Anti-periodic Solutions of Impulsive Cohen-Grossberg SICNNs on Time Scales

Meng Hu (Corresponding author) & Lili Wang
Department of Mathematics, Anyang Normal University
Anyang, Henan 455000, China
E-mail: humeng2001@126.com

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Abstract

By using the method of coincidence degree and constructing suitable Lyapunov functional, some sufficient conditions are established for the existence and global exponential stability of anti-periodic solutions for a kind of impulsive Cohen-Grossberg shunting inhibitory cellular neural networks (CGSICNNs) on time scales. An example is given to illustrate our results.

Keywords: Anti-periodic solution, Coincidence degree, CGSICNNs, Impulse, Time scales

1. Introduction

Since Bouzerdout and Pinter in [1993] described SICNNs as a new cellular neural networks, SICNNs have been extensively applied in psychophysics, perception, robotics, adaptive pattern recognition, vision and image processing, etc. It is shown that the applicability and efficiency of such networks hinge upon their dynamics, and therefore the analysis of dynamic behaviors is a preliminary step for any practical design and application of the networks. In particular, considerable effort has been devoted to the study of dynamic behaviors on the existence and stability of the equilibrium point, periodic and almost periodic solutions of SICNNs with time-varying delays and continuously distributed delays in the literature (see, e.g., [X.S. Yang, 2009; Y.H. Xia, 2007; L. Chen, 2008; Y.Q. Li, 2008] and the references therein).

Arising from problems in applied sciences, the existence of anti-periodic solutions plays a key role in characterizing the behavior of nonlinear differential equations (see [J.Y. Shao, 2009; J.Y. Shao, 2008; Y.K. Li, 2009; G.Q. Peng, 2009; Q.Y. Fan, 2009]). Since SICNNs can be analog voltage transmission which is often an anti-periodic process, it is worth continuing the investigation of the existence and stability of anti-periodic solutions of SICNNs. To the best of the authors' knowledge, nevertheless, there is no published paper considering the anti-periodic solutions of impulsive CGSICNNs.

Motivated by all above mentioned, we consider the following impulsive CGSICNNs on time scales

$$\begin{cases} x_{ij}^{\Delta}(t) = -a_{ij}(x_{ij}(t))\Big[b_{ij}(x_{ij}(t)) + \sum\limits_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t) f(x_{kl}(t-\tau_{kl}(t)))x_{ij}(t) - L_{ij}(t)\Big], \\ \qquad\qquad\qquad\qquad t\in\mathbb{T}^+, t\neq t_h, h\in\mathbb{N}, \\ \Delta x_{ij}(t_h) = x_{ij}(t_h^+) - x_{ij}(t_h^-) = I_{ijh}(x_{ij}(t_h)), t = t_h, i = 1,\dots,m, j = 1,\dots,n, \end{cases} \tag{1}$$

where $\mathbb{T}$ is an $\frac{\omega}{2}$-periodic time scale which has the subspace topology inherited from the standard topology on $\mathbb{R}$. C_{ij} denotes the cell at the (i, j) position of the lattice, the r-neighborhood $N_r(i, j)$ of C_{ij} is given by $N_r(i, j) = \{C_{ij} : \max\{|k - i|, |l - j|\} \le r, 1 \le k \le m, 1 \le l \le n\}$. x_{ij} acts as the activity of the cell C_{ij}, $L_{ij}(t)$ is the external input to C_{ij}, $a_{ij}(x_{ij}(t)) > 0$ and $b_{ij}(x_{ij}(t))$ represent an amplification function at time t and an appropriately behaved function at time t, respectively; $C_{ij}^{kl}(t) \ge 0$ is the connection or coupling strength of postsynaptic activity of the cell transmitted to the cell C_{ij}, and the activity function $f(\cdot)$ is a continuous function representing the output or firing rate of the cell C_{kl}; $x_{ij}(t_h^+)$, $x_{ij}(t_h^-)$ represent the right and left limit of $x_{ij}(t_h)$ in the sense of time scales, $\{t_h\}$ is a sequence of real numbers such that $0 < t_1 < t_2 < \dots < t_h \to \infty$ as $h \to \infty$. There exists a positive integer p such that $t_{h+p} = t_h + \frac{\omega}{2}$, $I_{ij(h+p)}(x_{ij}(t_{h+p})) = -I_{ijh}(-x_{ij}(t_h))$, $h \in \mathbb{N}$. Without loss of generality, we also assume that $[0, \frac{\omega}{2})_{\mathbb{T}} \cap \{t_h : h \in \mathbb{N}\} = \{t_1, t_2, \dots, t_q\}$. Let $\mathbb{R}^+ = (0, +\infty)$, $\mathbb{T}^+ = \mathbb{R}^+ \cap \mathbb{T}$.

The main purpose of this paper is to study the existence and global exponential stability of the anti-periodic solutions of (1) by applying the method of coincidence degree and constructing suitable Lyapunov functional. The methods used in this paper are different from those of the references listed above and our results can be applied to a large of neural networks.

Let $x(t) = (x_{11}(t), x_{12}(t), \dots, x_{1n}(t), \dots, x_{m1}(t), x_{m2}(t), \dots, x_{mn}(t))^T \in C(\mathbb{T}, \mathbb{R}^{mn})$, we define the norm $\| x \|=$

$\sum_{i=1}^{m}\sum_{j=1}^{n}\max_{t\in[0,\omega]_{\mathbb{T}}}|x_{ij}(t)|$. The initial conditions associated with system (1) are of the form

$$x_{ij}(t)=\varphi_{ij}(t),\quad t\in[-\tau,0]_{\mathbb{T}},\quad \tau=\max_{1\le k\le m,1\le l\le n}\sup_{t\in\mathbb{T}}\{\tau_{kl}(t)\},\tag{2}$$

where $\varphi_{ij}(t), i=1,2,\ldots,m, j=1,2,\ldots,n$ are continuous functions on $[-\tau,0]_{\mathbb{T}}$.

For the sake of convenience, we introduce some notations

$$\overline{L}_{ij}=\max_{t\in[0,\omega]_{\mathbb{T}}}|L_{ij}(t)|,\quad \overline{C}_{ij}^{kl}=\max_{t\in[0,\omega]_{\mathbb{T}}}|C_{ij}^{kl}(t)|,\quad \|g\|_2=\left(\int_0^\omega|g(t)|^2\Delta t\right)^{1/2},$$

where g is an ω-periodic function.

