Extended Cesaro Operator from A_ϕ^∞ to Bloch Space

Mingzhu Yang

School of Tianjin University of Finance and Economics

25 Zhu Jiang Street, Tianjin 300222, China

E-mail: mzhyang@mail.nankai.edu.cn

Abstract

Let g be a holomorphic function of the unit ball B in several complex variables, and denote by T_g the induced extended Cesaro operator. This paper discussed the boundedness and compactness of T_g acting from A_ϕ^∞ to Bloch space in the unit ball.

Keywords: Cesaro operator, Unit ball, Bounded, Compact, Bloch space

1. Introduction

Let B be the unit ball of C^n, and $H(B)$ denotes the class of analytic functions in B. Let H^p be the standard Hardy space on the unit disc D. For $f(z) = \sum_{j=0}^\infty a_j z^j$ in H^p, the classical Cesaro operator acting on f is given by the formula

$$C[f](z) = \sum_{j=0}^\infty \left(\frac{1}{j+1} \sum_{k=0}^j a_k \right) z^j.$$

The study of Cesaro operator has become a major driving force in the development of modern complex analysis. The recent papers are good sources for information on much of the developments in the theory of Cesaro operators up to the middle of last decade. In the recent years, boundedness and compactness of extended Cesaro operator between several spaces of holomorphic functions have been studied by many mathematicians. It is well known that the operator C is bounded on the usual Hardy spaces H^p and Bergman space, as well as the Dirichlet space. Basic results facts on Hardy spaces can be found in Durn(1970). For $0 < p < \infty$, Siskakis (1987) studied the spectrum of C, as a by-product he obtained that C is bounded on $H^p(D)$. For $p = 1$, the boundedness of C was given also by Siskakis (1990) by a particularly elegant method, independent of spectrum theory, a different proof of the result can be found in Giang and Morricz(1995)}. After that, for $0 < p < 1$, Miao(1992) proved C is also bounded. For $p = \infty$, the boundedness of C was given by Danikas and Sisakis(1993).

A little calculation shows $C[f](z) = \frac{1}{z} \int_0^z f(t)(\ln \frac{1}{1-t})' dt$. From this point of view, if $g \in H(B)$, it is natural to consider the extended Cesaro operator defined by

$$T_g(f)(z) = \int_0^1 f(tz) Rg(tz) \frac{dt}{t} \quad \text{Where} \quad Rg(z) = \sum_{j=0}^n z_j \frac{\partial g}{\partial z_j}(z).$$

It is easy to see that T_g take $H(B)$ into itself. In general, there is no easy way to determine when an extended Cesaro operator is bounded or compact.

The boundedness and compactness of this operator on weighted Bergman, mixed norm , Bloch, and Dirichlet spaces in the unit ball have been studied by Xiao and Hu. In this paper, we continue this line of research.

Now we introduce some spaces first. We define Bloch space Bloch as the space of holomorphic functions $f \in H(B)$ such that $\|f\|_{Bloch} = |f(0)| + \sup_{z \in B}(1 - |z|^2)|Rf(z)|$.

Let ϕ denote a strictly decreasing continuous function $\phi : [0,1] \to R$, with $\phi(0) = 1$. For $z \in B$, $\phi(z)$ will denote $\phi(|z|)$. The Banach space of all analytic functions $f \in H(B)$ which satisfy $\|f\|_\phi := \sup_{z \in B} \phi(z)|f(z)|$ will be denoted by A_ϕ^∞. For example, $\phi(r) = (1 - r^2)^\alpha$ with $\alpha > 0$. When $\phi = 1$, it becomes the classical bounded function space.

2. Some Lemmas

In the following, we will use the symbol C or M to denote a finite positive number which does not depend on variable z and may depend on some norms and parameters f, not necessarily the same at each occurrence.

By Montel theorem and the definition of compact operator, the following lemma follows.

2.1 Assume that $g \in H(B)$. Then $T_g : A_\phi^\infty \to Bloch$ is compact if and only if T_g is bounded and for any bounded sequence $f_k (k \in N)$ in A_ϕ^∞ which converges to zero uniformly on compact subsets of B as $k \to \infty$, $\| T_g f_k \|_{Bloch} \to 0$ as $k \to \infty$.

Proof. Assume that T_g is compact and suppose $f_k (k \in N)$ is a sequence in A_ϕ^∞ with $\sup_{n \in N} \| f \|_\phi < \infty$ and $f_k \to 0$ uniformly on compact subsets of B. By the compactness of T_g we have that $T_g f_k$ has a subsequence $T_g f_{km}$ which converges in A_ϕ^∞, say, to h. We have that for any compact set $K \subset B$, there is a positive constant C_K independent of f such that $| T_g f_k(z) - h(z) | \leq C_K \| T_g f_k - h \|_{Bloch}$ for all $z \in K$. This implies that $T_g f_k(z) - h(z) \to 0$ uniformly on compact sets of B. Since K is a compact subset of B, by the hypothesis and the definition of T_g, $T_g f_k(z)$ converges to zero uniformly on K. It follows from the arbitrary of K that the limit function h is equal to 0. Since it's true for arbitrary subsequence of f_k, we see that $T_g f_k$ converges to 0 in $Bloch$.

