Formulas for Solution of the Linear Differential Equations of the Second Order with the Variable Coefficients

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Abstract

In this paper we obtained the formula for the common solution of the linear differential equation of the second order with the variable coefficients in the more common case. We also obtained the formula for the solution of the Cauchy problem.

Keywords: The linear differential equation, The second order, The variable coefficients, The formula for the common solution, Cauchy problem

1. Introduction

We consider the equation

$$L[y] = y'' + p(t)y' + q(t)y = f(t), t \in I, \tag{1}$$

where $I = (t_1, t_2)$, $t_1 < t_2$, $p(t), q(t)$ and $f(t)$ are known continuous functions on I.

Many works are dedicated to the determination of the common solutions of the linear and nonlinear ordinary differential equations. But in common case any formulas for the decision of the linear differential equations haven't obtained. It is well known that if $p(t) = p_0 = const$, $q(t) = q_0 = const$, then depending on the sign of discriminant $D = p_0^2 - 4q_0$ the common solution of the equation (1) will be written by three formulas. In this theme the equation (1) is investigated in the more common cases.

2. Formulas for solution of the equations (1)

Depending on the correlation between $p(t)$ and $q(t)$ formulas for the determination of the common solution of this equation were obtained.

Theorem 1. *Let*

$$q(t) = (\varphi'(t))^2 e^{\int \alpha(t)\varphi'(t)dt} \left\{ b'(t)(\varphi'(t))^{-1} - b^2(t)e^{\int \alpha(t)\varphi'(t)dt} + \left[p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right](\varphi'(t))^{-1}b(t) - \right.$$
$$\left. - \alpha(t)b(t)\right\} + (\varphi'(t))^2 \left\{\alpha'(t)(\varphi'(t))^{-1} - \alpha^2(t) + \alpha(t)\left[p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right](\varphi'(t))^{-1}\right\}, \tag{2}$$

$$\beta(t) = e^{-\int \alpha(t)\varphi'(t)dt}\Big\{ - \alpha'(t)(\varphi'(t))^{-1} + \alpha^2(t) -$$
$$-\alpha(t)\left[p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right](\varphi'(t))^{-1} + q(t)(\varphi'(t))^{-2}\Big\}, t \in I, \tag{3}$$

where $\varphi'(t)$ and $\varphi''(t)$ are respectively first and second derivatives of the function $\varphi(t)$, $b'(t)$ and $\alpha'(t)$ are respectively the derivatives of the functions $b(t)$ and $\alpha(t)$. $b(t) \neq 0$ and $\varphi'(t) \neq 0$ for all $t \in I$. Then the common solution of the equation (1) will be written in the next form

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + y_3(t), t \in I \tag{4}$$

where c_1 and c_2 are arbitrary constants,

$$y_1(t) = exp\left\{-\int\left[\alpha(t) + b(t)e^{\int \alpha(t)\varphi'(t)dt}\right]\varphi'(t)dt\right\}, \tag{5}$$

$$y_2(t) = e^{-\int \alpha(t)\varphi'(t)dt}\left[-e^{-\int \frac{\beta(t)}{b(t)}\varphi'(t)dt} - u(t)\right], \tag{6}$$

$$u(t) = exp\left[-\int b(t)e^{\int \alpha(t)\varphi'(t)dt}\varphi'(t)dt\right]\int \frac{\beta(t)}{b(t)} exp\left\{\int\left[b(t)e^{\int \alpha(t)\varphi'(t)dt} - \frac{\beta(t)}{b(t)}\right]\varphi'(t)dt\right\}\varphi'(t)dt, \tag{7}$$

$$y_3(t) = e^{-\int \alpha(t)\varphi'(t)dt}\left[e^{-\int \frac{\beta(t)}{b(t)}\varphi'(t)dt}\int e^{\int \frac{\beta(t)}{b(t)}\varphi'(t)dt}\frac{f(t)dt}{b(t)\varphi'(t)} - v(t)\right], \tag{8}$$

$$\begin{aligned} v(t) \; = \; & exp\left[-\int b(t)e^{\int \alpha(t)\varphi'(t)dt}\varphi'(t)dt\right]\left\{\int \frac{f(t)}{b(t)\varphi'(t)} \times\right. \\ & \times exp\left[\int b(t)e^{\alpha(t)\varphi'(t)dt}\varphi'(t)dt\right]dt - \int exp\left\{\int\left[b(t)e^{\alpha(t)\varphi'(t)dt} - \frac{\beta(t)}{b(t)}\right]\varphi'(t)dt\right\} \times \\ & \left.\times \frac{\beta(t)}{b(t)}\left[\int e^{\int \frac{\beta(t)}{b(t)}\varphi'(t)dt}\frac{f(t)}{b(t)\varphi'(t)}dt\right]\varphi'(t)dt\right\}. \end{aligned} \tag{9}$$

