On Multi-valued Weak Contraction Mappings

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Received: December 21, 2010 Accepted: January 7, 2011 doi:10.5539/jmr.v3n2p151

Abstract

In this paper, we study fixed point theorems for multi-valued weak contractions. We show that the Picard projection iteration converges to a fixed point, give a rate of convergence and generalize Collage theorem. This work includes results on multi-valued contraction mappings studied by (Kunze, H.E., La Torre, D. & Vrscay, E.R., 2007) and on multi-valued Zamfirescu mappings intrudeced by (Kaewkhao, An. & Neammanee, K., 2010).

Keywords: Fixed point, Collage theorem, Multi-valued mapping, Picard projection iteration, Weak contraction and Zamfirescu mapping

1. Introduction

Let (X, d) be a metric space and $T : X \to X$ a mapping. We say that $x \in X$ is a fixed point of T if $Tx = x$. T is said to be an a-contraction mapping if there exists a constant $a \in (0, 1)$, called a contraction factor, such that

$$d(Tx, Ty) \leq ad(x, y) \quad \text{for all } x, y \in X.$$

There are some well-known results on fixed point Theorems for contraction mappings. For instance,

Theorem 1.1 [Banach theorem](Banach, S., 1922) Let (X, d) be a complete metric space and $T : X \to X$ be an a-contraction mapping. Then T has a unique fixed point.

Theorem 1.2 [Collage Theorem](Barnsley, M.F., 1989) Let (X, d) be a complete metric space and $T : X \to X$ be an a-contraction mapping. Then for any $x \in X$,

$$d(x, x^*) \leq \frac{1}{1-a} d(x, Tx),$$

where x^* is the fixed point of T.

Theorem 1.3 [Continuty of Fixed Points](Centore, P. & Vrscay, E.R., 1994) Let (X, d) be a complete metric space and $T_1, T_2 : X \to X$ be contraction mappings with contraction factors a_1 and a_2 and fixed points x_1^* and x_2^*, respectively. Then

$$d(x_1^*, x_2^*) \leq \frac{1}{1 - \max\{a_1, a_2\}} d_\infty(T_1, T_2),$$

where $d_\infty(T_1, T_2) = \sup\limits_{x \in X} d(T_1 x, T_2 x)$.

In this paper, we concern with multi-valued mapping $T : X \to \mathcal{P}(X)$, i.e., a set-valued mapping from a space X to its power set $\mathcal{P}(X)$.

Let $\mathcal{P}(X)$ be the family of all nonempty subsets of X and let T be a set-valued mapping from X to $\mathcal{P}(X)$. An element $x \in X$ such that $x \in Tx$ is called a fixed point of T. We denote by F_T the set of all fixed points of T, i.e., $F_T = \{x \in X : x \in Tx\}$. Let (X, d) be a metric space and let $\mathcal{CB}(X)$ denote the family of all nonempty bounded closed subsets of X. For $x \in X$, $A, B \in \mathcal{CB}(X)$, we write

$d(x, A) = \inf\{d(x, a) : a \in A\}$, the distance between x and A,

$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$, the distance between A and B,

$h(A, B) = \sup\{d(a, B) : a \in A\}$ and

$H(A, B) = \max\{h(A, B), h(B, A)\}$, the Hausdorff-Pompeiu metric on $\mathcal{CB}(X)$ induced by d.

The study of fixed point theorems for multi-valued mapping has been initiated by (Markin, J.T., 1968, Nadler, S.B., 1969). Since then, extensive literatures have been developed. They consist of many theorems dealing with fixed points for multi-valued mappings, see (Mizoguchi, N. & Takahashi, W., 1989, Ciric, L.B., 2003). Most of these cases require the range of each point to be closed and bounded, in others words, to be compact, as we shall assume throughout this work.

Given a point $x \in X$ and a compact set $A \subset X$. We know that there exists $a^* \in A$ such that $d(x, a^*) = d(x, A)$. We call a^* the *projection* of x on the set A and denote by $a^* = \pi_x A$. Note that a^* is not unique but we choose one of them.

We say that $T : X \to P(X)$ is a compact multi-valued mapping if Tx is compact for all $x \in X$ and define the *projection* associated with T by $Px = \pi_x(Tx)$. For $x_0 \in X$, we define $x_{n+1} = Px_n$, $n = 0, 1, 2 \ldots$ and call the sequence $\{x_n\}_{n=0}^{\infty}$ the *Picard projection iteration sequence* of T.