Conversely, $\{f_k\} \in K_r = B_{A_\phi^\infty}(0, r)$, where $B_{A_\phi^\infty}(0, r)$ is a ball in A_ϕ^∞, then by Montel's Lemma, f_k is a normal family, therefore there is a subsequence f_{km} which converges uniformly to $f \in H(B)$ on compact subsets of B.

Denote $B_k = B(0, 1 - \frac{1}{k}) \subset C^n$, and since $\{f_k\} \in K_r = B_{A_\phi^\infty}(0, r)$ then

$\phi(z) | f(z) | = \lim_{m \to \infty} \phi(z) | f_{km}(z) | < r$. Hence the sequence $\{f_{km} - f\}$ is such that

$\| f_{km} - f \|_\phi \leq 2r < \infty$ and converges to 0 on compact subsets of B, by the hypothesis of this lemma, we have that $T_g f_{km} \to T_g f$ in $Bloch$. Thus the set $T_g(K_r)$ is relatively compact, so T_g is compact, finishing the proof.

2.2 Let $g \in H(B)$, then $RT_g f(z) = f(z) Rg(z)$ for any $f \in H(B)$ and $z \in B$.

Proof: Suppose the holomorphic function fRg has the Taylor expansion

$$(fRg)(z) = \sum_{|\alpha| \geq 1} a_\alpha z^\alpha \cdot$$

Then we have

$$R(T_g f)(z) = R \int_0^1 f(tz) Rg(tz) \frac{dt}{t} = R \int_0^1 \sum_{|\alpha| \geq 1} a_\alpha (tz)^\alpha \frac{dt}{t}$$

$$= R[\sum_{|\alpha| \geq 1} \frac{a_\alpha z^\alpha}{|\alpha|}] = \sum_{\alpha \geq 1} a_\alpha z^\alpha = (fRg)(z) \cdot$$

3. Main Theorem

3.1 Suppose $g \in H(B)$, then $T_g : A_\phi^\infty \to Bloch$ is bounded if and only if

$$\sup_{z \in B} \frac{(1 - |z|^2)}{\phi(z)} | Rg(z) | < +\infty \cdot$$

Proof: We proof the sufficiency frist. Since $T_g f(0) = 0$ and

$$\sup_{z \in B} (1 - |z|^2) | RT_g f(z) | = \sup_{z \in B} (1 - |z|^2) | f(z) | | Rg(z) | \leq \| f \|_\phi \sup_{z \in B} \frac{(1 - |z|^2)}{\phi(z)} | Rg(z) |$$

Therefore, $\sup_{z \in B} \frac{(1 - |z|^2)}{\phi(z)} | Rg(z) | < +\infty$ implies that T_g is bounded.

Now we turn to the necessity. Setting the test function , for any $w \in B$, let $f_w(z) = \dfrac{1}{\phi(w)} \dfrac{1-|w|}{1-<z,w>}$, Then it is easy

to see that $\sup\limits_{w \in B} \| f_w \|_\phi = M < \infty$.

Then $M \parallel T_g \parallel \geq \parallel T_g f_w \parallel_{Bloch} = \sup\limits_{z \in B}(1-|z|^2)|RT_g f_w(z)|$

$$\geq (1-|w|^2)|f_w(w)| \cdot |Rg(w)| \geq \frac{1}{2}\frac{(1-|w|^2)}{\phi(w)}|Rg(w)|$$

Since $w \in B$ is arbitrary, we get the necessity.

Remark: note that by take the test function $f = 1$,we can get $g \in Bloch$.

3.2 Suppose $g \in H(B)$, and $\phi(z) = 1-|z|^2$, then $T_g : A_\phi^\infty \to Bloch$ is bounded if and only if $\sup\limits_{z \in B} |Rg(z)| < \infty$, that

is Rg belongs to the class of bounded holomorphic functions.

Proof: It is obvious from the 3.1.

