An Efficient Polynomial Approximation to the Normal Distribution Function and Its Inverse Function

Winston A. Richards
The Department of Mathematics and Statistics
The Pennsylvania State University, Harrisburg, USA

Robin Antoine & Ashok Sahai (Corresponding author)
Department of Mathematics & Computer Science
The University of The West Indies, St. Augustine
Trinidad & Tobago, West Indies
E-mail: sahai.ashok@gmail.com

M. Raghunadh Acharya
Department of Statistics and Computer Science
Aurora’s Post Graduate College, Osmania University
Hyderabad, India

Abstract
We propose approximations to the normal distribution function and to its inverse function using single polynomials in each case. The absolute error of these approximations is significantly less than those of other approximations available in the literature. We compare all the polynomial approximations empirically by calculating their respective percentage absolute relative errors.

Keywords: Normal distribution, Probabilistic polynomial approximation operator

1. Introduction
The problem of approximation arises in many areas of science and engineering in which numerical analysis and computing are involved. The modern history of the subject may be said to have begun in 1885 with Weierstrass’s celebrated approximation theorem on the approximation of continuous functions by polynomials. Later on, Bernstein gave a constructive proof of Weierstrass’s Theorem by furnishing explicitly, for every $f \in C[0, 1]$, a sequence of polynomials (the Bernstein polynomials) that converge to $f$

$$B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right)$$  \hspace{1cm} (1.1)

Polynomial functions are, of course, extremely well-behaved. Thus an approximation to the normal distribution function which employs only a single polynomial is likely to be more efficient than existing approximations and easy to calculate.

Let $X$ be a standard normal random variable and let $F$ be the distribution function of $X$. We aim to construct single polynomial approximations to both $F$ and to the inverse function $F^{-1}$ of $F$.

For the standard normal distribution, we know that
Now, to devise the weight function is and intervals. Let \( x_i = i/n \) for \( i = 0, 1, \ldots, n \). Consider a point \( x \in [0, 1] \). Then, if \( X \) is a random point in \([0, 1]\), \( P(X \leq x) = x \) and \( P(X > x) = 1 - x \). Thus, of \( n \) randomly chosen points, the expected number of points that are less than or equal to \( x \) is \( nx \) and the expected number greater than \( x \) is \( n(1-x) \).

Now, to devise the weight function \( A_k(x) \) associated with the node \( x_k \), we simply place it in the same shoes as \( x \). We know that, using \((n+1)\) equidistant nodes, for any node \( x_k \), there are \( k \) nodes to the left of the node \( x_k \) and \((n-k)\) nodes to the right of \( x_k \). Consequently, in this ‘probabilistic’ setup, the probability that the node \( x_k \) is chosen is

\[
\binom{n}{k} \frac{p(1-x)}{n-k} = \binom{n}{n-k} = A_k(x)
\]  

(2.2)

This equation may be expressed in terms of the Gamma function in order to accommodate any real value of \( x \in [0, 1] \).

Therefore, Sahai(2004)’s ‘probabilistic polynomial approximation’ for the distribution function \( F(x) \) is simply

\[
F(x) = 0.5 + \int_{0}^{1} \sum_{k=0}^{n} A_k(x)f(x_k)dx \tag{2.3}
\]

where

\[
f(x_k) = 3\left(\frac{1}{\sqrt{2\pi}}\right)\exp(-4.5x_k^2) \tag{2.4}
\]

The last integral in Eq. 2.1 has no closed form expression. Most statistical books give the values of this integral in normal tables. These tables may also be used to find the value of \( x \) when \( \Phi(x) \) is known. Several authors give approximations using polynomials (Chokri, (2003); Johnson (1994); Bailey (1981); Polya (1945)). These approximations give quite high accuracy but require significant amounts of computation and have a maximum absolute error of more than 0.003. Only the Polya approximation

\[
F(x) = 0.5\left[1 + \sqrt{1 - \exp(-\frac{x^2}{\pi})}\right] \tag{2.5}
\]

has one term to calculate. The others require more than one term. They are reviewed in Johnson et al (1994) and are as follows:

\[
(1) \quad F_1(x) = 1 - 0.5(a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5)^{-16}, \tag{2.6}
\]

in which \( a_0 = 0.9999998582, a_1 = 0.487385796, a_2 = 0.02109811045, a_3 = 0.003372948927, a_4 = 0.00005172897742 \) and \( a_5 = 0.0000856957942 \).

