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# Steady-state Analysis of the GI/M/1 Queue with Multiple Vacations and Set-up Time

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**Abstract**

In this paper, we consider a GI/M/1 queueing model with multiple vacations and set-up time. We derive the distribution and the stochastic decomposition of the steady-state queue length, meanwhile, we get the waiting time distributions.

**Keywords:** Multiple vacations, Set-up time, Stochastic decomposition

## Introduction

Vacation queues servers to stop the customers' service at some periods, and the time during which the service is interrupted is called the vacation time. Vacation queue research originated from Levy and Yechial, then many researchers on queuing theory deal with this fields. So far, the theory frame whose core is the stochastic decomposition is developed and vacation queues have been applied successfully to many fields, such as computer systems, communication networking, electronic and call centers. Details can be seen in the surveys of Doshi and the monographs of Tian. For GI/M/1 type queues with server vacations, Tian used the matrix geometric solution method to analyze and obtained the expressions of the rate matrix and proved the stochastic decomposition properties for queue length and waiting time in a GI/M/1 vacation model with multiple exponential vacations.

## 1. Description of the model

Consider a classical GI/M/1 queue, inter-arrival times are i.i.d.r.vs. Let $A(x)$ and $a^*(s)$ be the distribution function and L.S transform of the inter-arrival time $A$ of customers. The mean inter-arrival time is $E(A) = -a^{*'}(0) = \lambda^{-1}$,Service times during service period, vacation times and set-up times are assumed to be exponentially distributed with rate $\mu, \theta, \beta$, respectively. We assume that the service discipline is FCFS.

Suppose $\tau_n$ be the arrival epoch of nth customers with $\tau_0$=0. Let $L_n = L_v(\tau_n^-)$ be the number of the customers before the nth arrival. Define

$$J_n = J(\tau_n) = \begin{cases} 0, \text{the nth arrival occurs during a service period,} \\ 1, \text{the nth arrival occurs during a set-up period,} \\ 2, \text{the nth arrival occurs during a vacation period.} \end{cases}$$

The process $\{(L_n, J_n), n \geq 1\}$ is a Markov chain with the state space

$$\Omega = \{(0,2) \cup (k, j), k \geq 1, j = 0,1,2\}.$$

We introduce the expressions below

$$a_k = \int_0^\infty \frac{(\mu t)^k}{k!} e^{-\mu t} dA(t), k \geq 0, \quad b_k = \int_0^\infty \int_0^t \beta e^{-\beta x} \frac{[\mu(t-x)]^k}{k!} e^{-\mu(t-x)} dx dA(t), k \geq 0,$$

$$c_k = \int_0^\infty \int_0^t \int_0^{t-y} \theta e^{-\theta x} \beta e^{-\beta y} \frac{[\mu(t-x-y)]^k}{k!} e^{-\mu(t-x-y)} dx dy dA(t), k \geq 0.$$

First, the transition from $(i,0)$ to $(j,0)$ occur if $i+1-j$ services complete during an inter-arrival time. Therefore, we have

$$p_{(i,0)(j,0)} = a_{i+1-j}, i \geq 1, j = 1, \cdots, i+1.$$

Similarly,

$$p_{(i,1)(j,0)} = b_{i+1-j}, i \geq 1, j = 1, \cdots, i+1. \qquad p_{(i,1)(i+1,1)} = \int_0^\infty e^{-\beta t} dA(t) = a^*(\beta) = \gamma_2 .$$

$$p_{(i,2)(i+1,2)} = \int_0^\infty e^{-\theta t} dA(t) = a^*(\theta) = \gamma_3 \cdot \qquad p_{(i,2)(j,0)} = c_{i+1-j}, i \geq 0, j = 1, \cdots, i+1.$$

$$p_{(i,2)(i+1,1)} = \int_0^\infty \int_0^t \theta e^{-\theta x} e^{-\beta(t-x)} dA(t) = \theta(a^*(\beta) - a^*(\theta)) \big/ (\theta - \beta) = \alpha(\gamma_2 - \gamma_3) \cdot$$