Proof. We show that

$$L[y_1] = 0, L[y_2] = 0, L[y_3] = f(t), t \in I.$$

1) At first we proof $L[y_1] = 0$. In fact if we differentiate (5) we shall obtain

$$y_1'(t) = -\left[\alpha(t) + b(t)e^{\int \alpha(t)\varphi'(t)dt}\right]\varphi'(t)y_1(t), \tag{10}$$

$$\begin{aligned} y_1''(t) \; = \; & \left\{\left[\alpha(t) + b(t)e^{\int \alpha(t)\varphi'(t)dt}\right]^2(\varphi'(t))^2 - \right. \\ & - \left[\alpha'(t) + b'(t)e^{\int \alpha(t)\varphi'(t)dt} + b(t)\alpha(t)\varphi'(t)e^{\int \alpha(t)\varphi'(t)dt}\right]\varphi'(t) - \\ & \left. - \left[\alpha(t) + b(t)e^{\int \alpha(t)\varphi'(t)dt}\right]\varphi''(t)\right\}y_1(t). \end{aligned} \tag{11}$$

Then taking into account (10), (11) and (2) we have

$$\begin{aligned} L[y_1] \; = \; & \left\{\left[\alpha^2(t) + 2\alpha(t)b(t)e^{\int \alpha(t)\varphi'(t)dt} + b^2(t)e^{2\int \alpha(t)\varphi'(t)dt}\right](\varphi'(t))^2 - \alpha'(t)\varphi'(t) - \right. \\ & - b'(t)\varphi'(t)e^{\int \alpha(t)\varphi'(t)dt} - b(t)\alpha(t)(\varphi'(t))^2e^{\int \alpha(t)\varphi'(t)dt} - \alpha(t)\varphi''(t) - b(t)\varphi''(t) \times \\ & \left. \times e^{\int \alpha(t)\varphi'(t)dt}\right\}y_1(t) + p(t)\left[-\alpha(t)\varphi'(t) - b(t)\varphi'(t)e^{\int \alpha(t)\varphi'(t)dt}\right]y_1(t) + q(t)y_1(t) = \\ & y_1(t)\left\{q(t) - (\varphi'(t))^2e^{\int \alpha(t)\varphi'(t)dt}\left\{b'(t)(\varphi'(t))^{-1} - b(t)\alpha(t) - \right.\right. \\ & \left. - b^2(t)e^{\int \alpha(t)\varphi'(t)dt} + b(t)(\varphi'(t))^{-1}\left[p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right]\right\} - (\varphi'(t))^2 \times \\ & \left. \times \left\{\alpha'(t)(\varphi'(t))^{-1} - \alpha^2(t) + \alpha(t)(\varphi'(t))^{-1}\left[p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right]\right\}\right\} = 0, t \in I. \end{aligned}$$

Thus it is proved $L[y_1] = 0$.

2) We show $L(y_2) = 0$. If we differentiate (6) and (7) we shall have

$$y_2'(t) = -\alpha(t)\varphi'(t)y_2(t) + e^{-\int \alpha(t)\varphi'(t)dt}\left[\frac{\beta(t)}{b(t)}\varphi'(t)e^{-\int \frac{\beta(t)}{b(t)}\varphi'(t)dt} - u'(t)\right], \tag{12}$$

$$u'(t) = -b(t)e^{\int \alpha(t)\varphi'(t)dt}\varphi'(t)u(t) + \frac{\beta(t)}{b(t)}\varphi'(t)e^{-\int \frac{\beta(t)}{b(t)}\varphi'(t)dt}. \tag{13}$$

By substituting (13) in (12) we obtain

$$y_2'(t) = -\alpha(t)\varphi'(t)y_2(t) + b(t)\varphi'(t)u(t). \tag{14}$$

Differentiating (14) we obtain

$$y_2''(t) = [-\alpha'(t)\varphi'(t) - \alpha(t)\varphi''(t)]\, y_2(t) - \alpha(t)\varphi'(t)y_2'(t) + [b'(t)\varphi'(t) + b(t)\varphi''(t)]\, u(t) + b(t)\varphi'(t)u'(t).$$

Hence taking into account (13) and (14) we have

$$\begin{aligned} y_2''(t) \;=\; & \left[-\alpha'(t)\varphi'(t) - \alpha(t)\varphi''(t) + \alpha^2(t)(\varphi'(t))^2\right] y_2(t) + \\ & + \left[b'(t)\varphi'(t) + b(t)\varphi''(t) - \alpha(t)b(t)(\varphi'(t))^2 - b^2(t)(\varphi'(t))^2 e^{\int \alpha(t)\varphi'(t)dt}\right] u(t) + \\ & + \beta(t)(\varphi'(t))^2 e^{-\int \frac{\beta(t)}{b(t)}\varphi'(t)dt}. \end{aligned} \tag{15}$$