\[
(2) \quad F_2(x) = \exp(2y)/(1 + \exp(2y)), \text{ where } y = 0.7988x(1 + 0.04417x^2). \tag{2.7}
\]

\[
(3) \quad F_3(x) = 1 - 0.5\exp[-(83x + 351)x + 562]/(703/x + 165)] \tag{2.8}
\]

\[
(4) \quad F_4(x) = 0.5[1 + \sqrt{1 - \exp(-\frac{8}{\pi}x^2)}] \tag{2.9}
\]
The probability standard normal variates. This will have many applications in practical situations. One such application will be in generating random x-values for standard normal distribution.

In this section we compare the exact value of \( F(x) \) with its approximate ones. We make this comparison for the values \( x = 0.1, 0.3, 0.6, 1.0 \text{ and } 2.0 \). These values are tabulated in Table A1 given in the Appendix. The following table, Table A2, gives the values of the Absolute Percentage Relative Error (APRE) for each of the various approximating functions \( F(.) \). We calculate \( \text{APRE}[F(J)(x)] \), where

\[
\text{APRE}[F(J)(x)] = \left| \frac{F(J)(x) - F(x)}{F(x)} \right| \times 100\% \tag{3.1}
\]

The most favourable APRE value has been highlighted.

The last approximation (2.9) was proposed by Aludaat and Alodat (2008) as an improvement on that of Polya’s in (2.5). The others require substantial computation, since their inverse functions are quite intricate. Using our probabilistic approximation with \( n = 8 \), we get the 8th degree polynomial:

\[
\begin{align*}
F_8(x) &= 0.5 + \sum_{k=0}^{8} a_k(x^k)\frac{d}{dx} F(x) \\
&= 0.5 + \sum_{k=0}^{8} a_k x^k \\
&= 0.5 + 1.196826841 x - 0.00723280282x^2 - 1.1695709047x^3 - 0.5893704135x^4 \\
&+ 0.264946944x^5 - 3.026777282x^6 \\
&- 0.762187586x^7 + 1.616585528x^8 - 0.4984396913x^9
\end{align*}
\tag{2.11}
\]

Now, we consider the approximation of the inverse function \( F^{-1}(p) \), since \( F(x) = p \iff F^{-1}(p) = x \), where \( 0 \leq p \leq 1 \). This will have many applications in practical situations. One such application will be in generating random x-values for standard normal variates.

The probability \( p \) may be generated using a random number generator with the uniform distribution \( U[0,1] \). If we have generated \( p_1, p_2, \ldots, p_n \) then the inverse distribution function may be used to generate the normal variates \( (x_\alpha; \alpha = 1, 2, \ldots, n) \). We now consider approximations to \( F^{-1}(p) \). As \( F^{-1}(x) \) in (2.6) would have infinite terms, it could not be expressed in a closed form via a finite degree polynomial. In the absence of a closed form it would be very tedious to generate a good approximation to the inverse function. Hence we consider only the approximations to the inverse functions given above. These are:

\[
F^{-1}[2](p) = \text{Real root between 0 and 2 of the equation} \\
0.7988x(1 + 0.04417x^2) = |\log(p) - \log(1 - p)|/2
\tag{2.13}
\]

\[
F^{-1}[3](p) = \text{Real Root between 0 and 2 of the equation} \\
(83x + 351)x + 562 + ((703/x) + 165)(\log(2 - 2p)) = 0
\tag{2.14}
\]

\[
F^{-1}[4](p) = \sqrt{-\log(1 - (2p - 1)^2)}/\sqrt{\pi/8}
\tag{2.15}
\]

and

\[
F^{-1}[5](p) = \text{Real Root between 0 and 2 of the equation} \\
F_5(x) - 0.5 = p
\tag{2.16}
\]

as in (2.12).

