG$	of normal orthogroups,
$Clifford$	of clifford semigroups.

2. The Main Result

Let $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ be a strong semilattice of semigroups. If $\rho_\alpha \in \text{Con}(S_\alpha)$, then for every $\beta \in Y$ we can define a new relation $\rho_\alpha^* = \cup_{\beta \in Y} \rho_\alpha^* |_{S_\beta}$ as follows:

$$\rho_\alpha^* |_{S_\beta} = \begin{cases} \rho_\alpha & \beta = \alpha \\ \{(a\varphi_{\alpha,\beta}, b\varphi_{\alpha,\beta}) \in S_\beta \times S_\beta \mid (a,b) \in \rho_\alpha\} \cup 1_{S_\beta} & \beta < \alpha \quad (1^*) \\ \{(u,v) \in S_\beta \times S_\beta \mid (u\varphi_{\beta,\alpha}, v\varphi_{\beta,\alpha}) \in \rho_\alpha\} & \beta > \alpha \end{cases}$$

Theorem 2.1. Let $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ be a strong semilattice of semigroups, if $\rho_\alpha \in \text{Con}(S_\alpha)$, then $\rho_\alpha^* |_{S_\beta} \in \text{Con}(S_\beta)$.

Proof. We need prove from two parts as follows:

(1) Assume $\beta > \alpha, u, v \in S_\beta$ and $(u, v) \in \rho_\alpha^* |_{S_\beta}$, then by the definition

of ρ_α^* we have $(u\varphi_{\beta,\alpha}, v\varphi_{\beta,\alpha}) \in \rho_\alpha^* |_{S_\beta} = \rho_\alpha^*$. For any $c \in S_\beta, c\varphi_{\beta,\alpha} \in S_\alpha$, since $\rho_\alpha \in \text{Con}(S_\alpha)$, this imply that

$$(u\varphi_{\beta,\alpha})(c\varphi_{\beta,\alpha})\rho_\alpha(v\varphi_{\beta,\alpha})(c\varphi_{\beta,\alpha}) \Rightarrow (uc)\varphi_{\beta,\alpha}\rho_\alpha(vc)\varphi_{\beta,\alpha} \Rightarrow (uc, vc) \in \rho_\alpha^* |_{S_\beta}.$$

Similarly, we can show that $(cu, cv) \in \rho_\alpha^* |_{S_\beta}$. Thus $\rho_\alpha^* |_{S_\beta} \in \text{Con}(S_\beta)$.

(2) Assume $\beta < \alpha, m, n, u, v \in S_\beta$, and $(m, n), (u, v) \in \rho_\alpha^* |_{S_\beta}$. Form the definition of ρ_α^* , there exist $a, b, c, d \in S_\alpha$, so that $b\varphi_{\alpha,\beta} = v, d\varphi_{\alpha,\beta} = n$, and $(a, b) \in \rho_\alpha^*, (c, d) \in \rho_\alpha^*$. Since $\rho_\alpha \in \text{Con}(S_\alpha)$, we have $(ac, bd) \in \rho_\alpha$,

$$\begin{aligned} &\Rightarrow ((ac)\varphi_{\alpha,\beta}, (bd)\varphi_{\alpha,\beta}) \in \rho_\alpha^* |_{S_\beta} \\ &\Rightarrow (a\varphi_{\alpha,\beta})(c\varphi_{\alpha,\beta})\rho_\alpha^* |_{S_\beta} (b\varphi_{\alpha,\beta})(d\varphi_{\alpha,\beta}) \\ &\Rightarrow (um, vn) \in \rho_\alpha^* |_{S_\beta}. \end{aligned}$$

Thus $\rho_\alpha^* |_{S_\beta} \in \text{Con}(S_\beta)$. From (1) and (2), we conclude that $\rho_\alpha^* |_{S_\beta} \in \text{Con}(S_\beta)$ for any $\beta \in Y$.

We have immediately the following corollary and it's proofs are omitted.

Corollary 2.2. Let $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ be a strong semilattice of semigroups, and $a, b \in S_\beta$. If $(a, b) \in \rho_\alpha^*$, then $(a\varphi_{\beta,\gamma}, b\varphi_{\beta,\gamma}) \in \rho_\alpha$ for any $\gamma \leq \beta$.

Theorem 2.3. If $S = [Y; S_\alpha, \varphi_{\alpha,\beta}]$ be a strong semilattice of semigroups, then S is a P -semigroup.

Proof. We need prove for every $a \in Y$ every congruence on S_α can be extended to a congruence on S , that is to say, we only need prove $\rho_\alpha^* \in \text{Con}(S)$. Let $(a, b) \in \rho_\alpha^*$. By the definition of ρ_α^* as (1□), we know a is in the same subsemigroup of S with b . Assume that $a, b \in S_\beta$, then for any $c \in S$, let $c \in S_\gamma$, by Definition 1.4., we have

$$ac = (a\varphi_{\beta,\beta\gamma})(c\varphi_{\beta,\beta\gamma}), \quad bc = (b\varphi_{\beta,\beta\gamma})(c\varphi_{\beta,\beta\gamma}).$$

Since $(a, b) \in \rho_\alpha^* |_{S_\beta}$, thus

$$\begin{aligned} &\Rightarrow (a\varphi_{\beta,\beta\gamma}, b\varphi_{\beta,\beta\gamma}) \in \rho_\alpha^* |_{S_{\beta\gamma}} \in \text{Con}(S_{\beta\gamma}) \\ &\Rightarrow (a\varphi_{\beta,\beta\gamma})(c\varphi_{\beta,\beta\gamma})\rho_\alpha^* |_{S_{\beta\gamma}} (b\varphi_{\beta,\beta\gamma})(c\varphi_{\beta,\beta\gamma}) \\ &\Rightarrow (ac, bc) \in \rho_\alpha^* |_{S_{\beta\gamma}} \\ &\Rightarrow (ac, bc) \in \rho_\alpha^*. \end{aligned}$$

Similarly, we may show that $(ac, bc) \in \rho_\alpha^*$. This show that $\rho_\alpha^* \in \text{Con}(S)$. By arbitrariness of α , we get S is P -semigroup.

From Theorem 2.3. we know the sufficient condition of which a semigroup is the P -semigroup. If $S \in CR$, then we have immediately the following corollary.

Corollary 2.4. Let $S \in CR$. If $S = [Y; S_\alpha, \varphi_{\alpha,\beta}] \in NBG$, then S is P -completely regular semigroup.

From Corollary 2.4., it is obvious that all the subclass of NBG , i.e. $NB, ONBG, Clifford$, is P -

completely regular semigroup.

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