Throughout this paper, we assume that

(H_1) $C_{ij}^{kl}(t)\ge 0, L_{ij}(t)\in C(\mathbb{T},\mathbb{R}), C_{ij}^{kl}(t+\frac{\omega}{2})=C_{ij}^{kl}(t), \tau_{ij}(t+\frac{\omega}{2})=\tau_{ij}(t), L_{ij}(t+\frac{\omega}{2})=-L_{ij}(t), i=1,2,\ldots,m, j=1,2,\ldots,n$;

(H_2) $a_{ij}\in C(\mathbb{R},\mathbb{R}^+)$, $a_{ij}(-u)=a_{ij}(u)$ and there exist positive constants $\overline{a}_{ij}$, $\underline{a}_{ij}$ such that $0<\underline{a}_{ij}\le a_{ij}(u)\le\overline{a}_{ij}$ for all $u\in\mathbb{R}$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$;

(H_3) $b_{ij}\in C(\mathbb{R},\mathbb{R})$ are delta differentiable, $b_{ij}(0)=0$, $b_{ij}(-u)=-b_{ij}(u)$ and there exist positive constants ϱ_{ij}, δ_{ij} such that $0<\varrho_{ij}\le b_{ij}^{\Delta}(u)\le\delta_{ij}$ for all $u\in\mathbb{R}$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$;

(H_4) $f_{ij}\in C(\mathbb{R},\mathbb{R})$, $f_{ij}(-u)=f_{ij}(u)$, and there exist positive constants M_f and L_f such that $|f_{ij}(u)|\le M_f$, $|f(u)-f(v)|\le L_f|u-v|$;

(H_5) $I_{ijh}\in C(\mathbb{R},\mathbb{R})$ and there exist positive constants ρ_{ijh} such that $|I_{ijh}(u)-I_{ijh}(v)|\le\rho_{ijh}|u-v|$ for all $u,v\in\mathbb{R}$, $h\in\mathbb{N}$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$.

The organization of this paper is as follows. In Section 2, we introduce some definitions and lemmas. In Section 3, by using the method of coincidence degree, we establish sufficient conditions for the existence of the anti-periodic solutions of system (1). In Section 4, by constructing Lyapunov functional, we shall derive sufficient conditions for the global exponential stability of the anti-periodic solutions of system (1). An example is given to illustrate the effectiveness of our results in Section 5.

2. Preliminaries

In this section, we shall first recall some basic definitions, lemmas which are used in what follows.

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma,\rho:\mathbb{T}\to\mathbb{T}$ and the graininess $\mu:\mathbb{T}\to\mathbb{R}^+$ are defined, respectively, by

$$\sigma(t)=\inf\{s\in\mathbb{T}:s>t\},\quad \rho(t)=\sup\{s\in\mathbb{T}:s<t\}\quad\text{and}\quad \mu(t)=\sigma(t)-t.$$

A point $t\in\mathbb{T}$ is called left-dense if $t>\inf\mathbb{T}$ and $\rho(t)=t$, left-scattered if $\rho(t)<t$, right-dense if $t<\sup\mathbb{T}$ and $\sigma(t)=t$, and right-scattered if $\sigma(t)>t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k=\mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k=\mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k=\mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k=\mathbb{T}$.

A function $f:\mathbb{T}\to\mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$.

For $y:\mathbb{T}\to\mathbb{R}$ and $t\in\mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^{\Delta}(t)$, to be the number (if it exists) with the property that for a given $\varepsilon>0$, there exists a neighborhood U of t such that $\left|[y(\sigma(t))-y(s)]-y^{\Delta}(t)[\sigma(t)-s]\right|<\varepsilon|\sigma(t)-s|$ for all $s\in U$. If y is continuous, then y is right-dense continuous, and y is delta differentiable at t, then y is continuous at t. Let y be right-dense continuous, if $Y^{\Delta}(t)=y(t)$, then we define the delta integral by $\int_a^t y(s)\Delta s=Y(t)-Y(a)$.

A function $p:\mathbb{T}\to\mathbb{R}$ is called regressive provided $1+\mu(t)p(t)\neq 0$ for all $t\in\mathbb{T}^k$.

Similarly in [*E.R.Kaufmann*, 2006], we shall first give the definition of anti-periodic function on time scales as following:

Definition 2.1 A time scale $\mathbb{T}$ is periodic if there exists $p>0$ such that if $t\in\mathbb{T}$, then $t\pm p\in\mathbb{T}$. For $\mathbb{T}\neq\mathbb{R}$, the least positive p is called the period of the time scale. Let $\mathbb{T}\neq\mathbb{R}$ be a periodic time scale with period p. A function $f:\mathbb{T}\to\mathbb{R}$ is $\frac{\omega}{2}$-anti-periodic if there exists a natural number n such that $\frac{\omega}{2}=np$, $f(t+\frac{\omega}{2})=-f(t)$ for all $t\in\mathbb{T}$ and $\frac{\omega}{2}$ is the least number such that $f(t+\frac{\omega}{2})=-f(t)$. If $\mathbb{T}=\mathbb{R}$, we say that f is $\frac{\omega}{2}$-anti-periodic if $\frac{\omega}{2}$ is the least positive number such that $f(t+\frac{\omega}{2})=-f(t)$ for all $t\in\mathbb{T}$.

Lemma 2.1 [M. Bohner, 2001] Let p, q be regressive functions on $\mathbb{T}$. Then

(a) $e_0(t,s) \equiv 1$ and $e_p(t,t) \equiv 1$; (b) $e_p(\sigma(t),s) = (1+\mu(t)p(t))e_p(t,s)$;

(b) $e_p(t,s)e_p(s,r) = e_p(t,r)$; (d) $e_p^{\Delta}(\cdot,s) = pe_p(\cdot,s)$.

Lemma 2.2 [M. Bohner, 2001] Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^k$, then $(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t))$.

Lemma 2.3 [M. Bohner, 2006] Let $t_1, t_2 \in [0,\omega]_{\mathbb{T}}$. If $x : \mathbb{T} \to \mathbb{R}$ is ω-periodic, then $x(t) \le x(t_1) + \int_0^{\omega} |x^{\Delta}(s)|\Delta s$ and $x(t) \ge x(t_2) - \int_0^{\omega} |x^{\Delta}(s)|\Delta s$.

Lemma 2.4 [M. Bohner, 2001] Let $a, b \in \mathbb{T}$. For rd-continuous functions $f, g : [a,b]_{\mathbb{T}} \to \mathbb{R}$, we have $\int_a^b |f(t)||g(t)|\Delta t \le \left(\int_a^b |f(t)|^2\Delta t\right)^{1/2}\left(\int_a^b |g(t)|^2\Delta t\right)^{1/2}$.