In 2007, Kunze, La Torre and Vrscay extended Theorem 1.1-Theorem 1.3 to a compact multi-valued a-contraction mapping. A multi-valued mapping $T : X \to P(X)$ is called an a-contraction mapping if there exists a constant $a \in (0, 1)$ such that

$$H(Tx, Ty) \leq ad(x, y) \text{ for all } x, y \in X.$$

Theorem1.4-Theorem1.6 are results of (Kunze, H.E., La Torre, D. & Vrscay, E.R., 2007).

Theorem 1.4 (Kunze, H.E., La Torre, D. & Vrscay, E.R., 2007) Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a compact multi-valued a-contraction mapping. Then for any $x_0 \in X$, the Picard projection iteration $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in F_T$.

Theorem 1.5 (Kunze, H.E., La Torre, D. & Vrscay, E.R., 2007) Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a compact multi-valued a-contraction mapping. Then

$$d(x_0, F_T) \leq \frac{1}{1-a} d(x_0, Tx_0),$$

for all $x_0 \in X$.

Theorem 1.6 (Kunze, H.E., La Torre, D. & Vrscay, E.R., 2007) Let (X, d) be a complete metric space and $T_1, T_2 : X \to \mathcal{CB}(X)$ be compact multi-valued contraction mappings with contraction factors a_1 and a_2, respectively. If F_{T_1} and F_{T_2} are compact, then

$$H(F_{T_1}, F_{T_2}) \leq \frac{d_\infty(T_1, T_2)}{1 - \max\{a_1, a_2\}},$$

where $d_\infty(T_1, T_2) = \sup_{x \in X} H(T_1 x, T_2 x)$.

In this work we extend Theorem 1.4-Theorem 1.6 to the case of multi-valued weak contraction introduced by (Berinde, M. & Berinde, V., 2007). It is known that a contraction mapping is a weak contraction. We prove that the Picard projection iteration converges to a fixed point, give a rate of convergence and generalize Collage theorem in Section2. We show that the multi-valued Zamfirescu mappings introduced by (Kaewkhao, An. & Neammanee, K., 2010), is a multi-valued weak contraction in final section. This means that the outcome of this work contain some results of (Kaewkhao, An. & Neammanee, K., 2010).

2. Multi-valued weak contraction mappings

In this section, we extend Theorem 1.4-Theorem 1.6 to the case of weak contraction and hence the results of (Kunze, H.E., La Torre, D. & Vrscay, E.R., 2007) are consequences of our work.

Definition 2.1 Let (X, d) be a metric space and $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping. T is said to be the multi-valued weak contraction or multi-valued (θ, L)-weak contraction if and only if there exist constants $\theta \in (0, 1)$ and $L \geq 0$ such that

$$H(Tx, Ty) \leq \theta d(x, y) + Ld(x, Ty), \text{for all } x, y \in X.$$

Note that an a-contraction mapping is a $(a, 0)$-weak contraction.

We now state properties of metrics d and H on X and $\mathcal{CB}(X)$, respectively, used in our next results.

$$d(x, A) \leq d(x, y) + d(y, A) \text{ for all } x, y \in X \text{ and } A \in \mathcal{CB}(X), \tag{1}$$

$$d(x, A) \leq d(x, y) + d(y, B) + H(A, B) \text{ for all } x, y \in X \text{ and } A, B \in \mathcal{CB}(X), \tag{2}$$

$$d(a, A) \leq d(x, B) + H(A, B) \text{ for all } x \in X \text{ and } A, B \in \mathcal{CB}(X). \tag{3}$$

Theorem 2.2 Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a compact multi-valued (θ, L)-weak contraction. Then

(a) $F_T \neq \emptyset$;

(b) for any $x_0 \in X$, the Picard projection iteration sequence $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in F_T$;

(c) the following estimates

$$d(x_n, x^*) \leq \frac{\theta^n}{1-\theta} d(x_0, x_1), \qquad n = 0, 1, 2, \ldots$$

$$d(x_n, x^*) \leq \frac{\theta}{1-\theta} d(x_{n-1}, x_n), \qquad n = 1, 2, , 3, \ldots$$

hold.