2.1 A Numerical Comparison of the Approximations to \( F(x) \) and \( F^{-1}(x) \)

In this section we compare the exact value of \( F(x) \) with its approximate ones. We make this comparison for the values \( x = 0.1, 0.3, 0.6, 1.0 \text{ and } 2.0 \). These values are tabulated in Table A1 given in the Appendix. The following table, Table A2, gives the values of the Absolute Percentage Relative Error (APRE) for each of the various approximating functions \( F(.) \). We calculate \( \text{APRE}[F(J)(x)] \), where

\[
\text{APRE}[F(J)(x)] = \left| \frac{F(J)(x) - F(x)}{F(x)} \right| \times 100\%
\tag{3.1}
\]

The most favourable APRE value has been highlighted.
It is quite evident that our proposed approximation $F_5$ is doing well and is consistently better than that of the Aludaat and Alodat (2008) approximation $F_4$.

Similarly, we compare the exact value of $F^{-1}(p)$ with its approximated ones. We compare the numerical approximations with the exact values of $F^{-1}(p)$ at the values $p = 0.5539828, 0.617911, 0.725747, 0.841345, 0.933193$ and $0.977250$. These are given in Table A3 in the Appendix.

The following table A4 displays the values of the APRE for various approximations to $F^{-1}(p)$. Once again, the most favourable value has been highlighted. Our approximation $F^{-1}[5](p)$ does quite well and is consistently better than that of Aludaat and Alodat (2008).

References


APPENDIX.

Table A.1. Values of Various Approximating Functions $F_o(x)$ & Actual Value of Normal Feller Function $F(x)$

<table>
<thead>
<tr>
<th>x-values</th>
<th>$F(1)(x)$</th>
<th>$F(2)(x)$</th>
<th>$F(3)(x)$</th>
<th>$F(4)(x)$</th>
<th>$F(5)(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.538972</td>
<td>0.539873</td>
<td>0.539872</td>
<td>0.539519</td>
<td>0.539823</td>
</tr>
<tr>
<td>0.3</td>
<td>0.615312</td>
<td>0.618028</td>
<td>0.617933</td>
<td>0.617088</td>
<td>0.617895</td>
</tr>
<tr>
<td>0.6</td>
<td>0.719751</td>
<td>0.725877</td>
<td>0.725693</td>
<td>0.724700</td>
<td>0.725733</td>
</tr>
<tr>
<td>1.0</td>
<td>0.830390</td>
<td>0.841331</td>
<td>0.841280</td>
<td>0.841184</td>
<td>0.841330</td>
</tr>
<tr>
<td>1.5</td>
<td>0.919689</td>
<td>0.933053</td>
<td>0.933172</td>
<td>0.934699</td>
<td>0.933179</td>
</tr>
<tr>
<td>2.0</td>
<td>0.966501</td>
<td>0.977240</td>
<td>0.977250</td>
<td>0.979181</td>
<td>0.977234</td>
</tr>
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</table>

Table A.2. Values of Abs. Per. Rel. Error [APRE] For Various Approximating Functions $F_o(x)$

