LOD-BOR-FDTD Algorithm for Analyzing Debye Dispersive Media by Bilinear Z Transforms

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Abstract

In order to reduce memory usage and improve efficiency, the unconditionally stable locally 1-D (LOD)-FDTD method for bodies of revolution (BOR) is extended to Debye dispersive media based on the bilinear *Z* transform (BZT) theory. The LOD-BOR-FDTD method is proposed. To validate the Higher efficiency and Lower memory usage of the proposed algorithm, two numerical examples are given. Compared with the 3-D FDTD and ADI-BOR-FDTD result, they show good agreement and at least 80% of computational time to the ADI counterparts.

Keywords: Body of revolution (BOR), Locally 1-D finite difference time-domain method (LOD-FDTD), Rotationally symmetric geometry, Dispersive media

1. Introduction

THE dispersive property of bodies of revolution is important for the analysis of wide-band electromagnetic characteristics. The body-of-revolution finite-difference time-domain (BOR-FDTD) method (A. Taflove and S. C. Hagness, 2005)(D. B. Davidson and R. W. Ziolkowski, 1994)(Y. Chen, R. Mittra, and P. Harms, 1996) is an effective means for simulating electromagnetic wave propagation in circularly symmetric structures. Nevertheless, a time step size of the BOR-FDTD is limited by the Courant-Friedrich-Levy (CFL) condition in the conventional FDTD. As a result, the computational time will be increased significantly, when the use of finer space discretization or a large mode number is needed. To conquer the constraint, (H.-L. Chen, B. Chen, Y. Yi, and D.-G. Fang, 2007) presents alternating-direction-implicit(ADI)-BOR-FDTD method. Recently, the locally 1-D (LOD) scheme has been introduced to BOR-FDTD(Jun Shibayama, Bungo Murakami, Junji Yamauchi, and Hisamatsu Nakano, 2009). The LOD method (J. Shibayama, M. Muraki, J. Yamauchi, and H. Nakano, 2005) provides a quite simple algorithm compared with the ADI algorithm. The LOD formulation is a simple type of split-step approach (W. Fu and E. L. Tan, 2004). Although it is first-order accurate in time (J. Shibayama, M. Muraki, R. Takahashi, J. Yamauchi, and H. Nakano, 2006), the numerical results is comparable to the ADI counterparts which has second-order accurate in time. Although the locally 1-D (LOD)-BOR-FDTD is developed, but no dispersive media was considered.

On the other hand, the use of *Z* transforms (D. M. Sullivan, 1992) to the treatment of dispersive media in FDTD is an attractive alternative, since it has the advantage that the complicated convolution integrals can simply be reduced to algebraic equations, and the relationship between the flux density and the electric field can readily be translated into finite-difference equations. But we found the conventional *Z* transforms could bring higher error for Dispersive Media. So the bilinear *Z* transforms (*BZT*) is used to avoid error for the trapezoidal integration is more accurate than rectangular integration (D. M. Sullivan, 2000). This fact motivates us to apply the bilinear *Z* transforms to the LOD-BOR-FDTD for a concise frequency-dependent formulation.

In this letter, we use the *BZT* method to build an extension of the LOD-BOR-FDTD to Debye dispersive media in order to reduce memory usage and improve efficiency.

2. Formulation

Bilinear Z Transforms

For Debye dispersive medium having p poles, the electric flux density D and the electric field E in the frequency domain are related as (A. Taflove and S. C. Hagness, 2005)

$$D(\omega) = \varepsilon_0 \left(\varepsilon_\infty + \sum_{p=1}^{N_p} \frac{\varepsilon_{s,p} - \varepsilon_{\infty,p}}{1 + j\omega\tau_p} \right) E(\omega) \tag{1}$$

where ε_0 is the free-space permittivity, $\varepsilon_{s,p}$ is the static or zero frequency relative permittivity, $\varepsilon_{\infty,p}$ is the relative permittivity at infinite frequency, τ_p is the pole relaxation time, and N_p is the number of poles in susceptibility response.

