

Blowing up Solution of Initial-Boundary Value Problem for a Kind of Nonlinear Evolution Equations

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Abstract

This dissertation is to discuss the initial-boundary value problem under the third nonlinear boundary condition for a kind of nonlinear evolution equations. To apply the maximum value theory and convex functional method it is proved that the blowing up of solution in the definite time under some assumed conditions. The conclusion popularizes the results of references [Zhongwei Zha(2003)-Zhongwei Zha(2010)].

Keywords: Nonlinear evolution equations, Initial-boundary value problem, Blow up of solution, Convex functional method

1. Introduction

This paper discusses the initial-boundary value problem under the third nonlinear boundary condition for a kind of nonlinear evolution equations.

$$(I) \begin{cases} \frac{\partial u}{\partial t} = \nabla(h(u)\nabla u) + f(x, t, u, \nabla u) & D \times (0, T) \\ h(u)\frac{\partial u}{\partial \nu} + u = g(x, t, u, \nabla u) & \partial D \times (0, T) \\ u(x, 0) = \varphi(x) & \bar{D} \end{cases} \quad (1.1)$$

$$(1.2)$$

$$(1.3)$$

where D is a smooth boundary area in R^n , $x = (x_1, x_2, \dots, x_n)$, T is a positive constant, $h(u) > 0$ is a monotonous decreasing continuous function, $\nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n})$ is a gradient operator and $\frac{\partial u}{\partial \nu}$ is the exterior normal derivative for ∂D .

The solution losing regularity and occurring blow condition in finite time study for nonlinear partial differential equations (Nonlinear Evolution Equations) related to time variable t has important practical significance [Chaohao Gu(1993), Daqian Li(1989)], which are studied in [Pao C.V.(1980)-Mizoguchi N.(1997)]. However, due to the wide equation range and various characteristics of nonlinear equations, the existed results are obtained for finite solutions of specific physical problems, such as papers [Zhongwei Zha(2003)-Zhongwei Zha(2010)] are special cases of (I).

For convenience, denote $f'_{\nabla u} = (\frac{\partial f}{\partial q_1}, \frac{\partial f}{\partial q_2}, \dots, \frac{\partial f}{\partial q_n})$, where $q_1 = \frac{\partial u}{\partial x_1}, q_2 = \frac{\partial u}{\partial x_2}, \dots, q_n = \frac{\partial u}{\partial x_n}$. $\frac{\partial(\nabla u)}{\partial t} = (\frac{\partial q_1}{\partial t}, \frac{\partial q_2}{\partial t}, \dots, \frac{\partial q_n}{\partial t}) = q'_t$, and $f'_q \cdot q'_t = \frac{\partial f}{\partial q_1} \frac{\partial q_1}{\partial t} + \frac{\partial f}{\partial q_2} \frac{\partial q_2}{\partial t} + \dots + \frac{\partial f}{\partial q_n} \frac{\partial q_n}{\partial t}$ denotes inner product.

For initial-boundary value problem (I), the followings are supposed:

(i) When $x \in \bar{D}$, $\varphi(x) \geq 0$.

(ii) $f(x, t, p, q) \in C(R) \times C^1[0, T) \times C^1(R) \times C^1(R)$, f and f'_t are non-negative and the inner product $f'_q \cdot q'_t \geq 0$. Denote $H(u) = \int_0^{u(x,t)} h(\tau) d\tau$, $G(p) = \int_0^p H(\zeta) d\zeta$, then there exists $\beta > 1$ such that $GHf'_p - (\beta H^2 + GH')f \geq 0$ holds when $p \geq 0$.

(iii) $g(x, t, p, q) \in C(R) \times C^1[0, T) \times C^1(R) \times C^1(R)$ when $p \geq 0$ and $g \leq 0$. And when $p < 0$ and $g > 0$, g'_t and the inner product $g'_q \cdot q'_t$ are non-negative, while $g'_p < 1$.

(iv) When $x \in \bar{D}$, $\nabla(h(\varphi)\nabla\varphi) + f(x, 0, \varphi(x), \nabla\varphi) \geq 0$.

2. Non-negative property of solutions for problem (I)

In order to discuss the blowing up solutions for finite problem (I), we first prove two properties of solutions, that is

Lemma 1. If the conditions (i)-(iii) satisfy, the solution of (I) $u(x, t) \geq 0((x, t) \in \bar{D} \times [0, T))$.

Proof. If it isn't, then suppose the solution $u(x, t)$ of (I) can be taken negative number, there exists a point $P_0(x_0, t_0) \in \bar{D} \times [0, T)$ such that $u(P_0)$ is the negative minimum. From the initial condition (1.3) and condition (i), $t_0 \neq 0$. Noting the expression of $H(x, t)$ in condition (ii), it is obviously that $H(x, t)$ takes the negative minimum in P_0 too.

