Conditionally Permutable Subgroup and p-supersolubility of Finite Groups

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Abstract

In this paper, we research p-supersolubility of finite groups. We determine the structure of some groups by using the conditionally permutable subgroups. We obtain some sufficient or necessary and sufficient conditions of a finite group is p- supersolvabe.

Keywords: Conditionally permutable, Maximal subgroup, p-supersolvable

1. Introduction

All groups considered in this paper are finite.The product HT of subgroups H and T is still a subgroup if and only if $HT = TH$. Thus the permutability plays an important in the study of the structure of finite groups. For example, Ore O., 1939, P.431-460, proved that every permutable subgroups H of a group G is subnormal in G. However, for two subgroups H and T of a group G, maybe they are not permutable but there exists an element $x \in G$ such that $HT^x = T^xH$. Guo W.B., Shum K.P., Skiba A.N., 2004, P.128-133, 2005, P.493-510, introduced the concepts of conditionally permutable subgroups and completely conditionally permutable subgroups. With these concepts, some new elegant results, Hu Y.S., Guo X.Y., 2007, P.28-32, Hu Y.S., Wang L.L., 2007, P.1-4, Li C.W., Yu Q., 2007, P.8-10, Zhang X.M., Liu X., 2010, P.51-59, have been obtained. In this paper, we determine the structures of some groups by using the conditionally permutable subgroups. Some new criterions of p-supersolubility of some finite groups will be given and some known results are generalized.

We use "c-permutable" to denote "conditionally permutable". As usual, we denote a maximal subgroup M of G by $M < \cdot G$ and a minimal normal subgroup A of G by $A \cdot \lhd G$. All unexplained notions and terminologies are standard, see Refs. Guo W.B., 2000 and Xu M.Y., 1987.

2. Preliminaries

We cite here some known results which are useful in the later.

Definitions 2.1 (Guo W.B., Shum K.P., Skiba A.N.,2005, P.493-510) *Let G be a group. Suppose $H \le G$ and $T \le G$. Then*

(1)H is called c-permutable with T in G if there exists some $x \in G$ such that $HT^x = T^xH$.

(2) H is called c-permutable in G if for every subgroup K of G, there exists some $x \in G$ such that $HK^x = K^xH$.

Lemma 2.1 (Guo W.B., 2000; Theorem 1.9.4) *The following conditions are equivalent:*

(1) G is p- supersolvable;

(2) G is p- solvable and the index of every maximal subgroup of G either equal to p or be p'-number.

Lemma 2.2 (Guo W.B., 2000; Theorem 1.7.7) *Let G be π'-solvable group. Then there at least exists one π'-Hall subgroup $G_{\pi'}$ of G, and for every π'-subgroup A of G, there exists some $x \in G$ such that $A^x \subseteq G_{\pi'}$. In particular, any two π'-Hall subgroups of G conjugated in G.*

Lemma 2.3 (Guo W.B., 2000; Theorem 1.7.6) *Let G be π- solvable group. Then there at least exists one $\pi-$ Hall subgroup G_π of G, and for every π-subgroup A of G , there exists some $x \in G$ such that $A^x \subseteq G_\pi$. In particular, any two $\pi-$ Hall subgroups of G conjugated in G.*

Lemma 2.4 (Guo W.B., Shum K.P., Skiba A.N., 2004, P.128-133) *Let G be a group. Suppose that $N \lhd G$ and $H \le G$. Then*

(1) If $N \le T \le G$ and H is c-permutable with T in G, then HN/N is c-permutable with T/N in G/N;

(2) Assume that $N \le H$ and $T \le G$, if H/N is c-permutable with TN/N in G/N, then H is c-permutable with T in G;

(3) Assume that $T \le G$ and H is c-permutable with T in G, then H^x is c-permutable with T^x in G for any $x \in G$.