Proof: Assume that T is a compact multi-valued (θ, L)-weak contraction and let x_0 be arbitrary and $\{x_n\}_{n=0}^{\infty}$ the Picard projection iteration.

For each $n \in \mathbb{N}$ we see that,

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, Px_n) \\ &= d(x_n, Tx_n) \\ &\leq H(Tx_{n-1}, Tx_n) \\ &\leq \theta d(x_{n-1}, x_n) + Ld(x_n, Tx_{n-1}) \\ &= \theta d(x_{n-1}, x_n), \end{aligned} \tag{4}$$

hence, by (4), we obtain

$$d(x_n, x_{n+1}) \leq \theta^n d(x_0, x_1) \tag{5}$$

and

$$d(x_{n+k-1}, x_{n+k}) \leq \theta^k d(x_{n-1}, x_n), \quad \text{for} \quad k \in \mathbb{N} \cup \{0\} \tag{6}$$

by mathematical induction.

Then, for any $n, p \in \mathbb{N}$, (5) implies

$$\begin{aligned} d(x_n, x_{n+p}) &\leq \sum_{k=n}^{n+p-1} d(x_k, x_{k+1}) \\ &\leq \sum_{k=n}^{n+p-1} \theta^k d(x_0, x_1) \\ &\leq \frac{\theta^n(1-\theta^p)}{1-\theta} d(x_0, x_1) \\ &\leq \frac{\theta^n}{1-\theta} d(x_0, x_1). \end{aligned} \tag{7}$$

Since $0 \leq \theta < 1$, we have $\theta^n \to 0$ (as $n \to \infty$), so $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. Therefore $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in X$ by completeness property of (X, d). Combining (1) and (3), we get

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*) \\ &\leq d(x^*, x_{n+1}) + H(Tx_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + \theta d(x_n, x^*) + Ld(x^*, Tx_n) \\ &\leq d(x^*, x_{n+1}) + \theta d(x_n, x^*) + L[d(x^*, x_{n+1}) + d(x_{n+1}, Tx_n)] \\ &= d(x^*, x_{n+1}) + \theta d(x_n, x^*) + Ld(x^*, x_{n+1}), \end{aligned}$$

for all $n = 0, 1, 2, \ldots$ Thus we have $d(x^*, Tx^*) = 0$ by letting $n \to \infty$. Since Tx^* is closed, $x^* \in Tx^*$. Hence (a) and (b) hold. To prove (c), we note by (6) that

$$d(x_n, x_{n+p}) \leq (\theta + \theta^2 + \cdots + \theta^p) d(x_{n-1}, x_n) \leq \frac{\theta}{1-\theta} d(x_{n-1}, x_n). \tag{8}$$

As $p \to \infty$, we get

$$d(x_n, x^*) \leq \frac{\theta}{1-\theta} d(x_{n-1}, x_n). \tag{9}$$

Inequalities (7) and (9) prove (c).

Corollary 2.3 [Generalized Collage Theorem] Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a compact multi-valued (θ, L)-weak contraction mapping. Then

$$d(x_0, F_T) \leq \frac{1}{1-\theta} d(x_0, Tx_0),$$

for all $x_0 \in X$.

Proof : Let $x_0 \in X$. By Theorem 2.2(b), there exists a point $x^* \in F_T$ such that the Picard projection iteration sequence $\{x_n\}_{n=0}^{\infty}$ converges to x^* and hence by Theorem 2.2(c) we have

$$d(x_0, F_T) \leq d(x_0, x^*) \leq \frac{1}{1-\theta} d(x_0, x_1) = \frac{1}{1-\theta} d(x_0, Tx_0).$$

Theorem 2.4 Let (X, d) be a complete metric space and $T_1, T_2 : X \to \mathcal{CB}(X)$ be compact multi-valued weak contractions with parameters (θ_1, L_1) and (θ_2, L_2), respectively. If F_{T_1} and F_{T_2} are closed and bounded, then

$$H(F_{T_1}, F_{T_2}) \leq \frac{d_\infty(T_1, T_2)}{1 - \max\{\theta_1, \theta_2\}},$$

where $d_\infty(T_1, T_2) = \sup\limits_{x \in X} H(T_1 x, T_2 x)$.