If $P_0 \in D \times (0, T)$ and since $u(P_0)$ and $H(P_0)$ are the negative minimums, then $\Delta H(P_0) \geq 0$, $\frac{\partial H}{\partial x_i}|_{P_0} = 0 (i = 1, 2, \dots, n)$ and $\frac{\partial H}{\partial t}|_{P_0} \leq 0$. But for $h(u) > 0$ and from $\frac{\partial H}{\partial t}|_{P_0} = h(u)\frac{\partial u}{\partial t}|_{P_0}$, $\frac{\partial u}{\partial t}|_{P_0} \leq 0$ is gotten, therefore, $[\frac{\partial u}{\partial t} - \nabla(h(u))\nabla u]|_{P_0} = [\frac{\partial u}{\partial t} - \Delta H]|_{P_0} \leq 0$, which is a contradiction that f is nonnegative in condition (ii).

If $P_0 \in \partial D \times (0, T)$, $\frac{\partial u}{\partial \nu}|_{P_0} \leq 0$. From the left side of the boundary condition (1.2) $[h(u)\frac{\partial u}{\partial \nu} + u]|_{P_0} < 0$, which is contrast

with $g > 0$ in condition (iii). So for any $(x, t) \in \bar{D} \times [0, T)$, $u(x, t) \geq 0$. The proof is proven.

Lemma 2. If conditions (i)-(iv) satisfy, then $\frac{\partial u}{\partial t} \geq 0((x, t) \in \bar{D} \times [0, T))$.

Proof. Express the definite solution problem (I) by the following form:

$$\begin{cases} \frac{\partial u}{\partial t} = h'(u)(\nabla u)^2 + h(u)\Delta u = f(x, t, u, \nabla u) & D \times (0, T) \\ h(u)\frac{\partial u}{\partial \nu} + u = g(x, t, u, \nabla u) & \partial D \times (0, T) \\ u(x, 0) = \varphi(x) & \bar{D} \end{cases} \quad (1)$$

Take the derivation for variable t , denoted by $\frac{\partial u}{\partial t} = V(x, t)$, then we obtain the initial boundary problem what $V(x, t)$ satisfies:

$$(II) \begin{cases} \frac{\partial V}{\partial t} = h(V)(\Delta V) + 2h'(u)(\nabla u) \cdot (\nabla V) + [h''(u)(\nabla u)^2 + h'(u)(\Delta u) + f'_u]V \\ \quad + f'_t + f'_{\nabla u} \cdot \frac{\partial(\nabla u)}{\partial t} & D \times (0, T) \\ h(u)\frac{\partial V}{\partial \nu} + [h(u)\frac{\partial u}{\partial \nu} + 1 - g'_u]V = g'_t + g'_{\nabla u} \cdot \frac{\partial(\nabla u)}{\partial t} & \partial D \times (0, T) \\ V(x, 0) = \nabla(h(\varphi)\nabla\varphi) + f(x, 0, \varphi, \nabla\varphi) & \bar{D} \end{cases}$$

Similar with Lemma 1, we can prove that for any $(x, t) \in \bar{D} \times [0, T)$, the solution of mixed problem (II) $V(x, t) \geq 0$, therefore,

$$\frac{\partial u}{\partial t} \geq 0((x, t) \in \bar{D} \times [0, T))$$

The proof is proven.

In addition, we also need a conclusion in paper [Zhongwei Zha(1992)], that is:

Lemma 3. Suppose that $E(t)$ is a doubly differentiable function and $E(0) > 0$, $E'(0) < 0$, $E'' \leq 0$, then there exists T_0 such that $E(T_0) = 0$, where $0 < T_0 < -\frac{E(0)}{E'(0)}$.

3. Blow-up of solution in problem (I)

When the hypothesis conditions (i)-(iv) satisfy, the mixed problem (I) doesn't exist global smooth solution, i.e. the solution must appear blow-up within definite time, which is the following theorem:

Theorem If the conditions (i)-(iv) hold and $u(x, t)$ is the smooth solution of (I), then there exists time $T_0(0 < T_0 < +\infty)$, such that $\lim_{t \rightarrow T_0^-} \sup_{x \in \bar{D}} u(x, t) = \infty$.

Proof. Since $H(u) = \int_0^{u(x,t)} h(\tau)d\tau$, $H'(u) = h(u) > 0$. Noting that the definite problem (I) can be expressed the following:

$$(III) \begin{cases} \frac{\partial u}{\partial t} = \Delta H(u) + f(x, t, u, \nabla u) & D \times (0, T) & (3.1) \\ h(u)\frac{\partial u}{\partial \nu} + u = g(x, t, u, \nabla u) & \partial D \times (0, T) & (3.2) \\ u(x, 0) = \varphi(x) & \bar{D} & (3.3) \end{cases} \quad (2)$$

So we only need to prove the solutions of mixed problem (III) appear blow-up within definite time under the assumed condition.

Denote

$$G(u) = \int_0^u H(\zeta)d\zeta \quad (3)$$

$$F(t) = \int_D \frac{1}{1+\beta} [G(u)]^{1+\beta} dx \quad (4)$$

where β is a constant in condition (ii), then $G(u) > 0$, $F(t) < 0$ and

$$F(0) = \int_D \frac{1}{1+\beta} [G(\varphi)]^{1+\beta} dx > 0 \quad (5)$$

$$F'(t) = \int_D G^\beta H(u) \frac{\partial u}{\partial t} dx \quad (6)$$

$$F'(0) = \int_D G^\beta H(u) \frac{\partial u}{\partial t} \Big|_{t=0} > 0 \quad (7)$$

Take equation (3.1) into equation (6), we have

$$F'(t) = \int_D G^\beta H \Delta H dx + \int_D G^\beta H f dx \quad (8)$$

Apply integration by parts to the first integral of the right side in equation (8) and noting the boundary condition (3.2) in (III), we have

$$\begin{aligned} F'(t) &= \int_{\partial D} G^\beta H [h(u) \frac{\partial u}{\partial \nu}] ds - \int_D \nabla(G^\beta H) \cdot \nabla H dx + \int_D G^\beta H f dx \\ &= \int_{\partial D} G^\beta H (g - u) ds - \int_D \nabla(G^\beta H) \cdot \nabla H dx + \int_D G^\beta H f dx \end{aligned} \quad (9)$$

where ds is the area element of ∂D . Derivative the two sides of (9) about t , we have:

$$\begin{aligned} F''(t) &= \int_{\partial D} (\beta G^{\beta-1} H^2 + G^\beta H') (g - u) \frac{\partial u}{\partial t} ds + \int_{\partial D} G^\beta H [g'_t + g'_u \frac{\partial u}{\partial t} \\ &\quad + g'_{\nabla u} \frac{\partial(\nabla u)}{\partial t} - \frac{\partial u}{\partial t}] ds + \int_D (\beta G^{\beta-1} H^2 + G^\beta H') f \frac{\partial u}{\partial t} dx \\ &\quad - \int_D \{ [\beta(\beta - 1) G^{\beta-2} H^3 H' (\nabla u)^2 + 2\beta G^{\beta-1} G (\nabla H)^2] \frac{\partial u}{\partial t} \\ &\quad + [\beta G^{\beta-1} H^2 \nabla(\frac{\partial u}{\partial t}) + \beta G^{\beta-1} H \frac{\partial H}{\partial t} \nabla u + 2G^\beta \nabla(\frac{\partial H}{\partial t})] \cdot \nabla H \\ &\quad + \beta G^{\beta-1} H^2 (\nabla u) \cdot \nabla(\frac{\partial u}{\partial t}) \} dx + \int_D G^\beta H [f'_t + f'_u \frac{\partial u}{\partial t} + f'_{\nabla u} \cdot \frac{\partial(\nabla u)}{\partial t}] dx. \end{aligned} \quad (10)$$

On the other side, if derivating the two sides of (6) about t directly, we have:

$$F''(t) = \int_{\partial D} (\beta G^{\beta-1} H^2 + G^\beta H') (\frac{\partial u}{\partial t})^2 dx + \int_D G^\beta H \frac{\partial^2 u}{\partial t^2} dx. \quad (11)$$

We derivative the equation (3.1) about t , take the results into (11) and apply integration by parts, then

$$\begin{aligned} F''(t) &= \int_D (\beta G^{\beta-1} H^2 + G^\beta H') (\frac{\partial u}{\partial t})^2 dx + \int_D G^\beta H [f'_t + f'_u \frac{\partial u}{\partial t} + f'_{\nabla u} \frac{\partial(\nabla u)}{\partial t}] dx \\ &\quad + \int_{\partial D} G^\beta H [g'_t + g'_u \frac{\partial u}{\partial t} + g'_{\nabla u} \cdot \frac{\partial(\nabla u)}{\partial t} - \frac{\partial u}{\partial t}] ds \\ &\quad - \int_D (\beta G^{\beta-1} H^2 \nabla u + G^\beta \nabla H) \cdot \nabla(\frac{\partial H}{\partial t}) dx \end{aligned} \quad (12)$$