The transition matrix of $(L_n, J_n)$ can be written as the Block-Jacobi matrix

$$\widetilde{P} = \begin{bmatrix} B_{00} & A_{01} & & & & \\ B_1 & A_1 & A_0 & & & \\ B_2 & A_2 & A_1 & A_0 & & \\ B_3 & A_3 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where $B_{00} = 1 - c_0 - \alpha(\gamma_2 - \gamma_3) - \gamma_3, \quad A_{01} = (c_0, \alpha(\gamma_2 - \gamma_3), \gamma_3),$

$$A_0 = \begin{pmatrix} a_0 & 0 & 0 \\ b_0 & \gamma_2 & 0 \\ c_0 & \alpha(\gamma_2 - \gamma_3) & \gamma_3 \end{pmatrix}, \qquad A_k = \begin{pmatrix} a_k & 0 & 0 \\ b_k & 0 & 0 \\ c_k & 0 & 0 \end{pmatrix}, k \geq 1 \qquad B_k = \begin{pmatrix} 1 - \sum_{i=0}^k a_i \\ 1 - \sum_{i=0}^k b_i - \gamma_2 \\ 1 - \sum_{i=0}^k c_i - \alpha(\gamma_2 - \gamma_3) - \gamma_3 \end{pmatrix}, k \geq 1,$$

The matrix $\widetilde{P}$ is a GI/M/1 type matrix.

## 2. Steady-state queue length distribution

**Lemma1.** If $\rho = \lambda \mu^{-1} < 1$, $\theta, \beta > 0$, then $\delta > 0$, $\frac{\theta\beta}{\theta - \beta}(\delta - \Delta) > 0$.

where $$\delta = \frac{\gamma_1 - a^*(\beta)}{\beta - \mu(1 - a^*(\beta))} = \frac{\gamma_1 - \gamma_2}{\beta - \mu(1 - \gamma_2)}, \quad \Delta = \frac{\gamma_1 - a^*(\theta)}{\theta - \mu[1 - a^*(\theta)]} = \frac{\gamma_1 - \gamma_3}{\theta - \mu(1 - \gamma_3)} \cdot \quad (1)$$

**Theorem 1.** If $\rho < 1$, $\theta, \beta > 0$, then the matrix equation $R = \sum_{k=0}^\infty R^k A_k$ has the minimal non-negative solution

$$R=\begin{pmatrix}\gamma_1 & 0 & 0\\ \beta\delta & \gamma_2 & 0\\ \alpha\beta(\delta-\Delta) & \alpha(\gamma_2-\gamma_3) & \gamma_3\end{pmatrix}$$

where $\gamma_1$ is the unique roots in the range $0<z<1$ of the equation $z=a^*(\mu(1-z))$.

$\alpha=\theta/(\theta-\beta)$, $\delta$ and $\Delta$ are defined as in (1).

**Proof.** Because all $A_k, k\ge 0$ are lower triangular, we assume that $R$ has the same structure as

$$R=\begin{pmatrix}r_{11} & 0 & 0\\ r_{21} & r_{22} & 0\\ r_{31} & r_{32} & r_{33}\end{pmatrix}$$

we obtain

$$\begin{cases} r_{11}=\sum_{k=0}^{\infty} r_{11}^k a_k = a^*(\mu(1-r_{11})),\quad r_{22}=\gamma_2,\quad r_{33}=\gamma_3\\ r_{21}=\sum_{k=1}^{\infty} r_{21}\left(\sum_{i=0}^{k-1} r_{11}^i r_{22}^{k-1-i}\right)a_k+\sum_{k=0}^{\infty} r_{22}^k b_k,\quad r_{32}=\alpha(\gamma_2-\gamma_3)\\ r_{31}=\sum_{k=1}^{\infty} r_{31}\left(\sum_{i=0}^{k-1} r_{11}^i r_{33}^{k-1-i}\right)a_k+\sum_{k=2}^{\infty} r_{32}r_{21}\left(\sum_{i=0}^{k-2} r_{11}^i \sum_{j=0}^{k-2-i} r_{22}^j r_{33}^{k-2-i-j}\right)a_k\\ \quad +\sum_{k=1}^{\infty} r_{32}\left(\sum_{i=0}^{k-1} r_{22}^i r_{33}^{k-1-i}\right)b_k+\sum_{k=0}^{\infty} r_{33}^k c_k \end{cases} \tag{2}$$

As we known, if $\rho<1,\theta>0$, the first equation has the unique root $r_{11}=\gamma_1$ in the range $0<r_{11}<1$. We can compute

$$\sum_{k=0}^{\infty} r_{22}^k b_k=\frac{\beta[a^*(\mu(1-r_{22}))-r_{22}]}{\beta-\mu(1-r_{22})}=\frac{\beta[a^*(\mu(1-\gamma_2))-\gamma_2]}{\beta-\mu(1-\gamma_2)},\qquad 1-\sum_{k=1}^{\infty}\left(\sum_{i=0}^{k-1} r_{11}^i r_{22}^{k-1-i}\right)a_k=1-\sum_{k=1}^{\infty}\frac{\gamma_1^k-\gamma_2^k}{\gamma_1-\gamma_2}a_k=1-\frac{\gamma_1-a^*(\mu(1-\gamma_2))}{\gamma_1-\gamma_2}$$
$$=\frac{a^*(\mu(1-\gamma_2))-\gamma_2}{\gamma_1-\gamma_2},$$

Finally, we obtain $r_{21}=\beta\delta$, $r_{31}=\alpha\beta(\delta-\Delta)$ and the expression for $R$.

**Theorem 2.** The Markov chain $(L_n,J_n)$ is positive recurrent if and only if $\rho<1$, $\theta,\beta>0$.

**Proof.** Based on Neuts, the Markov chain $(L_n,J_n)$ is positive recurrent if and only if the spectral radius SP($R$)=max$\{\gamma_1,\gamma_2,\gamma_3\}$of $R$ is less than 1, and the matrix

$$B[R]=\begin{pmatrix}B_{00} & A_{01}\\ \sum_{k=1}^{+\infty}R^{k-1}B_k & \sum_{k=1}^{+\infty}R^{k-1}A_k\end{pmatrix}$$

has a positive left invariant vector. Evidently, SP($R$)=max$\{\gamma_1,\gamma_2,\gamma_3\}$<1. Substituting the expressions for $R, A_k$ and $B_k$ in $B[R]$,we obtain

$$B[R]=\begin{pmatrix}1-c_0 & c_0 & 0 & 0\\ \frac{a_0}{\gamma_1} & 1-\frac{a_0}{\gamma_1} & 0 & 0\\ 1-\frac{\beta\delta a_0}{\gamma_1\gamma_2}+\frac{b_0}{\gamma_2} & \frac{\beta\delta a_0}{\gamma_1\gamma_2}-\frac{b_0}{\gamma_2} & 0 & 0\\ A & B & \alpha\frac{\gamma_2-\gamma_3}{\gamma_3} & 1\end{pmatrix}$$

Where

$$A=-\alpha\beta\left(\frac{\delta-\Delta}{\gamma_1\gamma_3}+\frac{\delta(\gamma_2-\gamma_3)}{\gamma_1\gamma_2\gamma_3}\right)a_0-\alpha\frac{\gamma_2-\gamma_3}{\gamma_2\gamma_3}+\frac{c_0}{\gamma_3}-\alpha\frac{\gamma_2-\gamma_3}{\gamma_3}$$

$$B=\alpha\beta\left(\frac{\delta-\Delta}{\gamma_1\gamma_3}+\frac{\delta(\gamma_2-\gamma_3)}{\gamma_1\gamma_2\gamma_3}\right)a_0+\alpha\frac{\gamma_2-\gamma_3}{\gamma_2\gamma_3}-\frac{c_0}{\gamma_3}.$$

It can be verify that $B[R]$has the left invariant vector

$$\pi_0=K(1,\alpha\beta(\delta-\Delta),\alpha(\gamma_2-\gamma_3),\gamma_3). \tag{3}$$

Thus, if $\rho<1$, $\theta,\beta>0$, the Markov chain $(L_n,J_n)$ is positive recurrent.

If $\rho<1,\theta,\beta>0$, let $(L_v,J)$ be the stationary limit of the process $(L_n,J_n)$.Let

$$\pi_0=\pi_{02};\quad \pi_k=(\pi_{k0},\pi_{k1},\pi_{k2}),k\ge 1,$$

$$\pi_{kj}=P\{L=k,J=j\}=\lim_{n\to\infty}P\{L_n=k,J_n=j\},(k,j)\in\Omega.$$

**Theorem 3.** If $\rho<1$, the stationary probability distribution of $(L_v,J)$ is

$$\begin{cases}\pi_{k0}=K\alpha\beta\left(\dfrac{\gamma_1^k-\gamma_2^k}{\gamma_1-\gamma_2}\delta-\dfrac{\gamma_1^k-\gamma_3^k}{\gamma_1-\gamma_3}\Delta\right),k\ge 1,\\ \pi_{k1}=K\alpha(\gamma_2^k-\gamma_3^k),k\ge 1,\\ \pi_{k2}=K\gamma_3^k,k\ge 0,\end{cases}$$

where $K=\dfrac{(1-\gamma_1)(1-\gamma_2)(1-\gamma_3)}{\alpha\beta[(1-\gamma_3)\delta-(1-\gamma_2)\Delta]+\alpha(\gamma_2-\gamma_3)(1-\gamma_1)+(1-\gamma_1)(1-\gamma_2)}$

**Proof.** $(\pi_{02},\pi_{10},\pi_{11},\pi_{12})$ is given by the positive left invariant vector (3) and satisfies the normalizing condition

$$\pi_{02}+(\pi_{k0},\pi_{k1},\pi_{k2})(I-R)^{-1}e=1$$

Then, we get

$$K=\frac{(1-\gamma_1)(1-\gamma_2)(1-\gamma_3)}{\alpha\beta[(1-\gamma_3)\delta-(1-\gamma_2)\Delta]+\alpha(\gamma_2-\gamma_3)(1-\gamma_1)+(1-\gamma_1)(1-\gamma_2)}.$$

We obtain

$$\pi_{02}=K,\quad (\pi_{10},\pi_{11},\pi_{12})=K(\alpha\beta(\delta-\Delta),\alpha(\gamma_2-\gamma_3),\gamma_3).$$

We have

$$\pi_k=(\pi_{k0},\pi_{k1},\pi_{k2})=(\pi_{10},\pi_{11},\pi_{12})R^{k-1},\quad k\ge 1,$$

Finally, we obtain the theorem.

**Theorem 4.** If $\rho<1$, the stationary queue length $L_v$ can be decomposed into the sum of two independent random variables: $L_v=L+L_d$, where $L$ is the stationary queue length of a classical GI/M/1queue without vacation, follows a geometric distribution with parameter $\gamma_1$; $L_d$ follows the discrete PH distributions $(\varphi,T)$ of order 2,where

$$\varphi=\frac{K}{1-\gamma_1}\left(\alpha\frac{\gamma_2-\gamma_1+\beta\delta}{1-\gamma_2}\quad -\frac{(\alpha-1)(\gamma_3-\gamma_1)+\alpha\beta\Delta}{1-\gamma_3}\right),\varphi_3=\frac{K}{1-\gamma_1},$$

$$T=\begin{pmatrix}\gamma_2 & \\ & \gamma_3\end{pmatrix},T^0=\begin{pmatrix}1-\gamma_2\\1-\gamma_3\end{pmatrix}$$

**Proof.** The PGF of $L_v$ is as follows:

$$L_v(z)=\sum_{k=0}^{\infty}z^kP(L_v=k)$$

$$=K\alpha\left[\frac{z^k\gamma_3^k}{\alpha}+\left(\gamma_2^k-\gamma_3^k\right)z^k+\frac{\beta\delta\left(\gamma_1^k-\gamma_2^k\right)}{\gamma_1-\gamma_2}z^k-\frac{\beta\Delta\left(\gamma_1^k-\gamma_3^k\right)}{\gamma_1-\gamma_3}z^k\right]$$

$$=\frac{1-\gamma_1}{1-z\gamma_1}\frac{K}{1-\gamma_1}\left(\alpha\frac{1-z\gamma_1+z\beta\delta}{1-z\gamma_2}-\frac{(\alpha-1)(1-z\gamma_1)+z\alpha\beta\Delta}{1-z\gamma_3}\right)=L(z)L_d(z)$$

Where $L(z)$ is the PGF of $L$ of a classical GI/M/1queue without vacation.

$$L_d(z)=\frac{K}{1-\gamma_1}\left(\alpha\frac{1-z\gamma_1+z\beta\delta}{1-z\gamma_2}-\frac{(\alpha-1)(1-z\gamma_1)+z\alpha\beta\Delta}{1-z\gamma_3}\right) \tag{4}$$

$$\frac{1-z\gamma_1+z\beta\delta}{1-z\gamma_2}=\left(1-z\gamma_1+z\beta\delta\right)\sum_{k=0}^{\infty}\gamma_2^k z^k \quad =1+\left(\gamma_2+\beta\delta-\gamma_1\right)\sum_{k=1}^{\infty}\gamma_2^{k-1}z^k .$$

Substituting the above equation into (4),we obtain the distribution of $L_d$.

We can get means

$$E(L_v)=\frac{\gamma_1}{1-\gamma_1}+\frac{K}{1-\gamma_1}\left[\alpha\frac{\gamma_2-\gamma_1+\beta\delta}{(1-\gamma_2)^2}-\frac{(\alpha-1)(\gamma_3-\gamma_1)+\alpha\beta\Delta}{(1-\gamma_3)^2}\right].$$

**3. Waiting time distribution**

Let $W$ and $\widetilde{W}(s)$ be the steady-state waiting time and its LST, respectively. Firstly, let $H_0, H_1, H_2$ be the probability that the server is in the service(set-up, vacation) period when a new customer arrives. We can compute

$$H_0=\frac{\alpha\beta\left[(1-\gamma_3)\delta-(1-\gamma_2)\Delta\right]}{\alpha\beta\left[(1-\gamma_3)\delta-(1-\gamma_2)\Delta\right]+\alpha(\gamma_2-\gamma_3)(1-\gamma_1)+(1-\gamma_1)(1-\gamma_2)},$$

$$H_1=\frac{\alpha(\gamma_2-\gamma_3)(1-\gamma_1)}{\alpha\beta\left[(1-\gamma_3)\delta-(1-\gamma_2)\Delta\right]+\alpha(\gamma_2-\gamma_3)(1-\gamma_1)+(1-\gamma_1)(1-\gamma_2)},$$

$$H_2=\frac{(1-\gamma_1)(1-\gamma_2)}{\alpha\beta\left[(1-\gamma_3)\delta-(1-\gamma_2)\Delta\right]+\alpha(\gamma_2-\gamma_3)(1-\gamma_1)+(1-\gamma_1)(1-\gamma_2)}.$$

**Theorem 5.** If $\rho<1$, $\theta,\beta>0$, the LST of stationary waiting time $W$ is

$$\widetilde{W}(s)=H_1\frac{\beta}{\beta+s}\frac{(\mu+s)(1-\gamma_2)}{\mu(1-\gamma_2)+s}\frac{\mu(1-\gamma_3)}{\mu(1-\gamma_3)+s}+H_2\frac{\theta}{\theta+s}\frac{\mu(1-\gamma_3)}{\mu(1-\gamma_3)+s}\frac{\beta}{\beta+s}$$
$$+H_0\left[\mu+\frac{(\delta-\Delta)s}{\delta(1-\gamma_3)-\Delta(1-\gamma_2)}\right]\frac{(\mu+s)(1-\gamma_1)}{\mu(1-\gamma_1)+s}\frac{(1-\gamma_2)}{\mu(1-\gamma_2)+s}\frac{\mu(1-\gamma_3)}{\mu(1-\gamma_3)+s}$$

**Proof.** When a customer arrives, if there are $k$ customers and the server is in the service period, the waiting time equals $k$ service times by the rate $\mu$. Then, we have

$$\sum_{k=1}^{\infty}\pi_{k0}\widetilde{W}_{k0}(s)=K\alpha\beta\sum_{k=1}^{\infty}\left(\frac{\gamma_1^k-\gamma_2^k}{\gamma_1-\gamma_2}\delta-\frac{\gamma_1^k-\gamma_3^k}{\gamma_1-\gamma_3}\Delta\right)\left(\frac{\mu}{\mu+s}\right)^k$$
$$=H_0\left[\mu+\frac{(\delta-\Delta)s}{\delta(1-\gamma_3)-\Delta(1-\gamma_2)}\right]\frac{(\mu+s)(1-\gamma_1)}{\mu(1-\gamma_1)+s}\frac{(1-\gamma_2)}{\mu(1-\gamma_2)+s}\frac{\mu(1-\gamma_3)}{\mu(1-\gamma_3)+s} \tag{5}$$

When a customer arrives, if there are $k$ customers and the server is in the set-up period, the waiting time is the sum of the residual set-up time and $k$ service times by the rate $\mu$. Then, we have

$$\sum_{k=1}^{\infty}\pi_{k1}\widetilde{W}_{k1}(s)=K\alpha\frac{\beta}{\beta+s}\sum_{k=1}^{\infty}(\gamma_2^k-\gamma_3^k)\left(\frac{\mu}{\mu+s}\right)^k$$

$$=H_1\frac{\beta}{\beta+s}\frac{(\mu+s)(1-\gamma_2)}{\mu(1-\gamma_2)+s}\frac{\mu(1-\gamma_3)}{\mu(1-\gamma_3)+s}. \quad (6)$$