Proof: Let $x \in F_{T_1}$. By Corollary 2.3, we have

$$(1 - \theta_2) d(x, F_{T_2}) \leq d(x, T_2 x) \leq H(T_1 x, T_2 x) \leq d_\infty(T_1, T_2).$$

Take the supremum with respect to $x \in F_{T_1}$, we get

$$(1 - \theta_2) h(F_{T_1}, F_{T_2}) \leq d_\infty(T_1, T_2).$$

Next, upon interchanging F_{T_1} with F_{T_2} we obtain

$$H(F_{T_1}, F_{T_2}) \leq \frac{d_\infty(T_1, T_2)}{1 - \max\{\theta_1, \theta_2\}}.$$

Remark 2.5 Since an a-contraction is a $(\theta, 0)$-weak contraction, Theorem 1.4 -Theorem 1.6 can be deduced from Theorem 2.2- Theorem 2.4.

Corollary 2.6 Let (X, d) be a complete metric space and $T_n : X \to \mathcal{CB}(X)$ be a sequence of compact multi-valued weak contractions with weak contractivity constants θ_n such that $\sup_n \theta_n = \theta < 1$. Suppose that $T_n \to T$ with the metric d_∞ and T is a compact multi-valued weak contraction. Then $F_{T_n} \to F_T$ with the Hausdorff metric.

Proof: This is obvious by Theorem 2.4.

Corollary 2.6 implies the following corollary.

Corollary 2.7 (Kunze, H.E., La Torre, D. & Vrscay, E.R., 2007) Let (X, d) be a complete metric space and $T_n : X \to \mathcal{CB}(X)$ be a sequence of compact multi-valued contractions with contractivity constants a_n such that $\sup_n a_n = a < 1$. Suppose that $T_n \to T$ with the metric d_∞ and T is a compact multi-valued contraction. Then $F_{T_n} \to F_T$ with the Hausdorff metric.

3. Multi-valued Zamfirescu mappings

Let (X, d) be a metric space and $T : X \to \mathcal{CB}(X)$ be a multi-valued mapping. T is said to be a multi-valued Zamfirescu mapping if and only if there exist real numbers a, b and c satisfying $0 \leq a < 1$, $0 \leq b < \frac{1}{2}$ and $0 \leq c < \frac{1}{2}$, such that, for each $x, y \in X$ at least one of the following is true:

(z_1) $H(Tx, Ty) \leq a d(x, y)$

(z_2) $H(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)]$

(z_3) $H(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)]$.

Next Theorem is some result of (Kaewkhao, An. & Neammanee, K., 2010).

Theorem 3.1 (Kaewkhao, An. & Neammanee, K., 2010) Let (X, d) be a complete metric space and $T : X \to \mathcal{CB}(X)$ be a multi-valued Zamfirescu mapping with Tx is compact for all $x \in X$. Then

(1) $F_T \neq \emptyset$;

(2) for any $x_0 \in X$, the Picard projection iteration $\{x_n\}_{n=0}^{\infty}$ converges to some $x^* \in F_T$;

(3) the following estimates

$$d(x_n, x^*) \leq \frac{\alpha^n}{1-\alpha} d(x_0, x_1), \qquad n = 0, 1, 2, \ldots$$
$$d(x_n, x^*) \leq \frac{\alpha}{1-\alpha} d(x_{n-1}, x_n), \qquad n = 1, 2, , 3, \ldots$$

hold, for a certain constant $\alpha < 1$.

The next theorem shows that a multi-valued Zamfirescu mapping is a multi-valued weak contraction. Hence Theorem 3.1 follows immediately from Theorem 2.2.

Theorem 3.2 Let (X, d) be a metric space and $T : X \to \mathcal{CB}(X)$ be a multi-valued Zamfirescu mapping. Then T is a multi-valued weak contraction.

Proof: Let T be a multi-valued Zamfirescu mapping and $x, y \in X$. Then at least one of (z_1), (z_2) or (z_3) is true.

If x and y satisfy (z_2), then we have

$$\begin{aligned} H(Tx, Ty) &\leq b[d(x, Tx) + d(y, Ty)] \\ &\leq b[[d(x, Ty) + H(Tx, Ty)] + [d(y, x) + d(x, Ty)]] \\ &= bd(x, y) + 2bd(x, Ty) + bH(Tx, Ty) \end{aligned}$$

and hence $H(Tx, Ty) \leq \frac{b}{1-b} d(x, y) + \frac{2b}{1-b} d(x, