A Note on Two Theorems of C. Dong and J. Wang
Concerning Combinatorial Identities

Arnold R. Kräuter
Department of Mathematics and Information Technology
University of Leoben
Franz-Josef-Strasse 18, A-8700 Leoben, Austria
Tel: 43-3842-402-3803 E-mail: kraeuter@unileoben.ac.at

Abstract
In a recent paper C. Dong and J. Wang rederived three classical combinatorial identities by applying a special Vandermonde determinant. Two of their results, however, turn out to be incorrectly stated. This note presents counterexamples along with revised versions of the results mentioned.

Keywords: Vandermonde determinant, Identities involving binomial coefficients

1. Introduction
C. Dong and J. Wang applied a special Vandermonde determinant in order to establish a couple of well-known combinatorial identities in an elementary way (Dong & Wang, 2007). Unfortunately, two of these are incorrectly stated. In the following we present counterexamples as well as revised versions of the respective theorems in the cited paper.

Let $V_n$ denote the $n$-square Vandermonde matrix (cf. Lancaster & Tismenetsky, 1985, p. 35, Exercise 6) of the integers $1, 2, \ldots, n$,

$$V_n = \begin{bmatrix}
1 & 1 & 1 & \ldots & 1 \\
1 & 2 & 3 & \ldots & n \\
1 & 2^2 & 3^2 & \ldots & n^2 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2^{n-1} & 3^{n-1} & \ldots & n^{n-1}
\end{bmatrix},$$

and let $D_n = \text{det}(V_n)$ be the determinant of $V_n$. Furthermore, let $M_{ij}$ be the minor of the entry in the $n$th row and $j$th column of $V_n$, and let $S_j$ be the minor of the entry in the first row and $j$th column of $V_n$.

**Lemma 1** (Dong & Wang, 2007, p. 24, Eq. (1)).

$$M_{ij} = \left( \prod_{k=1}^{s+1} m_k \right) \binom{n-1}{n-j}, \quad j = 1, 2, \ldots, n. \quad (1)$$

**Lemma 2** (Dong & Wang, 2007, p. 24, Eq. (2)).

$$S_j = \left( \prod_{k=1}^{s+1} m_k \right) j, \quad j = 1, 2, \ldots, n. \quad (2)$$

Using (1) and (2) Dong and Wang obtained, among others, the following identities for positive integers $n$ (Dong & Wang, 2007, p. 25, Eqs. (3) and (6), resp.):

$$\sum_{j=1}^{s} (-1)^{s-j} j^{s-1} \binom{n}{j} = n!. \quad (3)$$
2. Counterexamples to Eqs. (3) and (4)

In their current form, Eqs. (3) and (4) turn out to be incorrect.

Example 1. Eq. (3) is false. For if, e. g., \( n = 4 \), (3) would imply

\[
\sum_{j=0}^{4-1} (-1)^j (n-j)^j \binom{n}{j} = 0, \quad 1 \leq i \leq n, \tag{4}
\]

Example 2. Eq. (4) is false in the case \( i = n \). For if, e. g., \( i = n = 4 \), (4) would imply

\[
\sum_{j=0}^{3} (-1)^j (4-j)^j \binom{4}{j} = 256 - 324 + 96 - 4 = 24 \neq 0.
\]

3. Restatements of Eqs. (3) and (4) with proofs

The original Eqs. (3) and (4) have to be replaced by the following statements.

Theorem 1 (cf. Gould, 1972, p. 2, Eq. (1.13), first part). For every nonnegative integer \( n \) we have

\[
\sum_{j=1}^{n} (-1)^{k+j} j^k \binom{n}{j} = n!.
\]  

\[
\sum_{j=1}^{n} (-1)^{k+j} j^k \binom{n}{j} = (n-1)!. \tag{5}
\]

Proof. According to the original proof (Dong & Wang, 2007, p. 25, Theorem 1) the following holds:

\[
D_n = \sum_{j=1}^{n} (-1)^{k+j} j^k \binom{n-1}{n-j} = \prod_{m=1}^{n-2} m!. \tag{5}
\]

This gives

\[
\sum_{j=1}^{n} (-1)^{k+j} j^k \binom{n-1}{n-j} = (n-1)!. \tag{6}
\]

Using the identity (cf. Gould, 1972, p. iv)

\[
\binom{n-1}{n-j} = \frac{j}{n} \binom{n}{j}, \quad 1 \leq j \leq n, \tag{6}
\]

we eventually obtain (5). \( \square \)

Theorem 2 (cf. Gould, 1972, p. 2, Eq. (1.13), second part). For every nonnegative integer \( n \) we have

\[
\sum_{j=0}^{n-1} (-1)^i (n-j)^i \binom{n}{j} = 0, \quad 1 \leq i \leq n-1. \tag{7}
\]

Proof. Let \( W_n^{(i)} \) denote the \( n \)-square matrix obtained from the Vandermonde matrix \( V_n \) by replacing the \( n \)th row by the \( i \)th row, \( 1 \leq i \leq n - 1 \).
Since $W_n^{(i)}$ is singular by construction, expansion of $\det(W_n^{(i)})$ by the $n$th row gives

\[
0 = \det(W_n^{(i)}) = \sum_{j=1}^{n} (-1)^{n+j} j^{-1} M_j.
\]

Using Lemma 1 and Eq. (6) we get

\[
0 = \sum_{j=1}^{n} (-1)^{n+j} j^{-1} \left( \prod_{m=1}^{n-2} m! \right) \left( \binom{n-1}{n-j} \right) = \\
= \sum_{j=1}^{n} (-1)^{n+j} j^{-1} \left( \prod_{m=1}^{n-2} m! \right) \left( \binom{n}{j} \right).
\]

Replacing $j$ by $n - j$ and by the symmetry of the binomial coefficients we eventually obtain (7). □

**Remark.** The arguments used in the proof of Theorem 2 still hold when $W_n^{(i)}$ is replaced by the matrix

\[
X_n^{(i)} = \begin{bmatrix}
1 & 2^{-1} & 3^{-1} & \cdots & n^{-1} \\
1 & 2 & 3 & \cdots & n \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 2^{n-2} & 3^{n-2} & \cdots & n^{n-2} \\
1 & 2^{n-1} & 3^{n-1} & \cdots & n^{n-1}
\end{bmatrix}, \quad 2 \leq i \leq n,
\]

followed by the expansion of $\det(X_n^{(i)})$ by the first row (which involves the numbers $S_j$ given in Lemma 2).

**References**