<table>
<thead>
<tr>
<th>x-values</th>
<th>$APREF(1)(x)$</th>
<th>$APREF(2)(x)$</th>
<th>$APREF(3)(x)$</th>
<th>$APREF(4)(x)$</th>
<th>$APREF(5)(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.158569</td>
<td>0.008336</td>
<td>0.008151</td>
<td>0.057240</td>
<td>0.000926</td>
</tr>
<tr>
<td>0.3</td>
<td>0.420611</td>
<td>0.018935</td>
<td>0.003560</td>
<td>0.133191</td>
<td>0.002589</td>
</tr>
<tr>
<td>0.6</td>
<td>0.826183</td>
<td>0.017913</td>
<td>0.007440</td>
<td>0.144265</td>
<td>0.001929</td>
</tr>
<tr>
<td>1.0</td>
<td>1.302082</td>
<td>0.001664</td>
<td>0.007726</td>
<td>0.161381</td>
<td>0.001783</td>
</tr>
<tr>
<td>1.5</td>
<td>1.447075</td>
<td>0.015002</td>
<td>0.002250</td>
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<td>0.001500</td>
</tr>
<tr>
<td>2.0</td>
<td>1.099923</td>
<td>0.001023</td>
<td>0.000000</td>
<td>0.197595</td>
<td>0.001637</td>
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Table A.3. Values of Approximating Inverse Functions $F^{-1}(o)(p)$ & Actual Value of The Inverse Function $F^{-1}(p)$

<table>
<thead>
<tr>
<th>p-values $\rightarrow$ Apxg. Fns. ↓</th>
<th>0.539828</th>
<th>0.617911</th>
<th>0.725747</th>
<th>0.841345</th>
<th>0.933193</th>
<th>0.977250</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F^{-1}(2)(p)$</td>
<td>0.099887</td>
<td>0.299694</td>
<td>0.599611</td>
<td>1.000057</td>
<td>1.501082</td>
<td>2.000184</td>
</tr>
<tr>
<td>$F^{-1}(3)(p)$</td>
<td>0.099889</td>
<td>0.299943</td>
<td>0.600163</td>
<td>1.000269</td>
<td>1.500158</td>
<td>2.000006</td>
</tr>
<tr>
<td>$F^{-1}(4)(p)$</td>
<td>0.100785</td>
<td>0.302171</td>
<td>0.603140</td>
<td>1.000658</td>
<td>1.486901</td>
<td>1.965099</td>
</tr>
<tr>
<td>$F^{-1}(5)(p)$</td>
<td>0.100013</td>
<td>0.300041</td>
<td>0.600043</td>
<td>1.000061</td>
<td>1.500110</td>
<td>2.000288</td>
</tr>
<tr>
<td>$F^{-1}(p)$-Values:</td>
<td>0.100000</td>
<td>0.299999</td>
<td>0.600000</td>
<td>1.000001</td>
<td>1.500002</td>
<td>2.000002</td>
</tr>
</tbody>
</table>


<table>
<thead>
<tr>
<th>p-values $\rightarrow$ Apxg. Fns. ↓</th>
<th>0.539828</th>
<th>0.617911</th>
<th>0.725747</th>
<th>0.841345</th>
<th>0.933193</th>
<th>0.977250</th>
</tr>
</thead>
<tbody>
<tr>
<td>APREF-1 (2)(p)</td>
<td>0.113000</td>
<td>0.101667</td>
<td>0.064833</td>
<td>0.005600</td>
<td>0.072000</td>
<td>0.009100</td>
</tr>
<tr>
<td>APREF-1 (3)(p)</td>
<td>0.111000</td>
<td>0.018667</td>
<td>0.027167</td>
<td>0.026800</td>
<td>0.010400</td>
<td>0.000200</td>
</tr>
<tr>
<td>APREF-1 (4)(p)</td>
<td>0.785000</td>
<td>0.724002</td>
<td>0.523333</td>
<td>0.065700</td>
<td>0.873399</td>
<td>1.745148</td>
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<tr>
<td>APREF-1 (5)(p)</td>
<td>0.013000</td>
<td>0.014000</td>
<td>0.007167</td>
<td>0.006000</td>
<td>0.007200</td>
<td>0.014300</td>
</tr>
<tr>
<td>$F^{-1}(p)$-Values:</td>
<td>0.100000</td>
<td>0.299999</td>
<td>0.600000</td>
<td>1.000001</td>
<td>1.500002</td>
<td>2.000002</td>
</tr>
</tbody>
</table>