On the strength of (14) and (15) it follows that

$$\begin{aligned} L[y_2] \;=\; & y_2(t)\left\{q(t) - (\varphi'(t))^2\left\{\alpha'(t)(\varphi'(t))^{-1} - \alpha^2(t) + \alpha(t)(\varphi'(t))^{-1}\left[p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right]\right\}\right\} + \\ & + u(t)\left\{b'(t)\varphi'(t) + b(t)\varphi''(t) - \alpha(t)b(t)(\varphi'(t))^2 - b^2(t)(\varphi'(t))^2 e^{\int \alpha(t)\varphi'(t)dt} + \right. \\ & \left. + p(t)b(t)\varphi'(t)\right\} + \beta(t)(\varphi'(t))^2 e^{-\int \frac{\beta(t)}{b(t)}\varphi'(t)dt}. \end{aligned} \tag{16}$$

By substituting (2) in (3) we have

$$\beta(t) = b'(t)(\varphi'(t))^{-1} - b^2(t)e^{\int \alpha(t)\varphi'(t)dt} + \left[p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right](\varphi'(t))^{-1}b(t) - \alpha(t)b(t), t \in I. \tag{17}$$

Taking into account (17) and (6) from (16) we obtain

$$\begin{aligned} L[y_2] \;=\; & y_2(t)\left\{q(t) - (\varphi'(t))^2\left\{\alpha'(t)(\varphi'(t))^{-1} - \alpha^2(t) + \alpha(t)(\varphi'(t))^{-1}\left[p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right]\right\}\right\} - \\ & - \beta(t)(\varphi'(t))^2 e^{\int \alpha(t)\varphi'(t)dt} e^{-\int \alpha(t)\varphi'(t)dt}\left[-e^{-\int \frac{\beta(t)}{b(t)}\varphi'(t)dt} - u(t)\right] = \\ & = y_2(t)\left\{q(t) - (\varphi'(t))^2\left[\alpha'(t)(\varphi'(t))^{-1} - \alpha^2(t) + \alpha(t)(\varphi'(t))^{-1}\left[p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right] - \right.\right. \\ & - e^{\int \alpha(t)\varphi'(t)dt}(\varphi'(t))^2\left\{b'(t)(\varphi'(t))^{-1} - b^2(t)e^{\int \alpha(t)\varphi'(t)dt} + \right. \\ & \left.\left.\left. + \left[p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right](\varphi'(t))^{-1}b(t) - \alpha(t)b(t)\right\}\right\} = 0, t \in I. \end{aligned}$$

Here we use the formula (2).

3)We are going to proof $L[y_3] = f(t), t \in I$. Differentiating (8) and (9) we have

$$\begin{aligned} y_3'(t) \;=\; & -\alpha(t)\varphi'(t)y_3(t) + e^{-\int \alpha(t)\varphi'(t)dt}\left\{\frac{f(t)}{b(t)\varphi'(t)} - \frac{\beta(t)}{b(t)}\varphi'(t) \times \right. \\ & \left. \times e^{-\int \frac{\beta(t)}{b(t)}\varphi'(t)dt}\int e^{\int \frac{\beta(t)}{b(t)}\varphi'(t)dt}\frac{f(t)}{b(t)\varphi'(t)}dt - v'(t)\right\}, \end{aligned} \tag{18}$$

$$\begin{aligned} v'(t) \;=\; & -b(t)e^{\int \alpha(t)\varphi'(t)dt}\varphi'(t)v(t) + exp\left[-\int b(t)e^{\int \alpha(t)\varphi'(t)dt}\varphi'(t)dt\right] \times \\ & \times\left\{\frac{f(t)}{b(t)\varphi'(t)}exp\left[\int b(t)e^{\int \alpha(t)\varphi'(t)dt}\varphi'(t)dt\right] - exp\left\{\int\left[b(t)e^{\int \alpha(t)\varphi'(t)dt} - \right.\right.\right. \\ & \left.\left.\left. - \frac{\beta(t)}{b(t)}\right]\varphi'(t)dt\right\}\left[\int e^{\int \frac{\beta(t)}{b(t)}\varphi'(t)dt}\frac{f(t)}{b(t)\varphi'(t)}dt\right]\frac{\beta(t)\varphi'(t)}{b(t)}\right\}. \end{aligned}$$

From here by simplifying we obtain

$$\begin{aligned} v'(t) \;=\; & -b(t)e^{\int \alpha(t)\varphi'(t)dt}\varphi'(t)v(t) + \frac{f(t)}{b(t)\varphi'(t)} - exp\left[-\int \frac{\beta(t)}{b(t)}\varphi'(t)dt\right] \times \\ & \times\left[\int e^{\int \frac{\beta(t)}{b(t)}\varphi'(t)dt}\frac{f(t)}{b(t)\varphi'(t)}dt\right]\frac{\beta(t)\varphi'(t)}{b(t)}. \end{aligned} \tag{19}$$

By substituting (19) in (18) we have

$$y_3'(t) = -\alpha(t)\varphi'(t)y_3(t) + b(t)\varphi'(t)v(t), t \in I. \tag{20}$$

Differentiating (20) we obtain

$$\begin{aligned} y_3''(t) &= \left[-\alpha'(t)\varphi'(t) - \alpha(t)\varphi''(t)\right] y_3(t) - \alpha(t)\varphi'(t)y_3'(t) + \\ &\quad + \left[b'(t)\varphi'(t) + b(t)\varphi''(t)\right] v(t) + b(t)\varphi'(t)v'(t). \end{aligned} \tag{21}$$

