The Properties of Relative Regularity and Compactness

Genglei Li
College of Science, Tianjin Polytechnic University
Tianjin, 300160, China
E-mail: lglxt@126.com

Huidong Wu
College of Business, Hebei Normal University
Shijiazhuang, 050000, China
E-mail: xtlqjz@sohu.com

Abstract
In this paper, some topological properties were studied, especially including relative Hausdorff, relative regular and relative strongly regular and the property of nearly-paracompact under the perfect mapping was also discussed.

Keywords: Y is Hausdorff in X, Y is regular in X, Y is strongly regular in X, Y is nearly-paracompact in X.

1. Introduction
Relative topological properties are extension of classic topological invariants. In 1989, the relative topological properties were discussed by A.V. Archangel'skii and H.M.M. Genecli in Note 1, and A.V. Arhangel'skii gave the first systematic text on relative topological properties in 1996. In recent years, some further new results of the relative topology were obtained respectively by A.V. Arhangel'skii, J. Tartir and W. Just, O. Pavlov and M. Matveer, I. Yaschenko, V.V. Tkachuk, M.G. Tkachenko and R.G. Wilson, etc.

In my paper, some relative topological properties were studied and some results were given.

2. The Properties of Relative Regularity
X is a space, Y ⊆ X, the concept of Y is Hausdorff, regular, superregular and strongly regular in X were introduced in Note 1. In this part, some properties of them were discussed, and I gave there results.

Definition 2.1  Y is Hausdorff in X: If for every two distinct points x and y of Y, there are disjoint open subsets u and v of X, such that: x ∈ u and y ∈ v.

Definition 2.2  Y is regular in X: If for each y of Y and each closed subset p of X, such that y ∉ p, there are disjoint open subsets u and v of X, such that: y ∈ u and p ∈ Y ∩ v.

Definition 2.3  Y is superregular in X: If for each y of Y and each closed subset p of X, such that y ∉ p, there are disjoint open subsets u and v of X, such that: y ∈ u and p ∈ Y.

Definition 2.4  Y is strongly regular in X: If for each x of X and each closed subset p of X, such that y ∉ p, there are disjoint open subsets u and v of X, such that: x ∈ u and p ∩ Y ⊆ v.

Theory 2.5  If Y is a dense subspace of a space X. Then, Y is Hausdorff in X if and only if Y is Hausdorff.

Proof. “⇒” Let y₁, y₂ are arbitrary points of Y. Since Y is Hausdorff in X, so there exist two disjoint open subsets u₁ and v₁ in X, such that: y₁ ∈ u₁ and y₂ ∈ v₁. We may assume that: u₁ = u₁ ∩ Y and v₁ = v₁ ∩ Y. Then, u and v are two open sets in Y and such that: u ∩ v = ∅, y₁ ∈ u and y₂ ∈ v. That is Y is Hausdorff.

“⇐” Let y₁, y₂ are arbitrary points of Y. Since Y is Hausdorff, so there exist two disjoint open subsets u₁ and v₁ in X, such that: y₁ ∈ u₁ and y₂ ∈ v₁. So there exist two open sets u and v in X, such that: u = u₁ ∩ Y, v = v₁ ∩ Y. As follow we will prove u ∩ v = ∅. We may also assume that u ∩ v ≠ ∅. Then u ∩ v is open in X since u and v are
open in $X$, and $Y$ is a dense subspace of $X$, so $(u \cap v) \cap Y \neq \emptyset$, therefore $u_i \cap v_i \neq \emptyset$. This is contradict with $u_i \cap v_i = \emptyset$. So, $u \cap v = \emptyset$. That is $Y$ is Hausdorff in $X$.

**Theorem 2.6** If $Y$ is closed-open subspace of $X$. Then, $Y$ is regular in $X$ if and only if $Y$ is superregular in $X$.

Proof. “$\Rightarrow$” Let $y$ is an arbitrary point of $Y$ and an arbitrary closed subset $p$ of $X$, such that $y \notin p$. Since $Y$ is regular in $X$, there are disjoint open subsets $u_i$ and $v_i$ of $X$, such that: $y \in u_i$ and $p \cap Y \subset v_i$. We may assume that $u = Y \cap u_i$, then $Y$ is a point of $Y$ and $Y$ is open in $X$, so $u$ is an open set of $X$, and such that: $y \in u$. Since $Y$ is closed in $X$, so $X/Y$ is an open set of $X$. We may also assume that: $v = v_i \cup (X/Y)$, then we can get: $p \subset v$ and $u \cap v = \emptyset$. That is $Y$ is superregular in $X$.

“$\Leftarrow$” Let $y$ is an arbitrary point of $Y$ and an arbitrary closed subset $p$ of $X$, such that $y \notin p$, there are disjoint open subsets $u$ and $v$ of $X$, such that: $y \in u$ and $p \subset v$. Obviously, $p \cap Y \subset v$. That is $Y$ is regular in $X$.

