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Chance-constrained Portfolio Selection with Birandom Returns

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Abstract

The aim of this paper is to solve the portfolio problem when security returns are birandom variables. Two types of portfolio selection based on chance measure are provided according to birandom theory. Since the proposed optimization problems are difficult to solve by traditional methods, a hybrid intelligent algorithm by integrating birandom simulation and genetic algorithm is designed. Finally, two numerical experiments are provided to illustrate the effectiveness of the algorithm.

Keywords: Portfolio selection, Birandom variable, Birandom simulation, Genetic algorithm

1. Introduction

The theory of portfolio selection was initially provided by Markowitz (1952, p.77) and has been greatly developed since then. It is concerned with selecting a combination of securities among portfolios containing large number of securities to reach the goal of obtaining satisfactory investment return. In his path-break work, Markowitz proposed a principle that when making investment decision, an investor should always strike a balance between maximizing the return and minimizing the risk, i.e., the investor maximize return for a given level of risk, or one should minimize risk for a predetermined return level. More importantly, Morkowitz initially quantified investment return as the expected value of returns of securities, and risk as variance from the expected value. After Maokowitz's work, scholars have been showing great enthusiasm in portfolio management, trying different mathematical approaches to develop the theory of portfolio selection. Traditionally, returns of individual securities are assumed to be stochastic variables, and many researchers were focused on extending Markowitz's mean-variance models and on developing new mathematical approaches to solve the problems of computation. Peng (2007, p. 433) proposed concept of birandom variable and the framework of birandom programming. However, investors may come across birandom returns in portfolio selection situations. For example, security returns are usually regarded to be normally distributed random variables, but the expected value may be still random variable, thus investors have to face random returns with random parameters, to deal with this type of uncertainty, we propose the security returns could be regarded as birandom variables. As a general mathematical description for this kind of stochastic phenomenon with incomplete statistical information, birandom variable is defined as a mapping with some kind of measurability from a probability space to a collection of random variables.

In general, there are three types of stochastic programming models for optimization problems in uncertain environment. The first is expected value model (EVM), which optimizes the expected objective function subject to some expected constraints. The second chance-constrained programming (CCP) was proposed by Charnes and Cooper (1965, p.73) and developed by many scholars as means of dealing with uncertainty by specifying a confidence level at which the uncertain constraints hold. We try to do something in this area. In this paper, returns of securities are assumed to be birandom parameters instead of stochastic ones. The portfolio will be selected according to the second type of programming models for optimization problems.

The rest of this paper is arranged as follows. After reviewing some necessary knowledge about birandom variable in section 2, in section 3, one type of single-objective birandom chance measure model and one type of birandom CCP model are proposed. To provide a general method for solving the new models, in section 4, a hybrid intelligent algorithm integrating genetic algorithm and birandom simulation is designed. To better illustrate the modeling idea of the paper and demonstrate the effectiveness of the proposed algorithm, two numerical examples are provided in section 5.

2. Preliminaries

Birandom variable theory was introduced by Peng in 2007. To better understand the chance measure and the CCP model for portfolio selection, let us briefly review some necessary knowledge about birandom variable.

Definition 1 A birandom variable ξ is a mapping from a probability space $(\Omega, \mathrm{A}, \Pr)$ to a collection of random variables such that for any Borel subset B of the real line R, the induced function $\Pr\{\xi(\omega)\in B\}$ is a measurable function with respect to ω.

Example 1 Let $\Omega=\{\omega_1,\omega_2\}$, and $\Pr\{\omega_1\}=\Pr\{\omega_2\}=1/2$. Assume that ξ is a function on $(\Omega, \mathrm{A}, \Pr)$ as follows.

$$\xi(\omega)=\begin{cases}\xi_1, & \omega=\omega_1\\ \xi_2, & \omega=\omega_2\end{cases}$$

where ξ_1 is a uniformly distributed random variable on $[0,1]$ and ξ_2 is a normally distributed random variable with mean 0 and variance 1, i.e., $\xi_1\sim U[0,1]$ and $\xi_2\sim N(0,1)$. Then ξ is a birandom variable according to the definition.

The definition of the primitive chance of birandom event as follows.