Taking bilinear Z transforms of(1)by $1/j\omega \to (1+z^{-1})/2(1-z^{-1})$, $1/(\alpha + j\omega) \to (1+z^{-1}e^{-\alpha\Delta t})/2(1-z^{-1}e^{-\alpha\Delta t})$ and $D = \varepsilon_0 \varepsilon_r(z)E(z)\Delta t$ in which the normalised expression of field components (A. Taflove and S. C. Hagness, 2005) is used yields the following equation in the Z domain:

$$D(z) = \varepsilon_0 \varepsilon_\infty E(z) + \sum_{p=1}^{N_p} S(z) \tag{2}$$

Where

$$\begin{aligned} S(z) &= \frac{\Delta t \varepsilon_0 \left(1 + z^{-1}e^{-\Delta t/\tau_p}\right)\left(\varepsilon_{s,p} - \varepsilon_{\infty,p}\right)/\tau_p}{2\left(1 - z^{-1}e^{-\Delta t/\tau_p}\right)} E(z) \\ &= e^{-\Delta t/\tau_p} z^{-1} S(z) + \frac{\Delta t \varepsilon_0 \left(\varepsilon_{s,p} - \varepsilon_{\infty,p}\right)\left(1 + z^{-1}e^{-\Delta t/\tau_p}\right)}{2\tau} E(z) \end{aligned} \tag{3}$$

in which Δt and z^{-1} represent the time period and the delay of l time periods, respectively. We readily translate (2), (3) into the finite difference equations in the time domain as

$$E^n = \frac{D^n - S^{n-1}\sum_{p=1}^{N} e^{-\Delta t/\tau_p} - \sum_{p=1}^{N} \frac{\Delta t \varepsilon_0}{2\tau_p}\left(\varepsilon_{s,p} - \varepsilon_{\infty,p}\right)e^{-\Delta t/\tau_p} E^{n-1}}{\varepsilon_0 \varepsilon_\infty + \sum_{p=1}^{N} \frac{\Delta t \varepsilon_0}{2\tau_p}\left(\varepsilon_{s,p} - \varepsilon_{\infty,p}\right)} \tag{4}$$

$$S^n = e^{-\Delta t/\tau_p} S^{n-1} + \frac{\Delta t \varepsilon_0 \left(\varepsilon_{s,p} - \varepsilon_{\infty,p}\right)}{2\tau} E^n + \frac{\Delta t \varepsilon_0 \left(\varepsilon_{s,p} - \varepsilon_{\infty,p}\right)e^{-\Delta t/\tau_p}}{2\tau} E^{n-1} \tag{5}$$

Combined Debye media with LOD-BOR-FDTD

We now solve (4),(5) combined with the following standard LOD-BOR-FDTD equations with the field normalization (Jun Shibayama, Bungo Murakami, Junji Yamauchi, and Hisamatsu Nakano, 2009). For simplicity, we will examine the case for mode number is 0:

$$E_\rho^{n+1/2} = E_\rho^n \tag{6a}$$

$$\frac{E_\phi^{n+1/2} - E_\phi^n}{\Delta t/2} = \frac{1}{\varepsilon}\left(\frac{\partial H_\rho^{n+1/2}}{\partial z} + \frac{\partial H_\rho^n}{\partial z}\right) \tag{6b}$$

$$\frac{E_z^{n+1/2} - E_z^n}{\Delta t/2} = \frac{1}{\varepsilon\rho}\left(\frac{\partial\left(\rho H_\phi^{n+1/2}\right)}{\partial\rho} + \frac{\partial\left(\rho H_\phi^n\right)}{\partial\rho}\right) \tag{6c}$$

$$\frac{H_\rho^{n+1/2} - H_\rho^n}{\Delta t/2} = \frac{1}{\mu}\left(\frac{\partial E_\phi^{n+1/2}}{\partial z} + \frac{\partial E_\phi^n}{\partial z}\right) \tag{6d}$$

$$\frac{H_\phi^{n+1/2} - H_\phi^n}{\Delta t/2} = \frac{1}{\mu}\left(\frac{\partial E_Z^{n+1/2}}{\partial\rho} + \frac{\partial E_Z^n}{\partial\rho}\right) \tag{6e}$$

