Rayleigh Instability of Plane-parallel Liquid Flows

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Abstract

Studying in this paper the stability of plane-parallel flows of an ordinary liquid can be naturally translated into the language of the theory of hydrodynamic resonances. Thus, resonant absorption of oscillations induces stability of the flows of an ideal liquid having a velocity profile without inflection points (Rayleigh theorem), while resonant emission leads to Rayleigh instability in the presence of an inflection point. The flow velocity profile has an inflection point. Thus, the presence of inflection points is a necessary condition for instability. If, however, the velocity profile has inflection points, the flow is stable (Rayleigh's theorem). Note that the sign of the jump depends on whether the neutral oscillations are regarded as the limiting cases of growing $(\text{Im}\,\omega > 0)$ or damped $(\text{Im}\,\omega < 0)$ oscillations.

Keywords: Rayleigh Instability, Plane-Parallel Liquid Flows

1. Introduction

Rayleigh has established in 1880 (Rayleigh, 1880), that plane-parallel liquid flows (Barston, 1991) of an ideal liquid, with velocity profiles that have no inflection points, are stable Rayleigh's theorem (Rayleigh, 1880).

Rayleigh instability (Khenner, et al., 1999; Wolf, 1970a, 1997b; Kumar, et al., 1994) occurs when a heavy fluid is supported by a lighter fluid. Any perturbation of the interface grows and leads to spikes of the heavier fluid penetrating into the lighter one.

It is known that vertical vibrations can lead both to the parametric excitation of waves at the interface and to the suppression of the Rayleigh–Taylor instability (Benjamin, et al., 1954; Miller, et al., 1983; Kumar, et al., 1994; Wolf, 1970), while the effects due to horizontal vibrations have been studied less. In experimental works by (Bezdeneznykh, et al., 1984; Piriz, et al., 2010) for a long horizontal reservoir filled with two immiscible viscous fluids, an interesting phenomenon was found at the interface: the horizontal vibrations lead to the formation of a steady relief. This formation mechanism has a threshold nature; it is noteworthy that such a wavy relief appears on the interface only if the densities of the two fluids are close enough, i.e. it does not appear for the liquid/gas interface.

Recently, (Dou, 2002; Kuznetsov, et al., 2011) proposed a new mechanism for flow instability and turbulent transition in parallel shear flows. In this paper, based on the previous work (Dou, 2002; Kuznetsov, et al., 2011), it is demonstrated that the stability of plane-parallel flows of an ordinary liquid can be naturally translated into the language of the theory of hydrodynamic resonances.

2. Basic Equations

Rayleigh's theorem deals with oscillations of an ideal liquid. The result of the theorem is valid for the flows of a real liquid as sufficiently large Reynolds numbers, when the influence of viscosity can be neglected. In this approximation, the equation of motion takes the following form

$$\frac{d\vec{V}}{dt} = -\frac{\nabla p}{\rho} \tag{1}$$

Where p is the pressure and ρ is the density of the medium.

If the liquid is incompressible, the velocity $\vec{V}$ must also satisfy the equation

$\nabla \cdot \vec{V} = 0$.

In the $xy-$plane the last equation will be satisfied if $\vec{V} = (\partial \psi / \partial y; -\partial \psi / \partial x)$, where ψ is the flow function.

Taking the z – component of the *curl* of (1) and expressing the velocity in terms of the flow current function, then

$$\frac{d}{dt}\Delta\psi = 0 \tag{2}$$

This equation is called the law of conservation of the velocity *curl* in an ideal incompressible liquid ($\nabla \times \vec{V} = z_0 \Delta\psi$).

Linearizing (2) in terms of small perturbations, then becomes

$$\left(\frac{\partial}{\partial t} + V_0(x)\frac{\partial}{\partial y}\right)\Delta\psi_1 - \frac{d^2 V_0}{d t^2}\frac{\partial\psi_1}{\partial y} = 0 \tag{3}$$

It is assumed here that a small perturbation characterized by ψ_1 is applied to a stationary flow having a velocity directed along y – axis and varying along x – axis.

By using the following perturbation

$$\psi_1(\vec{r},t) = \psi_1(x)\, e^{-i(\omega t - ky)}$$

Then we get

$$\psi_1'' + \left(\frac{kV_0''}{\omega - kV_0} - k^2\right)\psi_1 = 0 \tag{4}$$

This equation is called Rayleigh equation.

We shall follow (Lin, 1955). We multiply (4) by ψ_1^* and subtract from the product of the complex conjugate

$$\frac{d}{d x}\left(\psi_1^* \frac{d\psi_1}{d x} - \psi_1 \frac{d\psi_1^*}{d x}\right) = 2i\frac{\operatorname{Im}\omega kV_0''}{|\omega - kV_0|^2}|\psi_1|^2 \tag{5}$$

Integrating (5) from one boundary of the flow to the other

$$W\Big|_{x_1}^{x_2} = k\operatorname{Im}\omega\int_{x_1}^{x_2}\frac{kV_0''}{|\omega - kV_0|^2}|\psi_1|^2 d x \tag{6}$$

We have introduced in (6) the real quantity $W = (ik/2)[\psi_1(d\psi_1^*/dx) - \psi_1^*(d\psi_1/dx)]$. It defines the so-called Reynolds stress

$$\widetilde{W}_T = \rho W = -\rho\operatorname{Re}(V_{x_1}V_{y_1}^*) \tag{7}$$

in the oscillations.

Equation (5), which is the differential equivalent of (6), shows that at $\operatorname{Im}\omega = 0$ the function W is constant in the intervals and undergoes a jump at the resonant point. Note that the sign of the jump depends on whether the neutral oscillations are regarded as the limiting cases of growing ($\operatorname{Im}\omega > 0$) or damped ($\operatorname{Im}\omega < 0$) oscillations.

The normal velocity component $V_{x_1} = ik\psi_1$ and with it the left-hand side of (6) vanish on the solid walls that bound the flow. For $\operatorname{Im}\omega \neq 0$, however, the right-hand side of this equation can vanish only if V_0'' reverses sign in the interval (x_1, x_2), i.e., the flow velocity profile has an inflection point. Thus, the presence of inflection points is a necessary condition for instability. If, however, the velocity profile has now inflection points, the flow is stable (Rayleigh's theorem).

3. Conclusions

In this paper we study the basic laws determining the stability of plane-parallel flows of an ordinary liquid can be naturally translated into the language of the theory of hydrodynamic resonances. Thus, resonant absorption of oscillations induces stability of the flows of an ideal liquid having a velocity profile without inflection points (Rayleigh theorem), while resonant emission leads to Rayleigh instability in the presence of an inflection point.

Note that at first glance Equation (6) applies equally well to growing ($\operatorname{Im}\omega > 0$) and attenuating ($\operatorname{Im}\omega < 0$) oscillations. One would therefore conclude from (6) that if some natural oscillations are in fact

analysis of damped oscillations $(\text{Im}\,\omega < 0)$ calls for taking into account the viscosity, no matter how low, of the liquid.

Therefore all the conclusions concerning oscillations with $\text{Im}\,\omega < 0$, based on an analysis of the equation of an ideal liquid, are, generally speaking, incorrect.

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