Definition 2 Let $\xi=(\xi_1,\xi_2,\cdots,\xi_n)$ be a birandom vector on $(\Omega, \mathrm{A}, \Pr)$, and $f: R^n\to R^m$ be a vector-valued Borel measurable function. Then the primitive chance of birandom event characterized by $f(\xi)\le 0$ is a function from $(0,1]$ to $[0,1]$, defined as

$$\mathrm{ch}\{f(\xi)\le 0\}(\alpha)=\sup\{\beta \mid \Pr\{\omega\in\Omega \mid \Pr\{f(\xi(\omega))\le 0\}\ge\beta\}\ge\alpha\}.$$

The following are the definitions of optimistic value and pessimistic value of birandom variable ξ.

Definition 3 Let ξ be a birandom variable, and $\gamma,\delta\in(0,1]$, then

$$\xi_{\sup}(\gamma,\delta)=\sup\{x \mid \mathrm{ch}\{\xi\ge x\}(\gamma)\ge\delta\}$$

is called the (γ,δ)-optimistic value to ξ, and

$$\xi_{\inf}(r,\delta)=\inf\{x \mid \mathrm{ch}\{\xi\le x\}(\gamma)\ge\delta\}$$

is called the (γ,δ)-pessimistic value to ξ.

3. Birandom chance-constrained portfolio selection

Let x_i denote the investment proportions in security i, ξ_i the birandom return for the ith security, $i=1,2,\cdots,n$, respectively, if the investor wants to maximize the chance of the total investment return no less than R at the confidence level α, where R is the predetermined total return and α is the predetermined confidence level, then the model is

$$\begin{aligned}&\max_x \ \mathrm{Ch}\{\textstyle\sum_{i=1}^n x_i\xi_i\ge R\}(\alpha)\\ &\text{s. t.}\ \ \textstyle\sum_{i=1}^n x_i=1\\ &\qquad x_i\ge 0,\ i=1,2,\cdots,n\end{aligned}\qquad(1)$$

In the following, we provide a spectrum of birandom chance-constrained programming (CCP) of portfolio selection.

$$\begin{aligned}&\max\ \bar f\\ &\text{s.t.}\ \mathrm{Ch}\{x_1\xi_1+x_2\xi_2+\cdots+x_n\xi_n\ge\bar f\}(\gamma)\ge\delta\\ &\quad \mathrm{x}_1+x_2+\cdots+x_n=1\\ &\quad x_i\ge 0, i=1,2,\cdots,n\end{aligned}\qquad(2)$$

where γ and δ are the predetermined investment levels that the investor feels satisfactory, and $\max\bar f$ is the (γ,δ)-optimistic value to the security return with respect to primitive chance.

4. Hybrid intelligent algorithm

Since the two-fold uncertainty of birandom variable, it is difficult to analytically solve the models (1) and (2). To provide a general solution to the models, we design a hybrid intelligent algorithm integrating genetic algorithm (GA) and birandom simulation. Roughly speaking, in the proposed hybrid intelligent algorithm, the technique of birandom simulation is applied to compute the chance measure and (r,δ)-optimistic value of the return function, then birandom simulation and GA are integrated for solving the birandom models.

4.1 Birandom simulation

In this section, we mainly discuss the calculation of the chance measure of birandom variables and (r,δ)-optimistic value of birandom variables.

Let ξ_i be birandom variables and x_i decision variables, $i=1,2,\cdots,n$, respectively. Write

$$f(x,\xi)=\sum_{i=1}^{n} x_i\xi_i,$$

where $x=(x_1,x_2,\cdots,x_n), \xi=(\xi_1,\xi_2,\cdots,\xi_n)$. In order to solve the proposed models, we must handle the following two types of uncertain function.

$U_1: x \to \text{Ch}\{f(x,\xi)\ge R\}(\alpha)$

$U_2: x \to \text{Ch}\{f(x,\xi)\ge \bar{f}\}(\gamma)\ge\delta$

$U_1(x)$ may be estimated by the following procedure.

Algorithm 1 (birandom simulation for $U_1(x)$)

Step 1 Generate $\omega_1,\omega_2,\cdots,\omega_N$ from Ω according to the probability measure Pr.

Step 2 Compute the probability $\beta_n=\Pr\{f(x,\xi(\omega_n))\ge R\}$ for $n=1,2,\cdots,N$, respectively, by stochastic simulation.

Step 3 Set N' as the integer part of αN.

Step 4 Return the N'th largest element $\bar{\beta}$ in $\{\beta_1,\beta_2,\cdots,\beta_N\}$.