Lemma 2.5 [G. Guseinov, 2003] Let function f be continuous on $[a,b]_{\mathbb{T}}$ and delta differentiable on $[a,b)_{\mathbb{T}}$, then there exist $\xi, \varsigma \in [a,b)_{\mathbb{T}}$ such that $f^{\Delta}(\xi)(b-a) \le f(b) - f(a) \le f^{\Delta}(\varsigma)(b-a)$.

Definition 2.2 The anti-periodic solution $x^*(t)$ of system (1) with initial value $\varphi^*(t)$ is said to be globally exponentially stable if there exists a positive constant ϵ with $\epsilon \in \mathcal{R}^+$ such that for every $\alpha \in \mathbb{T}$, there exists $N = N(\alpha) \ge 1$ such that the solution $x(t)$ of (1) through $(\alpha, \varphi(\alpha))$ satisfies

$$\|x_{ij} - x_{ij}^*\| \le Ne_{\ominus\epsilon}(t,\alpha)\|\varphi - x^*\|, \forall\, t \in \mathbb{T}^+,\ i = 1,2,\ldots,m,\ j = 1,2,\ldots,n$$

where $\|\varphi - x^*\| = \sup\limits_{\alpha\in(-\tau,0]_{\mathbb{T}}} \max\limits_{(i,j)} |\varphi_{ij}(\alpha) - x_{ij}^*(\alpha)|$.

The following fixed point theorem of coincidence degree is crucial in the arguments of our main results.

Lemma 2.6 [D. O'Regan, 2006] Let $\mathbb{X}$, $\mathbb{Y}$ be two Banach spaces, $\Omega \subset \mathbb{X}$ be open bounded and symmetric with $0 \in \Omega$. Suppose that $L : D(L) \subset \mathbb{X} \to \mathbb{Y}$ is a linear Fredholm operator of index zero with $D(L) \cap \bar{\Omega} \ne \emptyset$ and $N : \bar{\Omega} \to \mathbb{Y}$ is L-compact. Further, we also assume that

(H) $Lx - Nx \ne \lambda(-Lx - N(-x))$ for all $D(L) \cap \partial\Omega$, $\lambda \in (0,1]$.

Then equation $Lx = Nx$ has at least one solution on $D(L) \cap \bar{\Omega}$.

3. Existence of anti-periodic solutions

Theorem 3.1 Assume that (H_1)-(H_5) hold. Suppose further that

(H_6) $E_{ij} > 0$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, where

$$E_{ij} = \underline{a}_{ij}\omega(1 - \overline{a}_{ij}\delta_{ij}\omega) - (\frac{1}{\varrho_{ij}} + \underline{a}_{ij}\omega)\Big[\overline{a}_{ij}\sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_f\omega + \sum_{h=1}^{2q}\rho_{ijh}\Big].$$

Then system (1) has at least one $\frac{\omega}{2}$-anti-periodic solution.

Proof. Let $C^k([0,\omega; t_1,t_2,\ldots,t_q,t_{q+1},\ldots,t_{2q}]_{\mathbb{T}}, \mathbb{R}^{mn}) = \{x : [0,\omega]_{\mathbb{T}} \to \mathbb{R}^{mn} | x^{(k)}(t)$ is a piecewise continuous map with first-class discontinuous points in $[0,\omega]_{\mathbb{T}} \cap \{t_h : h \in \mathbb{N}\}$ and at each discontinuous point it is continuous on the left$\}$, $k = 0,1$.

Take $\mathbb{X} = \{x \in C([0,\omega; t_1,t_2,\ldots,t_q,t_{q+1},\ldots,t_{2q}]_{\mathbb{T}}, \mathbb{R}^{mn}) : x(t+\frac{\omega}{2}) = -x(t)$ for all $t \in [0,\frac{\omega}{2}]_{\mathbb{T}}\}$ and $\mathbb{Y} = \mathbb{X} \times \mathbb{R}^{(mn)\times q}$ be two Banach spaces equipped with the norms $\|x\|_{\mathbb{X}} = \sum_{i=1}^{m}\sum_{j=1}^{n} |x_{ij}|_0$ and $\|y\|_{\mathbb{Y}} = \|x\|_{\mathbb{X}} + \|z\|$ for all $x \in \mathbb{X}$, $z \in \mathbb{R}^{(mn)\times q}$, in which $|x_{ij}|_0 = \max\limits_{t\in[0,\omega]_{\mathbb{T}}} |x_{ij}(t)|$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$, $\|\cdot\|$ is any norm of $\mathbb{R}^{(mn)\times q}$. Set $L : \mathrm{Dom}\,L \cap \mathbb{X} \to \mathbb{Y}$, $x \to (x^{\Delta}, \Delta x(t_1), \Delta x(t_2), \ldots, \Delta x(t_q))$, where $\mathrm{Dom}\,L = \{x \in C^1[0,\omega; t_1,t_2,\ldots,t_{2q}]_{\mathbb{T}} : x(t+\frac{\omega}{2}) = -x(t)$ for all $t \in [0,\frac{\omega}{2}]_{\mathbb{T}}\}$, and $N : \mathbb{X} \to \mathbb{Y}$,

$$Nx = \left(\begin{pmatrix} A_{11}(t) \\ \vdots \\ A_{1n}(t) \\ \vdots \\ A_{m1}(t) \\ \vdots \\ A_{mn}(t) \end{pmatrix}, \begin{pmatrix} I_{111}(x_{11}(t_1)) \\ \vdots \\ I_{1n1}(x_{1n}(t_1)) \\ \vdots \\ I_{m11}(x_{m1}(t_1)) \\ \vdots \\ I_{mn1}(x_{mn}(t_1)) \end{pmatrix}, \begin{pmatrix} I_{112}(x_{11}(t_2)) \\ \vdots \\ I_{1n2}(x_{1n}(t_2)) \\ \vdots \\ I_{m12}(x_{m1}(t_2)) \\ \vdots \\ I_{mn2}(x_{mn}(t_2)) \end{pmatrix}, \ldots, \begin{pmatrix} I_{11q}(x_{11}(t_q)) \\ \vdots \\ I_{1nq}(x_{1n}(t_q)) \\ \vdots \\ I_{m1q}(x_{m1}(t_q)) \\ \vdots \\ I_{mnq}(x_{mn}(t_q)) \end{pmatrix}\right),$$

where $A_{ij}(t) = -a_{ij}(x_{ij}(t))\Big[b_{ij}(x_{ij}(t)) + \sum\limits_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f(x_{kl}(t-\tau_{kl}(t)))x_{ij}(t) - L_{ij}(t)\Big]$, for $i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n$.