Taking into account (19) and (20) from (21) we have

$$\begin{aligned} y_3''(t) &= \left[-\alpha'(t)\varphi'(t) - \alpha(t)\varphi''(t) + \alpha^2(t)(\varphi'(t))^2\right] y_3(t) + \left[b'(t)\varphi'(t) + b(t)\varphi''(t) - \right. \\ &\quad \left. - \alpha(t)b(t)(\varphi'(t))^2 - b^2(t)(\varphi'(t))^2 e^{\int \alpha(t)\varphi'(t)dt}\right] v(t) + f(t) - \\ &\quad - \beta(t)(\varphi'(t))^2 exp\left[-\int \frac{\beta(t)}{b(t)}\varphi'(t)dt\right]\left[\int e^{\int \frac{\beta(t)}{b(t)}\varphi'(t)dt} \frac{f(t)}{b(t)\varphi'(t)}dt\right]. \end{aligned} \tag{22}$$

By substituting (20) and (22) in (1) we have

$$\begin{aligned} L[y_3] &= y_3(t)\left[q(t) - p(t)\alpha(t)\varphi'(t) - \alpha'(t)\varphi'(t) - \alpha(t)\varphi''(t) + \alpha^2(t)(\varphi'(t))^2\right] + v(t) \times \\ &\quad \times \left[p(t)b(t)\varphi'(t) + b'(t)\varphi'(t) + b(t)\varphi''(t) - \alpha(t)b(t)(\varphi'(t))^2 - b^2(t)(\varphi'(t))^2 e^{\int \alpha(t)\varphi'(t)dt}\right] + \\ &\quad + f(t) - \beta(t)(\varphi'(t))^2 exp\left[-\int \frac{\beta(t)}{b(t)}\varphi'(t)dt\right]\left[\int e^{\int \frac{\beta(t)}{b(t)}\varphi'(t)dt} \frac{f(t)}{b(t)\varphi'(t)}dt\right], t \in I. \end{aligned} \tag{23}$$

Then taking into account (17) from (23) we obtain

$$\begin{aligned} L[y_3] &= y_3(t)\left[q(t) - p(t)\alpha(t)\varphi'(t) - \alpha'(t)\varphi'(t) - \alpha(t)\varphi''(t) + \alpha^2(t)(\varphi'(t))^2\right] + f(t) + \\ &\quad + \left\{b'(t)\varphi'(t) - b^2(t)(\varphi'(t))^2 e^{\int \alpha(t)\varphi'(t)dt} + b(t)\varphi'(t)\left[p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right] - \right. \\ &\quad \left. -\alpha(t)b(t)(\varphi'(t))^2\right\}\left\{v(t) - exp\left[-\int \frac{\beta(t)}{b(t)}\varphi'(t)dt\right]\left[\int e^{\int \frac{\beta(t)}{b(t)}\varphi'(t)dt} \frac{f(t)}{b(t)\varphi'(t)}dt\right]\right\}. \end{aligned} \tag{24}$$

From (8) we have

$$\begin{aligned} v(t) &- exp\left[-\int \frac{\beta(t)}{b(t)}\varphi'(t)dt\right]\left[\int e^{\int \frac{\beta(t)}{b(t)}\varphi'(t)dt} \frac{f(t)}{b(t)\varphi'(t)}dt\right] = \\ &= -y_3(t)e^{\int \alpha(t)\varphi'(t)dt}, t \in I. \end{aligned} \tag{25}$$

By substituting (25) in (24) and taking into account (2) we obtain

$$\begin{aligned} L[y_3] \quad = \quad & f(t) + y_3(t)\left\{q(t) - e^{\int \alpha(t)\varphi'(t)dt}(\varphi'(t))^2\left[b'(t)(\varphi'(t))^{-1} - b^2(t)e^{\int \alpha(t)\varphi'(t)dt} + \right.\right. \\ & +b(t)(\varphi'(t))^{-1}\left(p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right) - \alpha(t)b(t)\Bigg] - (\varphi'(t))^2\left[\alpha'(t)(\varphi'(t))^{-1} + \right. \\ & \left.\left. +\alpha(t)(\varphi'(t))^{-1}\left(p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right) - \alpha^2(t)\right]\right\} = f(t), t \in I. \end{aligned}$$

Then $L[c_1y_1(t) + c_2y_2(t) + y_3(t)] = c_1L[y_1] + c_2L[y_2] + L[y_3] = f(t)$, $t \in I$, where c_1, c_2 are unrestricted constants. Theorem 1 has been proved.