Double(12) and then use it to subtract(10):

$$\begin{aligned} F''(t) &= 2\beta \int_D G^{\beta-1} H^2 (\frac{\partial u}{\partial t})^2 dx + 2 \int_D G^\beta H' (\frac{\partial u}{\partial t})^2 dx + \\ &\quad \int_D G^\beta H [f'_t + f'_{\nabla u} \cdot \frac{\partial(\nabla u)}{\partial t}] dx + \int_D G^{\beta-1} [GH f'_u - (\beta H^2 + GH') f] \frac{\partial u}{\partial t} dx \\ &\quad \int_{\partial D} G^\beta H [g'_t + g'_{\nabla u} \cdot \frac{\partial(\nabla u)}{\partial t}] ds + \int_{\partial D} G^\beta H (1 - g'_u) \frac{\partial u}{\partial t} ds \\ &\quad + \int_{\partial D} G^{\beta-1} (\beta H^2 + GH') (u - g) ds + \int_D \beta G^{\beta-1} H [H (\nabla H) \cdot \nabla(\frac{\partial u}{\partial t}) \\ &\quad + (\nabla H) \cdot (\nabla u) \frac{\partial H}{\partial t} - H (\nabla u) \cdot \nabla(\frac{\partial H}{\partial t})] dx. \end{aligned} \quad (13)$$

Noting the last integral in the right side of (13) can be expressed :

$$\int_D \beta G^{\beta-1} H [H (\nabla H) \cdot \nabla(\frac{\partial u}{\partial t}) + H'^2 (\nabla u)^2 \frac{\partial u}{\partial t} - HH'' (\nabla u)^2 \frac{\partial u}{\partial t}$$

$$-H(\nabla H) \cdot \nabla \left(\frac{\partial u}{\partial t} \right) dx = \int_D \beta G^{\beta-1} H(H'^2 - HH'') (\nabla u)^2 \frac{\partial u}{\partial t} dx \geq 0.$$

In fact, since $h(u)$ is monotone decreasing positive continuous function about u , and from the expression of $h(u)$ and $H(u) > 0$, $H'(u) = h(u) > 0$, while $H''(u) = h'(u) < 0$, therefore, in the above equation $H'^2 - HH'' > 0$.

Due to the conditions (ii)-(iii) and from the conclusions of Lemma 1 and Lemma 2, each integral in the right side of (13) is non-negative, so

$$F''(t) \geq 2\beta \int_D G^{\beta-1} H^2 \left(\frac{\partial u}{\partial t} \right)^2 dx. \quad (14)$$

The two sides of (14) both are multiplied by $F(t)$ and apply Schwarz inequality, we have:

$$\begin{aligned} F''(t)F(t) &\geq \frac{2\beta}{\beta+1} \int_D G^{\beta-1} \left(H \frac{\partial u}{\partial t} \right)^2 dx \cdot \int_D G^{\beta+1} dx \\ &\geq \frac{2\beta}{\beta+1} \left(\int_D G^{\beta} H \frac{\partial u}{\partial t} dx \right)^2 = \frac{2\beta}{\beta+1} [F'(t)]^2 \end{aligned} \quad (15)$$

Let $E(t) = [F(t)]^{-\frac{\beta-1}{\beta+1}} = [F(t)]^{-\beta_0}$, where $\beta_0 = \frac{\beta-1}{\beta+1} > 0$. From (5) and (7),

$$E(0) = [F(0)]^{-\beta_0} > 0, E'(0) = -\beta_0 [F(0)]^{-(\beta_0+1)} F'(0) < 0.$$

In addition, from inequality (15), $E''(t) \leq 0$. Based on Lemma 3, there exists $T_0 (0 < T_0 < -\frac{E(0)}{E'(0)})$ such that $E(T_0) = 0$, so $\lim_{t \rightarrow T_0^-} E(t) = \lim_{t \rightarrow T_0^-} [F(t)]^{-\beta_0} = 0$, i.e. $\lim_{t \rightarrow T_0^-} F(t) = \infty$. Noting the form of $F(t)$,

$$\lim_{t \rightarrow T_0^-} G(u(x, t)) = \infty \quad (16)$$

Therefore $\lim_{t \rightarrow T_0^-} \sup_{x \in \bar{D}} u(x, t) = \infty$.

In fact, if $\sup_{x \in \bar{D}} u(x, t) \leq M(\text{constant})$, then

$$G(u) = \int_0^u H(\zeta) d\zeta \leq \int_0^M H(\zeta) d\zeta \leq \int_0^M \left[\int_0^u h(\tau) d\tau \right] d\zeta \leq \int_0^M \left[\int_0^M h(\tau) d\tau \right] d\zeta \neq \infty,$$

which is a contradiction with (16).

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