It is easy to see that $\operatorname{Ker} L = \{0\}$ and $\operatorname{Im} L = \{z = (g, c_1, \ldots, c_q) \in \mathbb{Y} : \int_0^\omega g(s)\Delta s = 0\} \equiv \mathbb{Y}$. Thus $\dim \operatorname{Ker} L = 0 = \operatorname{codim} \operatorname{Im} L$, and L is a linear Fredholm operator of index zero.

Define the continuous projector $P : \mathbb{X} \to \operatorname{Ker} L$ and the averaging projector $Q : \mathbb{Y} \to \mathbb{Y}$ by $Px = \int_0^\omega x(s)\Delta s = 0$, and $Qz = Q(g, c_1, \ldots, c_q) = \Big(\frac{1}{\omega}\int_0^\omega g(s)\Delta s, 0, \ldots, 0\Big)$. Hence, $\operatorname{Im} P = \operatorname{Ker} L$ and $\operatorname{Ker} Q = \operatorname{Im} L = \operatorname{Im}(I - Q)$. Denoting by $L_P^{-1} : \operatorname{Im} L \to \operatorname{Dom}(L) \cap \operatorname{Ker} P$ the inverse of $L|_{D(L)\cap \operatorname{Ker} P}$, we have

$$L_P^{-1}z = \int_0^t g(s)\Delta s + \sum_{t_k<t} c_k - \frac{1}{2}\int_0^{\frac{\omega}{2}} g(s)\Delta s - \frac{1}{2}\sum_{k=1}^{q} c_k,$$

in which $c_{q+i} = -c_i$ for all $1 \le i \le q$.

Similar to [$Y.K.Li$, 2009], it is not difficult to show that $QN(\bar{\Omega})$, $L_P^{-1}(I - Q)N(\bar{\Omega})$ are relatively compact for any open bounded set $\Omega \subset \mathbb{X}$. Therefore, N is L-compact on $\bar{\Omega}$ for any open bounded set $\Omega \subset \mathbb{X}$.

In order to apply Lemma 2.6, we need to find an appropriate open bounded subset Ω in $\mathbb{X}$. Corresponding to the operator equation $Lx - Nx = \lambda(-Lx - N(-x))$, $\lambda \in (0, 1]$, we have

$$\begin{cases} x_{ij}^{\Delta}(t) = \frac{1}{1+\lambda}G_{ij}(t, x) - \frac{\lambda}{1+\lambda}G_{ij}(t, -x), \quad t \in \mathbb{T}^+,\ t \ne t_h,\ h \in \mathbb{N}, \\ \Delta x_{ij}(t_h) = \frac{1}{1+\lambda}I_{ijh}(x_i(t_h)) - \frac{\lambda}{1+\lambda}I_{ijh}(-x_{ij}(t_h)), \quad i = 1, 2, \ldots, m, j = 1, 2, \ldots, n, \end{cases} \tag{3}$$

where

$$G_{ij}(t, x) = -a_{ij}(x_{ij}(t))\Big[b_{ij}(x_{ij}(t)) + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f(x_{kl}(t-\tau_{kl}(t)))x_{ij}(t) - L_{ij}(t)\Big],$$

$$G_{ij}(t, -x) = -a_{ij}(-x_{ij}(t))\Big[b_{ij}(-x_{ij}(t)) + \sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f(-x_{kl}(t-\tau_{kl}(t)))(-x_{ij}(t)) - L_{ij}(t)\Big].$$

Set $t_0 = t_0^+ = 0$, $t_{2q+1} = \omega$, in view of (3), together with $(H_2) - (H_4)$, we obtain

$$\begin{aligned} \int_0^\omega \big|x_{ij}^{\Delta}(t)\big|\Delta t &= \sum_{h=1}^{2q+1}\int_{t_{h-1}^+}^{t_h} \big|x_{ij}^{\Delta}(t)\big|\Delta t + \sum_{h=1}^{2q}\big|\Delta x_{ij}(t_h)\big| \\ &\le \int_0^\omega \Big|\frac{1}{1+\lambda}G_{ij}(t, x) - \frac{\lambda}{1+\lambda}G_{ij}(t, -x)\Big|\Delta t + \sum_{h=1}^{2q}\Big|\frac{1}{1+\lambda}I_{ijh}(x_{ij}(t_h)) - \frac{\lambda}{1+\lambda}I_{ijh}(-x_{ij}(t_h))\Big| \\ &\le \overline{a}_{ij}\Big[\delta_{ij}\sqrt{\omega} \parallel x_{ij} \parallel_2 + \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_f \sqrt{\omega} \parallel x_{ij} \parallel_2 + \omega \overline{L}_{ij}\Big] \\ &\quad + \sum_{h=1}^{2q}\rho_{ijh}\big|x_{ij}\big|_0 + \sum_{h=1}^{2q}\big|I_{ijh}(0)\big|. \end{aligned} \tag{4}$$

Integrating (3) from 0 to ω, together with $(H_2) - (H_4)$, we can get

$$\begin{aligned} &\Big|\int_0^\omega \Big[\frac{a_{ij}(x_i(t))b_i(x_{ij}(t))}{1+\lambda} - \frac{\lambda a_{ij}(-x_{ij}(t))b_{ij}(-x_{ij}(t))}{1+\lambda}\Big]\Delta t\Big| \\ = &\Big|\int_0^\omega a_{ij}(x_{ij}(t))b_{ij}(x_{ij}(t))\Delta t\Big| \\ = &\Big| -\frac{1}{1+\lambda}\int_0^\omega a_{ij}(x_{ij}(t))\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f(x_{kl}(t-\tau_{kl}(t)))x_{ij}(t)\Delta t \\ &+\frac{\lambda}{1+\lambda}\int_0^\omega a_{ij}(-x_{ij}(t))\sum_{C_{kl}\in N_r(i,j)} C_{ij}^{kl}(t)f(-x_{kl}(t-\tau_{kl}(t)))(-x_{ij}(t))\Delta t \\ &+\frac{1}{1+\lambda}\int_0^\omega a_{ij}(x_{ij}(t))L_{ij}(t)\Delta t - \frac{\lambda}{1+\lambda}\int_0^\omega a_{ij}(-x_{ij}(t))L_{ij}(t)\Delta t \\ &+\frac{1}{1+\lambda}\sum_{h=1}^{2q} I_{ijh}(x_{ij}(t_h)) - \frac{\lambda}{1+\lambda}\sum_{h=1}^{2q} I_{ijh}(-x_{ij}(t_h))\Big| \\ \le &\ \overline{a}_{ij}\sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_f \sqrt{\omega} \parallel x_{ij} \parallel_2 + \overline{a}_{ij}\omega\overline{L}_{ij} + \sum_{h=1}^{2q}\rho_{ijh}\big|x_{ij}\big|_0 + \sum_{h=1}^{2q}\big|I_{ijh}(0)\big|, \end{aligned}$$

then, by Lemma 2.5 and (H_3), combining the above inequality, implies that

$$\begin{aligned}\left|\int_0^\omega a_{ij}(x_{ij}(t))x_{ij}(t)\Delta t\right| &\leq \frac{\overline{a}_{ij}}{\varrho_{ij}}\sum_{C_{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}M_f\sqrt{\omega}\parallel x_{ij}\parallel_2+\frac{\overline{a}_{ij}}{\varrho_{ij}}\omega\overline{L}_{ij}\\&\quad+\frac{1}{\varrho_{ij}}\sum_{h=1}^{2q}\rho_{ijh}\big|x_{ij}\big|_0+\frac{1}{\varrho_{ij}}\sum_{h=1}^{2q}\big|I_{ijh}(0)\big|. \end{aligned}\tag{5}$$

where $i=1,2,\ldots,m,\ j=1,2,\ldots,n$.

From Lemma 2.3, for any $\zeta_{ij},\eta_{ij}\in[0,\omega]_{\mathbb{T}}$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n$, we have

$$\int_0^\omega a_{ij}(x_{ij}(t))x_{ij}(t)\Delta t\leq\int_0^\omega a_{ij}(x_{ij}(t))x_{ij}(\zeta_{ij})\Delta t+\int_0^\omega a_{ij}(x_{ij}(t))\Big(\int_0^\omega\big|x_{ij}^\Delta(t)\big|\Delta t\Big)\Delta t,\tag{6}$$

and

$$\int_0^\omega a_{ij}(x_{ij}(t))x_{ij}(t)\Delta t\geq\int_0^\omega a_{ij}(x_{ij}(t))x_{ij}(\eta_{ij})\Delta t-\int_0^\omega a_{ij}(x_{ij}(t))\Big(\int_0^\omega\big|x_{ij}^\Delta(t)\big|\Delta t\Big)\Delta t,\tag{7}$$

where $i=1,2,\ldots,m,\ j=1,2,\ldots,n$.

Dividing by $\int_0^\omega a_{ij}(x_{ij}(t))\Delta t$ on the two sides of (6) and (7), respectively, we obtain

$$x_{ij}(\zeta_{ij})\geq\frac{1}{\int_0^\omega a_{ij}(x_{ij}(t))\Delta t}\int_0^\omega a_{ij}(x_{ij}(t))x_{ij}(t)\Delta t-\int_0^\omega\big|x_{ij}^\Delta(t)\big|\Delta t,\tag{8}$$

and

$$x_{ij}(\eta_{ij})\leq\frac{1}{\int_0^\omega a_{ij}(x_{ij}(t))\Delta t}\int_0^\omega a_{ij}(x_{ij}(t))x_{ij}(t)\Delta t+\int_0^\omega\big|x_{ij}^\Delta(t)\big|\Delta t,\tag{9}$$

where $i=1,2,\ldots,m,\ j=1,2,\ldots,n$.

Let $\bar{t}_{ij}$, $\underline{t}_{ij}\in[0,\omega]_{\mathbb{T}}$ such that $x_{ij}(\bar{t}_{ij})=\max_{t\in[0,\omega]_{\mathbb{T}}}x_{ij}(t)$, $x_{ij}(\underline{t}_{ij})=\min_{t\in[0,\omega]_{\mathbb{T}}}x_{ij}(t)$, then, together with (4), (5), (8) and (9), we obtain

$$\begin{aligned}x_{ij}(\underline{t}_{ij}) &\geq \frac{1}{\int_0^\omega a_{ij}(x_{ij}(t))\Delta t}\int_0^\omega a_{ij}(x_{ij}(t))x_{ij}(t)\Delta t-\int_0^\omega\big|x_{ij}^\Delta(t)\big|\Delta t\\&\geq -\frac{1}{\underline{a}_{ij}\omega}\Big[\frac{\overline{a}_{ij}}{\varrho_{ij}}\sum_{C_{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}M_f\sqrt{\omega}\parallel x_{ij}\parallel_2+\frac{\overline{a}_{ij}}{\varrho_{ij}}\omega\overline{L}_{ij}+\frac{1}{\varrho_{ij}}\sum_{h=1}^{2q}\rho_{ijh}\big|x_{ij}\big|_0\\&\quad+\frac{1}{\varrho_{ij}}\sum_{h=1}^{2q}\big|I_{ijh}(0)\big|\Big]-\overline{a}_{ij}\Big[\delta_{ij}\sqrt{\omega}\parallel x_{ij}\parallel_2+\sum_{C_{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}M_f\sqrt{\omega}\parallel x_{ij}\parallel_2\\&\quad+\omega\overline{L}_{ij}\Big]+\sum_{h=1}^{2q}\rho_{ijh}\big|x_{ij}\big|_0+\sum_{h=1}^{2q}\big|I_{ijh}(0)\big|\end{aligned}\tag{10}$$

and

$$\begin{aligned}x_{ij}(\bar{t}_{ij}) &\leq \frac{1}{\int_0^\omega a_{ij}(t)\Delta t}\int_0^\omega a_{ij}(t)x_{ij}(t)\Delta t+\int_0^\omega\big|x_{ij}^\Delta(t)\big|\Delta t\\&\leq \frac{1}{\underline{a}_{ij}\omega}\Big[\frac{\overline{a}_{ij}}{\varrho_{ij}}\sum_{C_{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}M_f\sqrt{\omega}\parallel x_{ij}\parallel_2+\frac{\overline{a}_{ij}}{\varrho_{ij}}\omega\overline{L}_{ij}+\frac{1}{\varrho_{ij}}\sum_{h=1}^{2q}\rho_{ijh}\big|x_{ij}\big|_0\\&\quad+\frac{1}{\varrho_{ij}}\sum_{h=1}^{2q}\big|I_{ijh}(0)\big|\Big]+\overline{a}_{ij}\Big[\delta_{ij}\sqrt{\omega}\parallel x_{ij}\parallel_2+\sum_{C_{kl}\in N_r(i,j)}\overline{C}_{ij}^{kl}M_f\sqrt{\omega}\parallel x_{ij}\parallel_2\\&\quad+\omega\overline{L}_{ij}\Big]+\sum_{h=1}^{2q}\rho_{ijh}\big|x_{ij}\big|_0+\sum_{h=1}^{2q}\big|I_{ijh}(0)\big|,\end{aligned}\tag{11}$$

where $i=1,2,\ldots,m,\ j=1,2,\ldots,n$.