Example. We consider the equation (1) for $p(t) = t, q(t) = 1, f(t) = 1, t \in I = (0, 1)$.
From (2) and (3) we have

$$\alpha(t) = 0, \varphi(t) = t, b(t) = t, \beta(t) = 1, t \in (0, 1).$$

Then from (5), (6), (7) and (8) we obtain

$$y_1(t) = e^{-\frac{t^2}{2}}, \quad y_2(t) = -\frac{1}{t} - e^{-\frac{t^2}{2}} \int \frac{1}{t^2} e^{\frac{t^2}{2}} dt, \quad y_3(t) = 1, t \in (0, 1).$$

Theorem 2. *Let* $t_0 \in I = (t_1, t_2)$,

$$q(t) = (\varphi'(t))^2 e^{\int_{t_0}^t \alpha(s)\varphi'(s)ds} \Big\{ b'(t)(\varphi'(t))^{-1} - b^2(t) e^{\int_{t_0}^t \alpha(s)\varphi'(s)ds} + \left[p(t) + \frac{\varphi''(t)}{\varphi'(t)} \right] (\varphi'(t))^{-1} b(t) - \\ - \alpha(t)b(t) \Big\} + (\varphi'(t))^2 \left\{ \alpha'(t)(\varphi'(t))^{-1} - \alpha^2(t) + \alpha(t)(\varphi'(t))^{-1} \left[p(t) + \frac{\varphi''(t)}{\varphi'(t)} \right] \right\}, \quad (26)$$

$$\beta(t) = e^{-\int_{t_0}^t \alpha(t)\varphi'(t)dt} \Big\{ -\alpha'(t)(\varphi'(t))^{-1} + \alpha^2(t) - \\ -\alpha(t) \left[p(t) + \frac{\varphi''(t)}{\varphi'(t)} \right] (\varphi'(t))^{-1} + q(t)(\varphi'(t))^{-2} \Big\}, t \in I, \quad (27)$$

where $b(t) \neq 0$ and $\varphi'(t) \neq 0$ for all $t \in I$. Then solution of the equation (1) with initial condition

$$y(t_0) = m, y'(t_0) = n, m, n \in R \quad (28)$$

will be written in the next form

$$y(t) = -\frac{1}{b(t_0)} \left[m\alpha(t_0) + \frac{n}{\varphi'(t_0)} \right] y_1(t) - \frac{1}{b(t_0)} \left[m\alpha(t_0) + b(t_0) + \frac{n}{\varphi'(t_0)} \right] y_2(t) + y_3(t), \quad (29)$$

where

$$y_1(t) = exp \left\{ -\int_{t_0}^t \left[\alpha(s) + b(s) e^{\int_{t_0}^s \alpha(\tau)\varphi'(\tau)d\tau} \right] \varphi'(s)ds \right\}, \quad (30)$$

$$y_2(t) = e^{-\int_{t_0}^t \alpha(s)\varphi'(s)ds} \left[-e^{-\int_{t_0}^t \frac{\beta(s)}{b(s)}\varphi'(s)ds} - u_1(t) \right], \quad (31)$$

$$u_1(t) = exp \left[-\int_{t_0}^t b(s) e^{\int_{t_0}^s \alpha(\tau)\varphi'(\tau)d\tau} \varphi'(s)ds \right] \int_{t_0}^t \frac{\beta(s)}{b(s)} \times \\ \times exp \left\{ \int_{t_0}^s \left[b(\tau) e^{\int_{t_0}^\tau \alpha(v)\varphi'(v)dv} - \frac{\beta(\tau)}{b(\tau)} \right] \varphi'(\tau)d\tau \right\} \varphi'(s)ds, \quad (32)$$

$$y_3(t) = e^{-\int_{t_0}^t \alpha(s)\varphi'(s)ds} \left[e^{-\int_{t_0}^t \frac{\beta(s)}{b(s)}\varphi'(s)ds} \int_{t_0}^t e^{\int_{t_0}^s \frac{\beta(\tau)}{b(\tau)}\varphi'(\tau)d\tau} \frac{f(s)}{b(s)\varphi'(s)} ds - v_1(t) \right], \quad (33)$$

$$v_1(t) = exp \left[-\int_{t_0}^t b(s) e^{\int_{t_0}^s \alpha(\tau)\varphi'(\tau)d\tau} \varphi'(s)ds \right] \Big\{ \int_{t_0}^t \frac{f(s)}{b(s)\varphi'(s)} \, exp \left[\int_{t_0}^s b(\tau) e^{\int_{t_0}^\tau \alpha(v)\varphi'(v)dv} \varphi'(\tau)d\tau \right] ds \\ - \int_{t_0}^t exp \left\{ \int_{t_0}^s \left[b(\tau) e^{\int_{t_0}^\tau \alpha(v)\varphi'(v)dv} - \frac{\beta(\tau)}{b(\tau)} \right] \varphi'(\tau)d\tau \right\} \times \\ \times \frac{\beta(s)}{b(s)} \left[\int_{t_0}^s e^{\int_{t_0}^\tau \frac{\beta(v)}{b(v)}\varphi'(v)dv} \frac{f(\tau)}{b(\tau)\varphi'(\tau)} d\tau \right] \varphi'(s)ds \Big\}. \quad (34)$$