Lemma 2.5 (Chen S.M., Chen G.Y., Zhang L.C.,2002, P.836-840; Theorem 1.8) *Let G be p-solvable and outer p-supersolvable group. Then $G = AN$ and $A \cap N = 1$, where $A < \cdot G$, $N \cdot \lhd G$ and $|N| = p^{\alpha}, \alpha > 1$.*

Lemma 2.6 (Qian G.H., Zhu P.T.,1999,P.15-17; Lemma 2) *Let G be a group, if there exist subgroups M and K of G such that $G = MK$, then $G = M^x K^y$ for any $x, y \in G$.*

Lemma 2.7 (Ballester-Bolinches A.,Cssey J. and Pedraza-Aguilera M.C.2001, P.3145-3152;Theorem 2) *If $G = AB$ is the product of two supersoluble subgroups A and B of G such that A permutes with every maximal subgroup of B and B permutes with every maximal subgroup of A,then G is solvable group.*

3. Main Result

Theorem 1. *Let G be a p-solvable group. Then the following conditions are equivalent:*

(i) G is p-supersolvable group;

(ii) Every maximal subgroup of G with the index of p^{α} is c-permutable in G, where α is an integer;

(iii) Every maximal subgroup of G with the index of p^{α} is c-permutable with every maximal subgroup of sylow p-subgroup of G in G;

(iv)Every maximal subgroup of G is c-permutable with every maximal subgroup of sylow p-subgroup of G in G;

Proof: *(i) $\Longrightarrow$ (ii)*

Let G be p-supersolvable group and M is a maximal subgroup of G, where $|G : M| = p^{\beta}$. It is clear that $|G : M| = p$ by Lemma 2.1. For any subgroup K of G, let $K = K_p K_{p'}$ and $M = M_p M_{p'} = M_p G_{p'}$, $K_p \in Sylp(K)$, $M_p \in Sylp(M)$, $K_{p'} \in Hall_{p'}(K)$, $M_{p'} \in Hall_{p'}(M)$ and $G_{p'} \in Hall_{p'}(G)$. By Lemma 2.2, there exists some $x \in G$ such that $K^x_{p'} \subseteq G_{p'} \subseteq M$. If $K^x_p \subseteq M$, then $MK^x = M = K^x M$. If $K^x_p \nsubseteq M$, then

$G = K^x_p M = K^x M = MK^x$.

All imply that M is c-permutable in G.

(ii) $\Longrightarrow$ (iii)

It is concluded from the definition of c-permutable subgroups.

(iii) $\Longrightarrow$ (iv)

Let G be a p-solvable group and every maximal subgroup of G with the index of p^{α} is c-permutable with every maximal subgroup of sylow p-subgroup of G in G.

For any maximal subgroup M of G, then $|G : M| = p^{\beta}$ or $|G : M|$ is a p'-number, where β is an integer. Set $P \in Sylp(G)$ and $P_1 < \cdot P$. If $|G : M| = p^{\beta}$, then M is c-permutable with P_1 in G by the hypothesis. If $|G : M|$ is a p'-number, then $M = M_p M_{p'} = G_p M_{p'}$, where $M_p \in Sylp(M)$, $G_p \in Sylp(G)$ and $M_{p'} \in Hall_{p'}(M)$. By Lemma 2.3, there exists some $y \in \langle M, P_1 \rangle = G$ such that $P^y_1 \subseteq G_p \subseteq M$. Hence $MP^y_1 = M = P^y_1 M$. All imply that M is c-permutable with P_1 in G.

(iv) $\Longrightarrow$ (i)

Let G be a p-solvable group and every maximal subgroup of G is c-permutable with every maximal subgroup of Sylow p-subgroup of G in G.