**Theorem 2.7** $Y$ is strongly regular in $X$ if and only for each point $x$ of $X$ and arbitrary open set $u$ of $X$, such that $x \in X$, there exist an open $u_x$ of $X$, such that $y \in u_x \subset u$ and $u_x \cap (Y\setminus u) = \emptyset$.

Proof. “$\Rightarrow$” Let $x$ is an arbitrary point of $Y$ and an arbitrary open subset $u$ of $X$, such that $x \in p$. Let $p = X \setminus u$, then $p$ is closed in $X$ and $y \notin p$. Since $Y$ is strongly regular in $X$, so there are disjoint open subsets $u_i$ and $v_i$ of $X$, such that: $x \in u_i$ and $p \cap Y = Y \setminus u \subset v_i$. We may let $u_x = u_i \cap u$, therefore, $x \in u_x \subset u$ and $u_x \cap (Y\setminus u) = \emptyset$.

“$\Leftarrow$” Let $x$ is an arbitrary point of $X$ and $p$ is an arbitrary closed subset of $X$, such that $x \notin p$. Let $u = X \setminus p$. Then $x \in u$. So there is an open set $u_x$ of $X$, such that: $x \in u_x \subset u$ and $u_x \cap (Y\setminus u) = \emptyset$. Since $Y \setminus u = p \cap Y$, So we can assume: $u_1 = u_x$ and $u_2 = X \setminus u_x$. Obviously, $x \in u_1$, $p \cap Y \subset u_2$ and $u_1 \cap u_2 = \emptyset$. That is $Y$ is strongly regular in $X$.

3. The Property of Relative Compactness under the Perfect Mapping.

The definition of nearly-paracompact was introduced in Note 3. Some properties of topological spaces under the perfect mapping were given in Note 6. In this part, I studied the property of nearly-paracompact under the perfect mapping, and gave a result about it.

**Definition 3.1** $Y$ is nearly-paracompact in $X$: If for each open covering $\mathcal{A}$ of $X$, there exists an open family covering $\mathcal{R}$ of $Y$, such that: $\mathcal{R}$ refines $\mathcal{A}$ and $\mathcal{R}$ is locally finite at each $y$ of $Y$.

**Theorem 3.2** Let $f: X \to Y$ is a perfect mapping. If $Y$ is nearly-paracompact in $Y$, then $f^{-1}(Y)$ is nearly-paracompact in $X$.

Proof. Let $\mathcal{A} = \{u_y : s \in S\}$ is an open covering of $X$. Since $f$ is a perfect mapping, so for each $y \in Y$, the fiber $f^{-1}(y)$ is a compact subset of $X$. Thus, there exists a finite subset $S(y)$ of $S$, such that: $f^{-1}(y) \subset \bigcup_{s \in S(y)} u_s = u_{y(s)}$. Since $f$ is a perfect mapping, by the TH1.4.13 in Note 6, there exists an open neighborhood $w_{y(s)}$ of $y$, such that: $f^{-1}(y) \subset f^{-1}(w_{y(s)}) \subset u_{y(s)}$. We may also assume that: $f^{-1}(w_{y(s)}) = v_{y(s)}$. That is $v_{y(s)} = f^{-1}(w_{y(s)})$. Then, it is obvious that $v_{y(s)}$ is open in $X$ and such that: $f^{-1}(y) \subset v_{y(s)} = f^{-1}(f(v_{y(s)})) \subset u_{y(s)}$ and $f(v_{y(s)})$ is an open subset of $Y$. Obviously, $\mathcal{R}_1 = \{f(v_{y(s)}) : y \in Y\}$ is an open covering of $Y$. Since $Y$ is nearly-paracompact in $Y$, so there exists an open family covering $\mathcal{R}_2 = \{v_a : a \in A\}$ of $Y$ by open subsets of $Y$, such that: $\mathcal{R}_2$ refines $\mathcal{R}_1$, and $\mathcal{R}_2$ is locally finite at each $y$ of $Y$. We may also assume that $f(v_{y(s)})$ which contains $v_a = f(v_a(s))$. Since $f$ is a perfect mapping, thus, $\mathcal{R}_3 = \{f^{-1}(v_a) : a \in A\}$ is an open family of $X$ and locally Finite each $x \in f^{-1}(Y)$. Obviously, $f^{-1}(v_a) \subset f^{-1}(f(v_{y(s)})) = v_{y(s)} = u_{y(s)}$. Let $\mathcal{R} = \{f^{-1}(v_a) \cap u_a \cap Y : a \in A, s \in S_a(y)\}$. Then, $\mathcal{R}$ is an open family covering of $Y$ and such that $\mathcal{R}$ refines $\mathcal{A}$ and $\mathcal{R}$ is locally finite at each $x \in f^{-1}(Y)$. That is $f^{-1}(Y)$ is nearly-paracompact in $X$. 

149
References