Proof. Differentiating (30), (31), (32), (33) and (34) we obtain

$$y_1'(t) = -\left[\alpha(t) + b(t) e^{\int_{t_0}^t \alpha(\tau)\varphi'(\tau)d\tau} \right] \varphi'(t) y_1(t), \quad (35)$$

$$y_2'(t) = -\alpha(t)\varphi'(t)y_2(t) + e^{-\int_{t_0}^t \alpha(s)\varphi'(s)ds} \left[\frac{\beta(t)}{b(t)} \varphi'(t) e^{-\int_{t_0}^t \frac{\beta(s)}{b(s)}\varphi'(s)ds} - u_1'(t) \right], \quad (36)$$

$$u_1'(t) = -b(t) e^{\int_{t_0}^t \alpha(s)\varphi'(s)ds} \varphi'(t) u_1(t) + \frac{\beta(t)}{b(t)} \varphi'(t) e^{-\int_{t_0}^t \frac{\beta(s)}{b(s)}\varphi'(s)ds}, \quad (37)$$

$$y_3'(t) = -\alpha(t)\varphi'(t)y_3(t) + e^{-\int_{t_0}^t \alpha(s)\varphi'(s)ds} \Big\{ -v_1'(t) - \frac{\beta(t)\varphi'(t)}{b(t)} \times \\ \times e^{-\int_{t_0}^t \frac{\beta(s)}{b(s)}\varphi'(s)ds} \int_{t_0}^t e^{\int_{t_0}^s \frac{\beta(\tau)}{b(\tau)}\varphi'(\tau)d\tau} \frac{f(s)}{b(s)\varphi'(s)} ds + \frac{f(t)}{b(t)\varphi'(t)} \Big\}, \quad (38)$$

$$
\begin{aligned}
v_1'(t) \;=\; & -b(t)e^{\int_{t_0}^{t}\alpha(s)\varphi'(s)ds}\varphi'(t)v_1(t)+\frac{f(t)}{b(t)\varphi'(t)}-\frac{\beta(t)}{b(t)}\varphi'(t)\times\\
& \times e^{-\int_{t_0}^{t}\frac{\beta(s)}{b(s)}\varphi'(s)ds}\int_{t_0}^{t}e^{\int_{t_0}^{\tau}\frac{\beta(v)}{b(v)}\varphi'(v)dv}\frac{f(\tau)}{b(\tau)\varphi'(\tau)}d\tau. \qquad (39)
\end{aligned}
$$

Taking into account (37) and (39) from (36) and (38) we have

$$y_2'(t)=-\alpha(t)\varphi'(t)y_2(t)+b(t)\varphi'(t)u_1(t), \qquad (40)$$

$$y_3'(t)=-\alpha(t)\varphi'(t)y_3(t)+b(t)\varphi'(t)v_1(t). \qquad (41)$$

Differentiating (35), (40) and (41) we obtain

$$
\begin{aligned}
y_1''(t) \;=\; & y_1(t)\Big\{-\alpha'(t)\varphi'(t)-b'(t)\varphi'(t)e^{\int_{t_0}^{t}\alpha(s)\varphi'(s)ds}-\alpha(t)b(t)(\varphi'(t))^2e^{\int_{t_0}^{t}\alpha(s)\varphi'(s)ds}-\\
& -\Big[\alpha(t)+b(t)e^{\int_{t_0}^{t}\alpha(s)\varphi'(s)ds}\Big]\varphi''(t)+\Big[\alpha(t)+b(t)e^{\int_{t_0}^{t}\alpha(s)\varphi'(s)ds}\Big]^2(\varphi'(t))^2\Big\}, \qquad (42)
\end{aligned}
$$

$$
\begin{aligned}
y_2''(t) \;=\; & \left[-\alpha'(t)\varphi'(t)-\alpha(t)\varphi''(t)\right]y_2(t)-\alpha(t)\varphi'(t)y_2'(t)+\\
& +\left[b'(t)\varphi'(t)+b(t)\varphi''(t)\right]u_1(t)+b(t)\varphi'(t)u_1'(t). \qquad (43)
\end{aligned}
$$

$$
\begin{aligned}
y_3''(t) \;=\; & \left[-\alpha'(t)\varphi'(t)-\alpha(t)\varphi''(t)\right]y_3(t)-\alpha(t)\varphi'(t)y_3'(t)+\\
& +\left[b'(t)\varphi'(t)+b(t)\varphi''(t)\right]v_1(t)+b(t)\varphi'(t)v_1'(t). \qquad (44)
\end{aligned}
$$

Taking into account formulas (40), (41) and (37) and (39) from (43) and (44) we have

$$
\begin{aligned}
y_2''(t) \;=\; & y_2(t)\Big[-\alpha'(t)\varphi'(t)-\alpha(t)\varphi''(t)+\alpha^2(t)(\varphi'(t))^2\Big]+u_1(t)\big[b'(t)\varphi'(t)+b(t)\varphi''(t)-\\
& -\alpha(t)b(t)(\varphi'(t))^2-b^2(t)(\varphi'(t))^2e^{\int_{t_0}^{t}\alpha(s)\varphi'(s)ds}\Big]+\\
& +\beta(t)(\varphi'(t))^2e^{-\int_{t_0}^{t}\frac{\beta(s)}{b(s)}\varphi'(s)ds}, \qquad (45)
\end{aligned}
$$

$$
\begin{aligned}
y_3''(t) \;=\; & y_3(t)\Big[-\alpha'(t)\varphi'(t)-\alpha(t)\varphi''(t)+\alpha^2(t)(\varphi'(t))^2\Big]+v_1(t)\big[b'(t)\varphi'(t)+b(t)\varphi''(t)-\\
& -\alpha(t)b(t)(\varphi'(t))^2-b^2(t)(\varphi'(t))^2e^{\int_{t_0}^{t}\alpha(s)\varphi'(s)ds}\Big]+f(t)-\\
& -\beta(t)(\varphi'(t))^2e^{-\int_{t_0}^{t}\frac{\beta(s)}{b(s)}\varphi'(s)ds}\int_{t_0}^{t}e^{\int_{t_0}^{s}\frac{\beta(\tau)}{b(\tau)}\varphi'(\tau)d\tau}\frac{f(s)}{b(s)\varphi'(s)}ds. \qquad (46)
\end{aligned}
$$

On the strength of formulas (35), (42) and (26) we have

$$
\begin{aligned}
L[y_1] \;=\; & y_1(t)\Big\{q(t)-(\varphi'(t))^2e^{\int_{t_0}^{t}\alpha(s)\varphi'(s)ds}\Big[p(t)b(t)(\varphi'(t))^{-1}+b'(t)(\varphi'(t))^{-1}-\\
& -\alpha(t)b(t)-b^2(t)e^{\int_{t_0}^{t}\alpha(s)\varphi'(s)ds}+\frac{b(t)\varphi''(t)}{(\varphi'(t))^2}\Big]-(\varphi'(t))^2\times\\
& \times\Big[p(t)\alpha(t)(\varphi'(t))^{-1}+\alpha'(t)(\varphi'(t))^{-1}+\alpha(t)\varphi''(t)(\varphi'(t))^{-2}-\alpha^2(t)\Big]\Big\}=0, t\in I.
\end{aligned}
$$

Taking into account formulas (40), (45) we obtain

$$
\begin{aligned}
L[y_2] \;=\; & y_2(t)\Big[q(t)-\alpha(t)p(t)\varphi'(t)-\alpha'(t)\varphi'(t)-\alpha(t)\varphi''(t)+\alpha^2(t)(\varphi'(t))^2\Big]+\\
& +u_1(t)\Big[b(t)p(t)\varphi'(t)+b'(t)\varphi'(t)+b(t)\varphi''(t)-\alpha(t)b(t)(\varphi'(t))^2-\\
& -b^2(t)(\varphi'(t))^2e^{\int_{t_0}^{t}\alpha(s)\varphi'(s)ds}\Big]+\beta(t)(\varphi'(t))^2e^{-\int_{t_0}^{t}\frac{\beta(s)}{b(s)}\varphi'(s)ds}, t\in I. \qquad (47)
\end{aligned}
$$

By substituting (26) in (27) we have

$$\beta(t) = b'(t)(\varphi'(t))^{-1} - b^2(t)e^{\int_{t_0}^t \alpha(s)\varphi'(s)ds} + \left(p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right)(\varphi'(t))^{-1}b(t) - \alpha(t)b(t), t \in I. \tag{48}$$

On the strength of (48) from (47) we obtain

$$\begin{aligned} L[y_2] =\ & y_2(t)\left\{q(t) - (\varphi'(t))^2\left[\alpha'(t)(\varphi'(t))^{-1} - \alpha^2(t) + \alpha(t)(\varphi'(t))^{-1}\left(p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right)\right]\right\} \\ & + (\varphi'(t))^2\left[b'(t)(\varphi'(t))^{-1} - b^2(t)e^{\int_{t_0}^t \alpha(s)\varphi'(s)ds} - \alpha(t)b(t) + \right. \\ & \left. + b(t)(\varphi'(t))^{-1}\left(p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right)\right]\left[u_1(t) + e^{-\int_{t_0}^t \frac{\beta(s)}{b(s)}\varphi'(s)ds}\right]. \end{aligned} \tag{49}$$

From (31) we have

$$u_1(t) + e^{-\int_{t_0}^t \frac{\beta(\tau)}{b(\tau)}\varphi'(\tau)d\tau} = -y_2(t)e^{\int_{t_0}^t \alpha(s)\varphi'(s)ds} \tag{50}$$

Taking into account (50) and (26) from (49) we obtain $L[y_2] = 0$ fot $t \in T$. Farther on the strength of formulas (41) and (46) we obtain

$$\begin{aligned} L[y_3] =\ & y_3(t)\left[-\alpha'(t)\varphi'(t) - \alpha(t)\varphi''(t) + \alpha^2(t)(\varphi'(t))^2 - p(t)\alpha(t)\varphi'(t) + q(t)\right] + \\ & + v_1(t)\left[b'(t)\varphi'(t) + b(t)\varphi''(t) - \alpha(t)b(t)(\varphi'(t))^2 - b^2(t)(\varphi'(t))^2 e^{\int_{t_0}^t \alpha(s)\varphi'(s)ds} + \right. \\ & \left. + p(t)b(t)\varphi'(t)\right] + f(t) - \beta(t)(\varphi'(t))^2 e^{-\int_{t_0}^t \frac{\beta(s)}{b(s)}\varphi'(s)ds}\int_{t_0}^t e^{\int_{t_0}^s \frac{\beta(\tau)}{b(\tau)}\varphi'(\tau)d\tau}\frac{f(s)}{b(s)\varphi'(s)}ds. \end{aligned} \tag{51}$$

On the strength of (48) from (51) we have

$$\begin{aligned} L[y_3] =\ & y_3(t)\left\{q(t) - (\varphi'(t))^2\left[\alpha'(t)(\varphi'(t))^{-1} - \alpha^2(t) + \alpha(t)(\varphi'(t))^{-1}\left(p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right)\right]\right\} + \\ & + (\varphi'(t))^2\left[b'(t)(\varphi'(t))^{-1} - b^2(t)e^{\int_{t_0}^t \alpha(s)\varphi'(s)ds} - \alpha(t)b(t) + b(t)(\varphi'(t))^{-1}\times \right. \\ & \left. \times\left(p(t) + \frac{\varphi''(t)}{\varphi'(t)}\right)\right]\left[v_1(t) - e^{-\int_{t_0}^t \frac{\beta(s)}{b(s)}\varphi'(s)ds}\int_{t_0}^t e^{\int_{t_0}^\tau \frac{\beta(\nu)}{b(\nu)}\varphi'(\nu)d\nu}\frac{f(\tau)}{b(\tau)\varphi'(\tau)}d\tau\right]. \end{aligned} \tag{52}$$

From (33) we obtain

$$v_1(t) - e^{-\int_{t_0}^t \frac{\beta(s)}{b(s)}\varphi'(s)ds}\int_{t_0}^t e^{\int_{t_0}^s \frac{\beta(\tau)}{b\tau}\varphi'(\tau)d\tau}\frac{f(s)}{b(s)\varphi'(s)}ds = -y_3(t)e^{\int_{t_0}^t \alpha(s)\varphi'(s)ds}. \tag{53}$$

By substituting (53) in (52) and taking into account formula (26) we have $L[y_3] = f(t)$, $t \in I$. Thus, the function determined by formula (29) is the solution of the equation (1). We show the realization of the initial condition (28) i.e. $y(t_0) = m$, $y'(t_0) = n$.

From (30), (31), (32), (33) and (34) we have

$$y_1(t_0) = 1, y_2(t_0) = -1, u(t_0) = 0, y_3(t_0) = 0, v_1(t_0) = 0. \tag{54}$$

Then from (29) we obtain

$$y(t_0) = -\frac{1}{b(t_0)}\left[m\alpha(t_0) + \frac{n}{\varphi'(t_0)}\right] + \frac{1}{b(t_0)}\left[m\alpha(t_0) + mb(t_0) + \frac{n}{\varphi'(t_0)}\right] = m$$

Differentiating (29) we have

$$y'(t) = -\frac{1}{b(t_0)}\left[m\alpha(t_0) + \frac{n}{\varphi'(t_0)}\right]y_1'(t) - \frac{1}{b(t_0)}\left[m\alpha(t_0) + mb(t_0) + \frac{n}{\varphi'(t_0)}\right]y_2'(t) + y_3'(t). \tag{55}$$

Taking into account (54) from (35), (40) and (41) we obtain

$$y_1'(t_0) = -[\alpha(t_0) + b(t_0)]\varphi'(t_0), \tag{56}$$

$$y_2'(t_0) = \alpha(t_0)\varphi'(t_0), y_3'(t_0) = 0. \tag{57}$$

Taking into account (56) and (57) from (55) we have

$$y'(t_0) = -\frac{1}{b(t_0)}\left[m\alpha(t_0) + \frac{n}{\varphi'(t_0}\right](-1)[\alpha(t_0) + b(t_0)]\varphi'(t_0) - \frac{1}{b(t_0)} \times$$
$$\times\left[m\alpha(t_0) + mb(t_0) + \frac{n}{\varphi'(t_0}\right]\alpha(t_0)\varphi'(t_0) = \frac{m\alpha(t_0)}{b(t_0)}[\alpha(t_0) + b(t_0)]\varphi'(t_0) +$$
$$+\frac{n}{b(t_0)}[\alpha(t_0) + b(t_0)] - \frac{m\alpha(t_0)}{b(t_0)}[\alpha(t_0) + b(t_0)]\varphi'(t_0) - \frac{n\alpha(t_0)}{b(t_0)} = n.$$

Theorem 2 has been proved.

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