Algorithm 2 (birandom simulation for $U_2(x)$)

Step 1 Generate $\omega_1,\omega_2,\cdots,\omega_N$ from Ω according to the probability measure Pr.

Step 2 Find the largest value $\bar{f}_n$ such that

$\Pr\{f(x,\xi(\omega))\ge \bar{f}_n\}\ge\delta$

for $n=1,2,\cdots,N$,respectively, by stochastic simulation.

Step 3 Set N' as the integer part of γN.

Step 4 Return the N'th largest element in $\{\bar{f}_1,\bar{f}_2,\cdots,\bar{f}_N\}$.

4.2 Genetic algorithm

Representation structure: A solution $x=(x_1,x_2,\cdots,x_n)$ is represented by the chromosome $V=(v_1,v_2,\cdots,v_n)$, where the genes $v_1,v_2,\cdots,v_n$ are randomly generated in the interval $[0,1]$, and the relation between x and V are formulated as follows:

$x_i=v_i/(v_1+v_2+\cdots+v_n), i=1,2,\cdots,n,$

which ensures that

$x_1+x_2+\cdots+x_n=1,\ x_i\ge 0, i=1,2,\cdots,n$ always holds.

Initialization process: *pop_size* number of chromosomes are initialized randomly by generating points $(v_1,v_2,\cdots,v_n)$ from the hypercube $[0,1]^n$ pop-size times. Since the constraint required that $x=(x_1,x_2,\cdots,x_n)$ satisfy $x_1+x_2+\cdots+x_n=1$, based on the relation between x and V, the feasibility of the randomly generated chromosomes is obvious.

Evaluation function: Evaluation function, denoted by $\text{Eva}(V)$, is to assign a probability of reproduction to each chromosome V so that its likelihood of being selected is proportional to its fitness relative to the other chromosomes in the population. That is, the chromosomes with higher fitness will have more chance to produce offspring by using roulette wheel selection.

One well-known evaluation function is based on allocation of reproductive trial according to rank rather than actual objective values. We can rearrange the pop-size chromosomes according to their objective values to make better chromosome take smaller ordinal number. That is, after rearrange, among pop-size chromosomes $V_1,V_2,\cdots,V_{pop\_size}$,

V_1 is the best chromosome, and V_{pop_size} the worst one, then a parameter $a \in (0,1)$ in the genetic system is given. We can define the rank-based evaluation function as follows:

$\mathrm{Eva}(V_i) = a(1-a)^i$, $i = 1,2,\cdots, pop\_size$.

Note that $i = 1$ means the best individual, $i = pop\_size$ the worst one.

Selection process: Firstly, calculate the cumulative probability q_i for each chromosome V_i,

$$q_0 = 0,\ q_i = \sum_{j=i}^{i} Eva(V_j),\ i = 1,2,\cdots, pop\_size$$

Secondly, generate a random number r in $(0, q_{pop\_size}]$, and select the chromosome V_i if r satisfies $q_{i-1} < r \le q_i$.

Repeat the second and third steps pop_size times and obtain pop_size copies of chromosome.

Crossover operation: A parameter p_c of a genetic system as the probability of crossover is defined first. The parents for crossover operation are selected by doing the following process repeatedly from $i = 1$ to pop_size: Generating a random number r from the interval $[0,1]$, the chromosome V_i is selected as a parent if $r < p_c$, the selected parents are denoted by $V_1', V_2', V_3', \cdots$ and divided into the pairs: $(V_1', V_2'), (V_3', V_4'), (V_5', V_6'), \cdots$. The crossover operation on each pair is illustrated by (V_1', V_2'). At first, we generate a random number c from the open interval $(0,1)$, then the operator on V_1' and V_2' will product two children X and Y as follows:

$X = cV_1' + (1-c)V_2'$, $Y = (1-c)V_1' + cV_2'$.

If both children are feasible, then we replace the parents with them. If not, we keep the feasible one if it exists, and then redo the crossover operator by regenerating a random number c until two feasible children are obtained or a given number of cycles is finished. In this case, we only replace the parents with the feasible child .