3.3 Suppose $g \in H(B)$, then $T_g : A_\phi^\infty \to Bloch$ is compact if and only if

$$\lim_{|z| \to 1} \frac{(1-|z|^2)}{\phi(z)}|Rg(z)| = 0 \cdot$$

Proof: We consider the sufficiency first. Assume the condition holds, then for any given $\varepsilon > 0$, there exists a

$\delta(0 < \delta < 1)$, such that $\dfrac{(1-|z|^2)}{\phi(z)}|Rg(z)| < \varepsilon$ when $|z| > \delta$. Let $K = \{z \in B : |z| \leq \delta\}$, and for any sequence f_k with

$\| f_k \|_\phi \leq C$ and f_k converges to 0 uniformly on compact subsets of B . Notice that $T_g f(0) = 0$, then

$$\| T_g f_k \|_{Bloch} = \sup_{z \in B}(1-|z|^2)|f_k(z)||Rg(z)| \leq \sup_{z \in K}(1-|z|^2)|f_k(z)||Rg(z)| + \sup_{z \in B-K}(1-|z|^2)|f_k(z)||Rg(z)|$$

$$\leq \| g \|_{Bloch} \bullet \sup_{z \in K}|f_k(z)| + \sup_{z \in B-K}\frac{1-|z|^2}{\phi(z)}|Rg(z)| \bullet \| f_k \|_\phi$$

With the uniform convergence of f_k we get $\| T_g f_k \| \to 0$ as $k \to \infty$. Owing to Lemma 1, T_g is compact.

Now we turn to the necessity. For the necessity ,we choose the test functions as follows. For any sequence $\{z_j\}$ with

$|z_j| \to 1$ as $j \to \infty$. We set $h_j(z) = \dfrac{1}{\phi(z_j)} \dfrac{1-|z_j|}{1-<z,z_j>}$. It is easy to check that $\sup\limits_{j \in N} \| h_j \|_\phi < \infty$ and h_j uniformly

converges to 0 in any compact subset of B . That is to say h_j satisfy the condition of lemma 2.1, then we have

$\| T_g h_j \|_{Bloch} \to 0$ as $j \to \infty$.

Therefore, $0 \leftarrow \| T_g h_j \|_{Bloch} = \sup\limits_{z \in B}(1-|z|^2)|RT_g h_j(z)| = \sup\limits_{z \in B}(1-|z|^2)|h_j(z)| \cdot |Rg(z)|$

$$\geq (1-|z_j|^2)|h_j(z_j)| \cdot |Rg(z_j)| \geq \frac{1}{2}\frac{1-|z_j|^2}{\phi(z_j)} \cdot |Rg(z_j)| \cdot$$

The conclusion follows by the arbitrary of the sequence $\{z_j\}$.

3.4 Suppose $g \in H(B)$, and $\phi(z) = 1-|z|^2$, then $T_g : A_\phi^\infty \to Bloch$ is compact if and only if g is a constant.

Proof: By 3.3 we can obtain that $\lim\limits_{|z| \to 1}|Rg(z)| = 0$, by the maximum module of principle we have $Rg \equiv 0$, and by the

formula that $f(z) - f(0) = \int_0^1 \dfrac{Rf(tz)}{t}dt$, we must have g is a constant.

References

A. Siskakis(1990).The cesaro operator is bounded on H^1 , Proc. Amer. Math. Soc., 110(1990):461-462.

A. Siskakis. (1987). Composition semigroups and the cesaro operator on $H^p(D)$, *J. London Math. Soc.* 36(1987): 153-164.

A. Siskakis. (1987). Semigroups of composition operators in Bergman spaces, *Bull. Austral. th. Soc.* 35 (1987): 397-406.

C.C.Cowen and B.D.MacCluer. (1995). Composition operators on spaces of analytic functions, CRC Press, Boca Raton, FL, 1995.

D. V. Giang and F. Morricz. (1995).The cesaro operator on Dirichlet is bounded on the Hardy space H^1, *Acta Sci. Math.* 61 (1995): 535-544.

J. Miao. (1992). The cesaro operator is bounded on H^p for 0<p<1, *Proc. Amer. Math. Soc.,* 116, 1077-1079.

J. Xiao. (2004). Riemann-Stieltjes operators on weighted Bloch and Bergman spaces of the unit ball, *J.London Math.Soc.*(2)70(2004) 199-214.

K. H. Zhu. (2004). Spaces of Holomorphic functions in the Unit Ball, Springer-Verlag (GTM 226), 2004.

N. Danikas and A. Siskakis. (1993). The cesaro operator on bounded analytic functions, *Analysis,* 13, 195-199.

P. Durn(1970). *Theory of H^p space*, Academic Press, New York, 1970.

P.Galanopoulos. (2001). The cesaro operator on Dirichlet spaces, *Acta Sci. Math*. 67 (2001), 441-420.

W.Rudin. (1980). Function theory in the unit ball of C^n, Springer-Verlag, New York, 1980.

Z. J. Hu. (2003). Extended cesaro operators on mixed norm space, *Proc. Amer. Math. Soc.,* 131, 2171-2179.

Z. J. Hu. (2003). Extended cesaro operators on the Bloch spaces in the unit ball of C^n, *Acta Math. Sci.,* 23, (2003), 561-566.

Zhong-Shan Fang and Ze-Hua Zhou. (2009). Extended Cesaro operators on Zygmund spaces in the unit balL, *Journal of Computational Analysis and Applications*, preprint.