$$H_z^{n+1/2} = H_z^n \tag{6f}$$

for the first step and

$$E_\rho^{n+1} + \frac{\Delta t}{2\varepsilon}\frac{\partial H_\phi^{n+1}}{\partial z} = E_\rho^{n+1/2} - \frac{\Delta t}{2\varepsilon}\frac{\partial H_\phi^{n+1/2}}{\partial z} \tag{7a}$$

$$E_\phi^{n+1} + \frac{\Delta t}{2\varepsilon}\frac{\partial H_z^{n+1}}{\partial\rho} = E_\phi^{n+1/2} - \frac{\Delta t}{2\varepsilon}\frac{\partial H_z^{n+1/2}}{\partial\rho} \tag{7b}$$

$$E_z^{n+1} = E_z^{n+1/2} \tag{7c}$$

$$H_\rho^{n+1} = H_\rho^{n+1/2} \tag{7d}$$

$$H_\phi^{n+1} + \frac{\Delta t}{2\mu}\frac{\partial E_\rho^{n+1}}{\partial z} = H_\phi^{n+1/2} - \frac{\Delta t}{2\mu}\frac{\partial E_\rho^{n+1/2}}{\partial z} \tag{7e}$$

$$H_z^{n+1} + \frac{\Delta t}{2\mu\rho}\frac{\partial\left(\rho E_\phi^{n+1}\right)}{\partial\rho} = H_z^{n+1/2} - \frac{\Delta t}{2\mu\rho}\frac{\partial\left(\rho E_\phi^{n+1/2}\right)}{\partial\rho} \tag{7f}$$

for the second step.

We now calculate the E and H from n time steps to $n+1$.

1) Substituting (7c) into (6c)(6e) equations, we get

$$\frac{D_z^{n+1} - D_z^n}{\Delta t/2} = \frac{1}{\varepsilon\rho}\left[\frac{\partial\left(\rho H_\phi^{n+1/2}\right)}{\partial\rho} + \frac{\partial\left(\rho H_\phi^n\right)}{\partial\rho}\right] \tag{6c*}$$

$$\frac{H_\phi^{n+1/2} - H_\phi^n}{\Delta t/2} = \frac{1}{\mu}\left(\frac{\partial E_z^{n+1}}{\partial\rho} + \frac{\partial E_z^n}{\partial\rho}\right) \tag{6e*}$$

Substituting (4),(6e*) into (6c*), and implicitly solve the resultant tridiagonal equation, we get E_z^{n+1}. And then explicitly solve (6e*), (4) and (5), getting D_z^{n+1}, $H_\phi^{n+1/2}$, S_z^{n+1}.

2) Substituting (7d) into (6b) (6d) equations, we get

$$\frac{H_\rho^{n+1}-H_\rho^n}{\Delta t/2}=\frac{1}{\mu}\left(\frac{\partial E_\phi^{n+1/2}}{\partial z}+\frac{\partial E_\phi^n}{\partial z}\right) \tag{6d*}$$

$$\frac{E_\phi^{n+1/2}-E_\phi^n}{\Delta t/2}=\frac{1}{\varepsilon}\left(\frac{\partial H_\rho^{n+1}}{\partial z}+\frac{\partial H_\rho^n}{\partial z}\right) \tag{6b*}$$

Substituting (6b*) into (6d*), we get H_ρ^{n+1}. And then explicitly solve (6b*), getting $E_\phi^{n+1/2}$.

3) Substituting (6a) into (7a)(7e) equations, we get

$$\frac{D_\rho^{n+1}-D_\rho^n}{\Delta t/2}=-\frac{1}{\varepsilon}\left(\frac{\partial H_\phi^{n+1}}{\partial z}+\frac{\partial H_\phi^n}{\partial z}\right) \tag{7a*}$$

$$\frac{H_\phi^{n+1}-H_\phi^{n+1/2}}{\Delta t/2}=-\frac{1}{\mu}\left(\frac{\partial E_\rho^{n+1}}{\partial z}+\frac{\partial E_\rho^n}{\partial z}\right) \tag{7e*}$$

Substituting (4), (7e*) into (7a*), we get E_ρ^{n+1}. And then explicitly solve (7e*), (4) and (5), getting D_ρ^{n+1}, H_ϕ^{n+1}, S_ρ^{n+1}.