Similarly,

$$\sum_{k=1}^{\infty}\pi_{k2}\tilde{W}_{k2}(s)=K\sum_{k=1}^{\infty}\left(\frac{\mu\gamma_3}{\mu+s}\right)^k\frac{\beta}{\beta+s}\frac{\theta}{\theta+s}=H_2\frac{\theta}{\theta+s}\frac{\mu(1-\gamma_3)}{\mu(1-\gamma_3)+s}\frac{\beta}{\beta+s}. \quad (7)$$

From (5)-(7), we have the result in Theorem 4.

With the structure in Theorem 4, we can get the expected waiting time

$$E(W)=H_2\left[\frac{1}{\theta}+\frac{1}{\mu(1-\gamma_3)}+\frac{1}{\beta}\right]+H_1\left[\frac{\gamma_2}{\mu(1-\gamma_2)}+\frac{1}{\mu(1-\gamma_3)}+\frac{1}{\beta}\right]$$

$$+H_0\left[\frac{\gamma_1}{\mu(1-\gamma_1)}-\frac{\delta-\Delta}{\mu[\delta(1-\gamma_3)-\Delta(1-\gamma_2)]}+\frac{1}{\mu(1-\gamma_2)}+\frac{1}{\mu(1-\gamma_3)}\right].$$

**4. Numerical examples**

In the above analysis,we obtain the expected queue length in the steady state. The difference of parameters may influence the queue length. So, we present numerical examples to explain.

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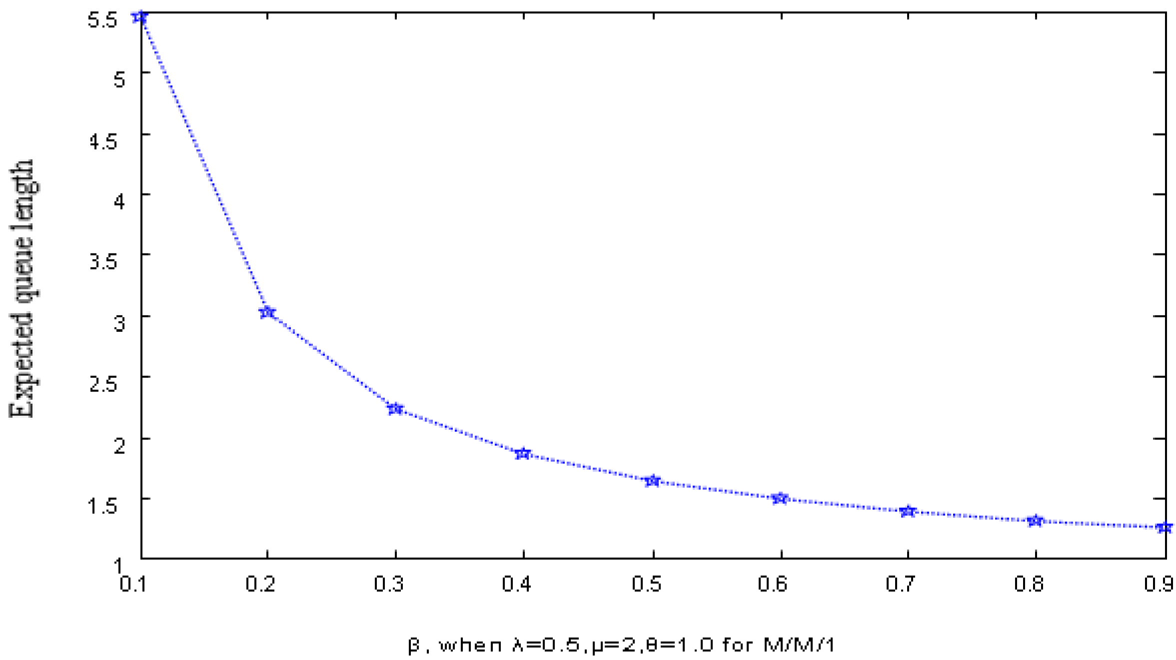


Figure 1. The relation of $E(L_v)$with $\beta$