Then, from (10) and (11), we can get

$$\begin{aligned}
\left|x_{ij}\right|_0 &= \max_{t\in[0,\omega]_{\mathbb{T}}} \left|x_{ij}(t)\right| \\
&\le \frac{1}{\underline{a}_{ij}\omega}\Big[\frac{\overline{a}_{ij}}{\varrho_{ij}} \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_f \sqrt{\omega} \parallel x_{ij} \parallel_2 + \frac{\overline{a}_{ij}}{\varrho_{ij}}\omega\overline{L}_{ij} + \frac{1}{\varrho_{ij}} \sum_{h=1}^{2q} \rho_{ijh}\left|x_{ij}\right|_0 \\
&\quad + \frac{1}{\varrho_{ij}} \sum_{h=1}^{2q} \left|I_{ijh}(0)\right|\Big] + \overline{a}_{ij}\Big[\delta_{ij}\sqrt{\omega} \parallel x_{ij} \parallel_2 + \frac{1}{\varrho_{ij}} \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_f \sqrt{\omega} \parallel x_{ij} \parallel_2 \\
&\quad + \omega\overline{L}_{ij}\Big] + \sum_{h=1}^{2q} \rho_{ijh}\left|x_{ij}\right|_0 + \sum_{h=1}^{2q} \left|I_{ijh}(0)\right|, \quad i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n. \qquad (12)
\end{aligned}$$

In addition, we have

$$\|x_{ij}\|_2 = \Big(\int_0^\omega \left|x_{ij}(s)\right|^2 \Delta s\Big)^{1/2} \le \sqrt{\omega} \max_{t\in[0,\omega]_{\mathbb{T}}} \left|x_{ij}(t)\right| = \sqrt{\omega}\left|x_{ij}\right|_0, \qquad (13)$$

where $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$.

Then, together with (12), (13), we obtain

$$\begin{aligned}
\underline{a}_{ij}\omega\left|x_{ij}\right|_0 &\le \Big[\frac{\overline{a}_{ij}}{\varrho_{ij}} \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_f \omega\left|x_{ij}\right|_0 + \frac{\overline{a}_{ij}}{\varrho_{ij}}\omega\overline{L}_{ij} + \frac{1}{\varrho_{ij}} \sum_{h=1}^{2q} \rho_{ijh}\left|x_{ij}\right|_0 \\
&\quad + \frac{1}{\varrho_{ij}} \sum_{h=1}^{2q} \left|I_{ijh}(0)\right|\Big] + \underline{a}_{ij}\omega\Big[\overline{a}_{ij}\delta_{ij}\omega\left|x_{ij}\right|_0 + \overline{a}_{ij} \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_f \omega\left|x_{ij}\right|_0 \\
&\quad + \overline{a}_{ij}\omega\overline{L}_{ij} + \sum_{h=1}^{2q} \rho_{ijh}\left|x_{ij}\right|_0 + \sum_{h=1}^{2q} \left|I_{ijh}(0)\right|\Big], \ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n.
\end{aligned}$$

That is,

$$\begin{aligned}
&\Big\{\underline{a}_{ij}\omega(1 - \overline{a}_{ij}\delta_{ij}\omega) - (\frac{1}{\varrho_{ij}} + \underline{a}_{ij}\omega)\Big[\overline{a}_{ij} \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_f \omega + \sum_{h=1}^{2q} \rho_{ijh}\Big]\Big\}\left|x_{ij}\right|_0 \\
\le\ & (\frac{1}{\varrho_{ij}} + \underline{a}_{ij}\omega)\Big[\sum_{h=1}^{2q} \left|I_{ijh}(0)\right| + \overline{a}_{ij}\omega\overline{L}_{ij}\Big] \\
:=\ & D_{ij}, \quad i = 1, 2, \ldots, m, n = 1, 2, \ldots, n. \qquad (14)
\end{aligned}$$

Denote $E_{ij} = \underline{a}_{ij}\omega(1 - \overline{a}_{ij}\delta_{ij}\omega) - (\frac{1}{\varrho_{ij}} + \underline{a}_{ij}\omega)\Big[\overline{a}_{ij} \sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_f \omega + \sum_{h=1}^{2q} \rho_{ijh}\Big]$. From (14) and (H_6), we can get $\left|x_{ij}\right|_0 \le \frac{D_{ij}}{E_{ij}} := M_{ij}, \quad i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.

Let $\Theta = \sum_{i=1}^{m}\sum_{j=1}^{n} M_{ij} + 1$, Θ is independent of λ. Then take $\Omega = \{x \in \mathbb{X} : \|x\|_{\mathbb{X}} < \Theta\}$. It is clear that Ω satisfies all the requirement in Lemma 2.6 and the condition (H) is satisfied. Then system (1) has at least one $\frac{\omega}{2}$-anti-periodic solution. This completes the proof. □

4. Global exponential stability of the anti-periodic solution

In this section, we will construct some suitable Lyapunov functions to study the global exponential stability of the anti-periodic solution of system (1).

Theorem 4.1 Assume that (H_1)-(H_6) hold. Suppose further that

(H_7) The impulsive operators $I_{ijh}(x_{ij}(t))$ satisfy $I_{ijh}(x_{ij}(t_h)) = -\gamma_{ijh}x_{ij}(t_h)$, $x_{ij}(t_h^-) = x_{ij}(t_h)$, $0 \le \gamma_{ijh} \le 2$, for all $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, $h \in \mathbb{N}$.

(H_8) There exist positive constants ν_{ij} such that $|a_{ij}(u) - a_{ij}(v)| \le \nu_{ij}|u - v|$, for all $u, v \in \mathbb{R}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.

(H_9) There exist $m \times n$ positive constants $\xi_{ij} > 0$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$, such that $\beta_{ij} = (\nu_{ij}\overline{L}_{ij} - \underline{a}_{ij}\varrho_{ij})\xi_{ij} + \overline{a}_{ij} \sum_{C_{kl} \in N_r(i,j)} \overline{C}_{ij}^{kl} M_f \xi_{ij} + \overline{a}_{ij} \sum_{C_{kl} \in N_r(i,j)} \overline{C}_{ij}^{kl} M_0 L_f \xi_{kl} < 0$.

where $M_0 = \max_{(i,j)} M_{ij}$, then the $\frac{\omega}{2}$-anti-periodic solution of system (1) is globally exponentially stable.