4) Substituting (6f) into (7f) (7b) equations, we get

$$\frac{H_z^{n+1}-H_z^n}{\Delta t/2}=-\frac{1}{\mu\rho}\left[\frac{\partial\left(\rho E_\phi^{n+1}\right)}{\partial\rho}+\frac{\partial\left(\rho E_\phi^{n+1/2}\right)}{\partial\rho}\right] \tag{7f*}$$

$$\frac{E_\phi^{n+1}-E_\phi^{n+1/2}}{\Delta t/2}=-\frac{1}{\varepsilon}\left(\frac{\partial H_z^{n+1}}{\partial\rho}+\frac{\partial H_z^n}{\partial\rho}\right) \tag{7b*}$$

Substituting (7b*) into (7f*) equations, we get H_z^{n+1}, E_ϕ^{n+1}. Then, the E and H are updated from n to $n+1$.

3. Numerical Results

To verify the proposed algorithm and accuracy, two numerical examples we analysed are shown. In all cases, the θ-polarized plane wave is introduced by defining a set of equivalent currents on a closed Huygen's surface (D. E. Merewether, R. Fisher, and F. W. Smith, 1980). The Computational domains are truncated by PML cells which have n=8 layers (Jun Shibayama, Bungo Murakami, Junji Yamauchi, and Hisamatsu Nakano, 2009).

As a first example (H.-L. Chen, B. Chen, Y. Yi, D.-G. Fang and Heng Liu, 2009), a typical first order Debye material is water with the parameters $\varepsilon_s=80$, $\varepsilon_\infty=5.27$, $\tau=1.0\times10^{-11}$. Here, the scattering from a water sphere with radius of 420 μm is calculated. The grid size is $\Delta\rho=\Delta z=10\,\mu m$ and mode number ranges from 0 to 6. Fig.1(a) shows the back scattered radar cross section (RCS) for different CFLN values, which is defined as $CFLN=\Delta t/\Delta t_{FDTD}$, where Δt_{FDTD} is the maximum FDTD time step. For the purpose of comparison, the result obtained from three-dimensional (3-D) FDTD method with grid size $\Delta x=\Delta y=\Delta z=10\,\mu m$ is also given. Fig. 1(b) shows the errors of the LOD versus ADI method. Compared with the theoretical value, they show good agreement. The errors in time domain are also tabulated, which are evaluated using the following expression:

$$Error(\%)=100\times\left\|E_{far}-E_{far}^{ref}\right\|/\left\|E_{far}^{ref}\right\| \tag{8}$$

where E_{far}^{ref} is the far field in time domain calculated using 3-D FDTD, $\|\bullet\|$ represents the Euclid norm operation.

The second example, we calculate the back scattered RCS of a cold plasma cylinder with height of 6cm and radius of 3cm illuminated by perpendicularly incident plane wave. The complex relative permittivity of the plasma medium is defined as

$$\varepsilon_r=1+\omega_p^2/\left(j\omega V_c-\omega^2\right) \tag{9}$$

where ω_p is the radian plasma frequency and V_c is the collision

frequency. It has been shown that (K. S. Kunz and R. J. Luebbers, 1993), with the transformation $\varepsilon_r=1+x_d+\sigma/\left(j\varepsilon_0\omega\right)$, the plasma medium behaves like a Debye medium with a conductivity, and a negative susceptibility $x_d=-\omega_p^2/V_c^2/\left(1+j\omega/V_c\right)$. The grid size is $\Delta\rho=\Delta z=0.5mm$ and mode number ranges from 0 to 9. The plasma considered has a plasma frequency ω_p of $4.0\times10^{10}\,rad/s$ and a collision frequency V_c of 20 GHz. Again, we compare the results of the proposed method with that of the conventional 3-D FDTD method with grid size $\Delta x=\Delta y=\Delta z=0.5mm$. As shown in Fig. 2(a), good agreements are achieved even if the time step of the ADI-BOR-FDTD is four times that used in the conventional FDTD method. Fig. 2(b) shows the errors of the LOD versus ADI method. Good agreement is expressed.