Assume that the proposition (i) is false and let G be a counterexample of a minimal order. Let $H \cdot \lhd G$, $M/H < \cdot G/H$, $P/H \in Sylp(G/H)$ and $P_1/H < \cdot P/H$. If $P_0 \in Sylp(P)$ and $P_2 \in Sylp(P_1)$, then $M < \cdot G$, $P_0 \in Sylp(G)$ and $P_2 < \cdot P_0$. Hence by the hypothesis M is c-permutable with P_2 in G. Clearly $P_2 H/H = P_1/H$ and $P_0 H/H = P/H$. By Lemma 2.4, P_1/H is c-permutable with M/H in G/H. This shows that the hypothesis holds on G/H.

Since G is p-solvable and outer p-supersolvable group, by Lemma 2.5, $G = AN$ and $A \cap N = 1$, where $A < \cdot G$, $N \cdot \lhd G$ and $|N| = p^{\alpha}, \alpha > 1$.

Let $N \in Sylp(G)$ and $N_1 < \cdot N$. By the hypothesis, A is c-permutable with N_1 in G. Hence By Lemma 2.4,there exists some $z \in \langle A, N_1 \rangle$ such that $D = N_1 A^z = A^z N_1$. If $D = G$, then $|G : A^z| = |N_1| = |G : A| = |N|$, this is a contradiction since $N_1 < \cdot N$. So $D \neq G$, and $N_1 A^z = A^z$ since $A^z < \cdot G$. Then $N_1^{z^{-1}} \subseteq A \cap N = 1$ and $|N_1| = 1, |N| = p$, this is a contradiction. This induces that N is not a Sylow p-subgroup of G.

Let $A_p \in Sylp(A)$, by Lemma 2.3 there exists some subgroup $P \in Sylp(G)$ such that $A_p \subset P$. And there exists some subgroup P_1 of P such that $P_1 < \cdot P$ and $A_p \subseteq P_1$. By the hypothesis, A is c-permutable with P_1 in G. So by Lemma 2.3,there exists some $w \in \langle A, P_1 \rangle$ such that $B = P_1 A^w = A^w P_1$. Since $G = AN$, then there exists some $a \in A$ and $n \in N \subseteq P$ such that $w = an$. Hence $B = P_1 A^n$ and $A^n_p \subseteq P^n_1 = P_1$ since $P_1 \lhd P$. If $B = G$, then

$P = P \cap P_1 A^n = P_1(P \cap A^n) = P_1 A^n_p = P_1$,

this is a contradiction. This implies that $B \neq G$. Thus $A^n < \cdot G$ and $B = A^n, P_1 \leq A^n$. So $|G : A^n| = |G : A| = p = |N|$. This contradiction completes the proof.

Theorem 2. *Let G be a p-solvable group, $G = AB$ and $A \in Sylp(G)$, $B \in Hall_{p'}(G)$. If B is c-permutable in G, then G is p-supersolvable.*

Proof: Assume that the assertion is false and G be a counterexample of a minimal order. Let $H \cdot \lhd G$. Then G/H is p-solvable group and $G/H = AH/H \cdot BH/H$ which $AH/H \in Sylp(G/H)$ and $BH/H \in Hall_{p'}(G/H)$. By the hypothesis and Lemma 2.4, BH/H is c-permutable in G/H. This shows that the hypothesis holds on G/H.

Since G is p-solvable and outer p-supersolvable group, $G = MN$ and $M \cap N = 1$ by Lemma 2.5, where $M < \cdot G$, $N \cdot \lhd G$ and $|N| = p^{\alpha}, \alpha > 1$. Hence $N \leq A$ and $A = A \cap G = A \cap NM = N(A \cap M)$. If $A \cap M = A$, then $N \leq A \subseteq M$, this is a contradiction. So $A \cap M \neq A$ and there exists some subgroup T of G such that $T < \cdot A$ and $A \cap M \subseteq T$. By the hypothesis, B is c-permutable in G. So there exists some $x \in G$ such that $BT^x = T^x B$. Hence

$$G = AB = N(A \cap M)B = (NT)B = (NT)^x B = NBT^x.$$

This implies that either $BT^x = G$ or BT^x is a supplement of N in G. If $BT^x = G$, then $G = BT^x = BT$ by Lemma 2.6 and $A = A \cap BT = T(A \cap B) = T$. If $BT^x \cap N = 1$, then $T^x \cap N = 1$ and $N = |A : T| = p$ since $A = NT$. This contradiction completes the proof.