Proof. According to Theorem 3.1, we know that system (1) has an $\frac{\omega}{2}$-anti-periodic solution $x^*(t) = (x^*_{11}(t), x^*_{12}(t), \ldots, x^*_{1n}(t), \ldots, x^*_{m1}(t), x^*_{m2}(t), \ldots, x^*_{mn}(t))^T$ with the initial value $\varphi^*(t) = (\varphi^*_{11}(t), \ldots, \varphi^*_{1n}(t), \ldots, \varphi^*_{m1}(t), \ldots, \varphi^*_{mn}(t))^T$ and $\left|x_{ij}\right|_0 \le M_0$, suppose that $x(t) = (x_{11}(t), x_{12}(t), \ldots, x_{1n}(t), \ldots, x_{m1}(t), x_{m2}(t), \ldots, x_{mn}(t))^T$ is an arbitrary solution of system (1) with $\varphi(t) = (\varphi_{11}(t), \ldots, \varphi_{1n}(t), \ldots, \varphi_{m1}(t), \ldots, \varphi_{mn}(t))^T$.

Let $y(t) = x(t) - x^*(t)$, then system (1) can be written as

$$\begin{cases} (y_{ij}(t))^\Delta &= -[a_{ij}(x_{ij}(t))b_{ij}(x_{ij}(t)) - a_{ij}(x^*_{ij}(t))b_{ij}(x^*_{ij}(t))] \\ & \quad -\Big[a_{ij}(x_{ij}(t)) \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t) f(x_{kl}(t - \tau_{kl}(t)))x_{ij}(t) \\ & \quad -a_{ij}(x^*_{ij}(t)) \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t) f(x^*_{kl}(t - \tau_{kl}(t)))x^*_{ij}(t)\Big] \\ & \quad +[a_{ij}(x_{ij}(t)) - a_{ij}(x^*_{ij}(t))]L_{ij}(t), \ t \ne t_h \\ \Delta y_{ij}(t_h) &= -\gamma_{ijh} y_{ij}(t_h), \quad i = 1, 2, \ldots, m, j = 1, 2, \ldots, n. \end{cases} \tag{15}$$

Also, $|y_{ij}(t_h + 0)| = |1 - \gamma_{ijh}||y_{ij}(t_h)| \le |y_{ij}(t_h)|$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$, $h \in \mathbb{N}$.

The initial conditions of system (15) is $\psi_{ij}(s) = \varphi_{ij}(s) - x^*_{ij}$, $s \in [-\tau, 0]_{\mathbb{T}}$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.

If (H_9) holds, it can always find a small enough constant $\epsilon > 0$, satisfying $\forall t \in \mathbb{T}, 1 - \mu(t)\epsilon > 0$, such that

$$\begin{aligned} &(\epsilon + \nu_{ij}\overline{L}_{ij} - \underline{a}_{ij}\varrho_{ij})\xi_{ij} + \overline{a}_{ij} \sum_{C_{kl} \in N_r(i,j)} \overline{C}_{ij}^{kl} M_f \xi_{ij} + \overline{a}_{ij} \sum_{C_{kl} \in N_r(i,j)} \overline{C}_{ij}^{kl} M_0 L_f e_\epsilon(t, t - \tau_{kl}(t))\xi_{kl} \\ &< 0, \ i = 1, 2, \ldots, m, j = 1, 2, \ldots, n. \end{aligned} \tag{16}$$

Define a Lyapunov function $V = (V^*_{11}(t), V^*_{12}(t), \ldots, V^*_{1n}(t), \ldots, V^*_{m1}(t), V^*_{m2}(t), \ldots, V^*_{mn}(t))^T$, where $V_{ij}(t) = e_\epsilon(t, \alpha)\left|y_{ij}(t)\right|$, $\alpha \in (-\tau, 0]_{\mathbb{T}}, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.

For $t \in \mathbb{T}^+$, $t \ne t_h$, $h \in \mathbb{N}$, calculating the upper right derivative of $V_{ij}(t)$ along the solution of system (15), we have

$$\begin{aligned} & D^+\left|V_{ij}(t)\right|^\Delta \\ = \ & \epsilon\left|y_{ij}(t)\right|e_\epsilon(t, \alpha) + e_\epsilon(\sigma(t), \alpha)\text{sign}\Big\{ - [a_{ij}(x_{ij}(t))b_{ij}(x_{ij}(t)) - a_{ij}(x^*_{ij}(t))b_{ij}(x^*_{ij}(t))] \\ & -\Big[a_{ij}(x_{ij}(t)) \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t) f(x_{kl}(t - \tau_{kl}(t)))x_{ij}(t) \\ & -a_{ij}(x^*_{ij}(t)) \sum_{C_{kl} \in N_r(i,j)} C_{ij}^{kl}(t) f(x^*_{kl}(t - \tau_{kl}(t)))x^*_{ij}(t)\Big] + [a_{ij}(x_{ij}(t)) - a_{ij}(x^*_{ij}(t))]L_{ij}(t)\Big\} \\ \le \ & (1 + \mu(t)\epsilon)\Big\{(\epsilon + \nu_{ij}\overline{L}_{ij} - \underline{a}_{ij}\varrho_{ij})V_{ij}(t) + \overline{a}_{ij} \sum_{C_{kl} \in N_r(i,j)} \overline{C}_{ij}^{kl} M_f V_{ij}(t) \\ & +\overline{a}_{ij} \sum_{C_{kl} \in N_r(i,j)} \overline{C}_{ij}^{kl} M_0 L_f e_\epsilon(t, t - \tau_{kl}(t)) V_{kl}(t - \tau_{kl}(t))\Big\}. \end{aligned} \tag{17}$$

Define the curve $\rho = \{w(l) : w_{ij} = \xi_{ij}l, l > 0, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n\}$ and the set $\Omega(w) = \{u : 0 \le u \le w, w \in \rho\}$, $S_{ij}(w) = \{u \in \Omega(w) : u_{ij} = w_{ij}, 0 \le u \le w\}$. It is obvious that if $l > \overline{l}$, then $\Omega(w(l)) \subset \Omega(w(\overline{l}))$.

We shall prove that the zero solution of (15) is exponential stable.

Let $\xi^M = \max_{1 \le i \le m, 1 \le j \le n}\{\xi_{ij}\}$, $\xi^m = \min_{1 \le i \le m, 1 \le j \le n}\{\xi_{ij}\}$, $l_0 = (1 - \alpha)|\psi_{ij}|_0/\xi^m$, where $-\alpha \ge 0$ is a constant, $|\psi_{ij}|_0 = \max_{\alpha \in [-\tau, 0]_{\mathbb{T}}} |\psi_{ij}(\alpha)|$. Then $\{|V| : |V| = e_\alpha(t, \alpha)|\psi(\alpha)|, -\tau \le t \le \alpha \le 0\} \subset \Omega(w(l_0))$, namely $|V_{ij}(\alpha)| = e_\alpha(t, \alpha)|\psi_{ij}(\alpha)| < \xi_{ij}l_0$, $-\tau \le \alpha \le 0$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$.