4. Conclusion

In this letter, the LOD-BOR-FDTD method has been extended to solve electromagnetic problems in Debye dispersive media using the bilinear *Z* transforms. The LOD-BOR-FDTD combine with the *BZT* offers quite simple algorithm with a subsequent reduction in the computational time, maintaining numerical results identical to the ADI counterparts. Numerical results indicate that the presented method is valid and efficient. Compared with the 3-D FDTD and ADI-BOR-FDTD results, they show good agreement and at least 45% and 80% of running time, respectively. For open region problems, efficient absorbing boundary conditions such as perfectly matched layers (I. Ahmed, E. Li, K. Krohne, 2007)(V. E. do Nascimento, B. H. V. Borges, and F. L. Teixeira, 2006) for LOD-BOR-FDTD are required. The proposed method has reduced memory usage and improved efficiency.

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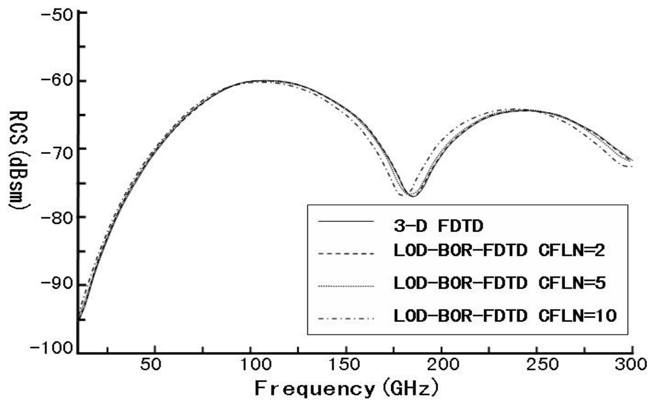
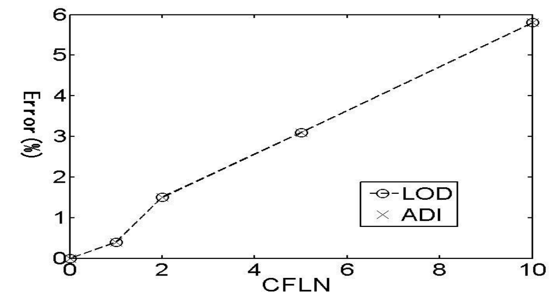
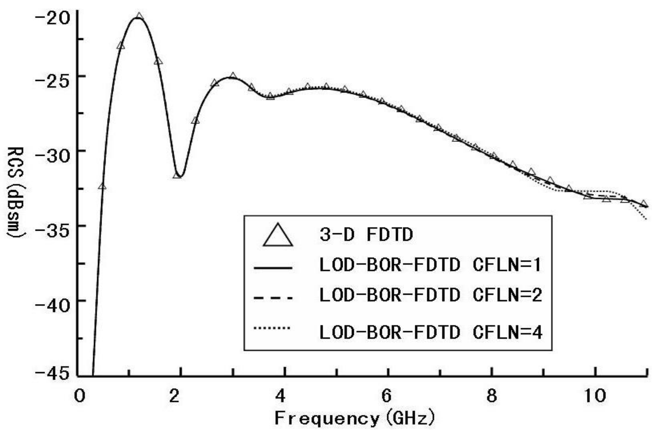


(a)

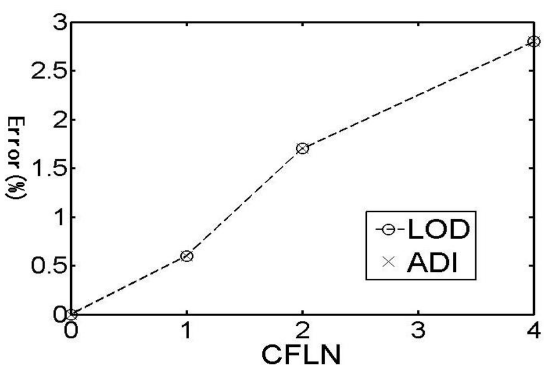


(b)

Figure 1. Backscattering from the water sphere is showed in(a) and errors of the LOD versus ADI method is showed in(b).



(a)



(b)

Figure 2. Back scattered RCS of the plasma cylinder is showed in(a) and errors of the LOD versus ADI method is showed in(b).