Theorem 3. *Let G be p-solvable group. $G = AB$ which A and B are p-supersolvable groups and $(|A|, |B|) = 1$. If A is c-permutable with every maximal subgroup of B in G, and B is c-permutable with every maximal subgroup of A in G, then G is p-supersolvable group.*

Proof: Suppose that the theorem is false and let G be a conterexample of minimalorder.

Let $H \cdot \lhd G$. Obviously, G/H is a p-solvable group and $G/H = AH/H \cdot BH/H$, where AH/H and BH/H are p-supersolvable groups. Since $(|A|, |B|) = 1$,

$$(|AH/H|, |BH/H|) = (|A|/|A \cap H|, |B|/|B \cap H|) = 1.$$

Let $T/H < \cdot AH/H$. Then there exists subgroup A_0 of G such that $A_0 < \cdot A$ and $A_0H/H = T/H$. By the hypothesis, B is c-permutable with A_0 in G. By Lemma 2.4, BH/H is c-permutable with $A_0H/H = T/H$ in G/H. Similarly, it can be proved that AH/H is c-permutable with every maximal subgroup of BH/H in G/H. Thus G/H satisfies the hypothesis and G/H is p-supersolvable.

Since G is a p-solvable and outer p-supersolvable group. By Lemma 2.5, $G = MN$ and $|N| = p^{\alpha}, \alpha > 1$,where $N \cdot \lhd G$ and $M < \cdot G$. Since $(|A|, |B|) = 1$, without loss of generality, we may assume that $N \subseteq A$ and $B \subseteq M$. Then $A = A \cap G = A \cap NM = N(A \cap M)$. If $A \cap M = A$, then $N \leq A \subseteq M$, this is a contradiction. Hence $A \cap M \neq A$ and there exists subgroup T of G such that $T < \cdot A$ and $A \cap M \subseteq T$. By the hypothesis, B is c-permutable with T in G and there exists some $x \in G$ such that $BT^x = T^x B$. Hence $G = AB = N(A \cap M)B = (NT)^x B = NBT^x$. Then $BT^x \cap N = 1$ since $N \cdot \lhd G$ and N is a abelian group. So $T^x \cap N = 1$ and $T \cap N = 1$. Then $|N| = |A : T| = p$ since $A = NT$, this is a contradiction. This implies that G is p-supersolvable group.

Corollary 4. *Let G be p-solvable group. $G = AB$ which A and B are p-nilpotent groups and $(|A|, |B|) = 1$. If A is c-permutable with every maximal subgroup of B in G and B is c-permutable with every maximal subgroup of A in G, then G is p-supersolvable group.*

Corollary 5 (Liu X., Li B.J., Yi X.L.,2008, P.79-86; Theorem 3.1) *A group G is supersoluble if and only if $G = AB$ is the product of two supersoluble subgroups A and B of coprime orders such that A permutes with every maximal subgroup of B and B permutes with every maximal subgroup of A.*

Proof: We only need to prove the sufficiency part as the necessity part is trivial. It is easy to see that a supersoluble group is also a p-supersolvable group and a permutable subgroup is also a c-permutable subgroup. Hence, we know that the corollary holds by our Theorem 3 and Lemma 2.7.

Corollary 6 (Liu X., Li B.J., Yi X.L.,2008, P.79-86; Corollary 3.3) *A group G is supersoluble if and only if $G = AB$ is the product of two supersoluble subgroups A and B of coprime orders such that every Sylow subgroup of B is permutable with every maximal subgroup of A and every Sylow subgroup of A is permutable with every maximal subgroup of B.*

Proof: Clearly, by our Theorem 3 and Lemma 2.7, the corollary holds.

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