We can claim that $|V_{ij}(t)| < \xi_{ij}l_0$, for $t \in \mathbb{T}^+$, $i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. If it is not true then there exist some $ij \in \{11, 12, \ldots, 1m, \ldots, m1, m2, \ldots, mn\}$ and $t_1 (t_1 \in \mathbb{T}^+)$ such that $|V_{ij}(t_1)| = \xi_{ij}l_0$, $[V_{ij}^\Delta(t_1)]^+ \ge 0$ and $|V_{ij}(t)| \le \xi_{ij}l_0$ for

$t \in [-\tau, t_1]_{\mathbb{T}}$, $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, n$. However, from (16) and (17), we have

$$\begin{aligned} D^+ V_{ij}^{\Delta}(t_1) &\leq (1+\mu(t)\epsilon)\Big\{(\epsilon + \nu_{ij}\overline{L}_{ij} - \underline{a}_{ij}\varrho_{ij})\xi_{ij} + \overline{a}_{ij}\sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_f \xi_{ij} \\ &\quad + \overline{a}_{ij}\sum_{C_{kl}\in N_r(i,j)} \overline{C}_{ij}^{kl} M_0 L_f e_\epsilon(t, t-\tau_{kl}(t))\xi_{kl}\Big\} l_0 \\ &< 0,\ t \in \mathbb{T}^+, t \neq t_h. \end{aligned}$$

This is a contradiction, so $|V_{ij}(t)| < \xi_{ij} l_0$, for $t \in \mathbb{T}^+, t \neq t_h, i = 1, 2, \ldots, m, j = 1, 2, \ldots, n$. Also

$$V_{ij}(t_h + 0) = e_\epsilon(t_h + 0, \alpha)|y_{ij}(t_h + 0)| \leq e_\epsilon(t_h, \alpha)|y_{ij}(t_h)| = V_{ij}(t_h),\ h \in \mathbb{N}.$$

Then

$$|y_{ij}(t)| < e_{\ominus\epsilon}(t, \alpha)\xi_{ij} l_0 = e_{\ominus\epsilon}(t, \alpha)\xi_{ij}(1-\alpha)|\psi_{ij}|_0/\xi^m,\ t \in \mathbb{T}^+,\ i = 1, 2, \ldots, m,\ j = 1, 2, \ldots, n,$$

which means that

$$\|y\| \leq \frac{\xi^M(1-\alpha)}{\xi^m} e_{\ominus\epsilon}(t, \alpha)\|\psi\| = N e_{\ominus\epsilon}(t, \alpha)\|\psi\|,\ t \in \mathbb{T}^+,$$

where $N = N(\alpha) = \frac{\xi^M}{\xi^m}(1-\alpha) > 1$. In view of $y(t) = x(t) - x^*(t), \psi(t) = \varphi(t) - x^*(t)$, then, we have

$$\|x - x^*\| \leq N e_{\ominus\epsilon}(t, \alpha)\|\varphi - x^*\|,\ t \in \mathbb{T}^+.$$

From Definition 2.2, the $\frac{\omega}{2}$-anti-periodic solution $x^*(t)$ of system (1) is globally exponentially stable. This completes the proof. □

5. An example

Consider the following CGSICNNs with impulses. Let

$$\begin{aligned} &(a_{ij})_{2\times 2} = \begin{pmatrix} 2.0 + 0.1\sin|u| & 2.0 - 0.2\cos|u| \\ 1.9 + 0.1\sin|u| & 1.9 - 0.2\cos|u| \end{pmatrix},\ b_{ij}(x_{ij}(t)) = x_{ij}(t),\ i, j = 1, 2, \\ &(C_{ij})_{2\times 2} = \begin{pmatrix} 0.6|\sin(16\pi t)| & 0.9|\sin(16\pi t)| \\ 0.8|\cos(16\pi t)| & 0.5|\cos(16\pi t)| \end{pmatrix},\ (L_{ij})_{2\times 2} = \begin{pmatrix} 0.06\sin(8\pi t) & 0.05\cos(8\pi t) \\ 0.07\cos(8\pi t) & 0.04\sin(8\pi t) \end{pmatrix}, \\ &f(u) = \frac{1}{20}|\sin u|,\ \Delta x_{ij}(t_h) = x_{ij}(t_h^+) - x_{ij}(t_h^-) = -0.025 x_{ij}(t_h), t = t_h, h = 1, 2, \end{aligned}$$

in (1), then, we have

$$\begin{aligned} &\underline{a}_{11} = 1.9, \underline{a}_{12} = \underline{a}_{21} = 1.8, \underline{a}_{22} = 1.7,\ \bar{a}_{11}(t) = \bar{a}_{22}(t) = 2.1, \bar{a}_{12}(t) = 2.2, \bar{a}_{21}(t) = 2.0, \\ &M_f = \frac{1}{20}, L_f = \frac{1}{20}, \rho_{11h} = \rho_{12h} = \rho_{21h} = \rho_{22h} = 0.025, h = 1, 2, \varrho_{ij} = \delta_{ij} = 1, \\ &\Sigma_{C^{kl}\in N_1(i,j)}\overline{C}_{11}^{kl} = \Sigma_{C^{kl}\in N_1(i,j)}\overline{C}_{12}^{kl} = \Sigma_{C^{kl}\in N_1(i,j)}\overline{C}_{21}^{kl} = \Sigma_{C^{kl}\in N_1(i,j)}\overline{C}_{22}^{kl} = 2.8, \\ &\bar{L}_{11} = 0.06, \bar{L}_{12} = 0.05, \bar{L}_{21} = 0.07, \bar{L}_{22} = 0.04. \end{aligned}$$

Computing by MATLAB, we can get

$$(E_{ij})_{2\times 2} = \begin{pmatrix} 0.1910 & 0.1634 \\ 0.1960 & 0.1684 \end{pmatrix},\ (D_{ij})_{2\times 2} = \begin{pmatrix} 0.1202 & 0.1232 \\ 0.1233 & 0.1012 \end{pmatrix}.$$

So, $M_0 = 0.7540$. Take $\nu_{11} = \nu_{21} = 0.1, \nu_{12} = \nu_{22} = 0.2$ and $\xi_{ij} = 1,\ i, j = 1, 2$, then

$$(\beta_{ij})_{2\times 2} = \begin{pmatrix} -1.3783 & -1.2498 \\ -1.3019 & -1.1763 \end{pmatrix}.$$

Now, we can see that (H_1)-(H_9) are all hold. By Theorem 3.1 and Theorem 4.1, system (1) has a $\frac{1}{8}$-anti-periodic solution which is global exponential stable.

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