Mutation operation: A parameter p_m of a genetic system as the probability of mutation is defined first. This probability gives us the expected number of $p_m \cdot pop\_size$ of chromosomes undergoing the mutation operations. We repeat the following steps from $i = 1$ to pop_size: Generating a random number r from the interval $[0,1]$, the chromosome V_i is selected as a parent if $r < p_m$. For each selected parent V_i, we mutate it in the following way. Let M be an appropriate large positive number. We choose a mutation direction d in R^n randomly. If $V + M \cdot d$ is not feasible, then we set M as a random number between 0 and M until it is feasible. If the above process cannot find a feasible solution in a predetermined number of iterations, then we set $M = 0$. Anyway, we replace the parent V_i with its feasible child $V + M \cdot d$.

The following is the hybrid intelligent algorithm integrating birandom simulation and genetic algorithm.

Algorithm 3 (hybrid intelligent algorithm)

Step 1 Initialize pop_size chromosomes.

Step 2 Calculate the objective values for all chromosomes by birandom simulation.

Step 3 Given the rank order of the chromosomes according to the objective values, and the values of the rank-based evaluation function of the chromosomes.

Step 4 Compute the fitness of each chromosome according to the rank-based evaluation function.

Step 5 Select the chromosomes by spinning the roulette wheel.

Step 6 Update the chromosomes by crossover and mutation operations.

Step 7 Repeat the second step to the sixth step for a given number of cycles.

Step 8 Take the best chromosomes as the solution of portfolio selection.

5. Numerical examples

To illustrate the modeling idea and to test the effectiveness of the designed hybrid intelligent algorithm, let us consider two numerical examples. The two examples are both performed on a personal computer by using C++ programming language. The parameters in the HIA are set as follows: the probability of crossover $p_c = 0.3$, the probability of mutation $p_m = 0.2$, the parameter $a = 0.05$ in the rank-based evaluation function.

Example 2 Assume that there are 5 securities, the returns of securities are all birandom variables.

$$\max_x \ \text{Ch}\{\sum_{i=1}^{5} x_i\xi_i \ge R\}(\alpha)$$

subject to

$$\sum_{i=1}^{5} x_i = 1$$

$$x_i \ge 0, \ \ i = 1,2,3,4,5$$

Where

$\xi_1 \sim N(\mu_1, 1)$ with $\mu_1 \sim N(1,1)$, $\xi_2 \sim N(\mu_2, 1)$ with $\mu_2 \sim N(2,2)$, $\xi_3 \sim N(\mu_3, 1)$ with $\mu_3 \sim N(3, 2^2)$,

$\xi_4 \sim N(\mu_4, 2)$ with $\mu_4 \sim N(4, 3^2)$, $\xi_5 \sim N(\mu_5, 1)$ with $\mu_5 \sim N(5, 3^2)$ and $R = 2.321$, $\alpha = 0.9$.

Here, $N(\mu, \sigma^2)$ represents the normally distributed random variable with mean μ and standard variance σ. A run of the hybrid intelligent algorithm with 3000 generations shows that among 5 securities, in order to gain maximum chance measure of the total securities return no less than the predetermined total return R at the confidence level α, the investor should assign his money according to the optimal solution:

$x_1^* = 0, \ x_2^* = 0.2125, x_3^* = 0.2456, \ x_4^* = 0.1672, \ x_5^* = 0.3747$.

Example 3 Consider the following CCP birandom portfolio selection,

$$\max \ \bar{f}$$

subject to

$$\text{Ch}\{x_1\xi_1 + x_2\xi_2 + x_3\xi_3 + x_4\xi_4 \ge \bar{f}\}(0.95) \ge 0.95$$

$$x_1 + x_2 + x_3 + x_4 = 1$$

$$x_i \ge 0, i = 1,2,3,4.$$

where $\xi_1, \xi_2, \xi_3, \xi_4$ are birandom variables representing the securities returns defined as follows,

$\xi_1 \sim N(\mu_1, 1)$ with $\mu_1 \sim U(0,1)$, $\xi_2 \sim N(\mu_2, 1)$ with $\mu_2 \sim U(1,2)$, $\xi_3 \sim N(\mu_3, 1)$ with $\mu_3 \sim U(2,3)$

$\xi_4 \sim N(\mu_4, 2)$ with $\mu_4 \sim N(0,1)$.

A run of the HIA 5000 cycles shows that the optimal solution is

$x_1^* = 0.1365, \ x_2^* = 0.3217, x_3^* = 0.4561, x_4^* = 0.0857$

The corresponding (0.95, 0.95) optimistic value is

$\bar{f}^* = 1.9782$.

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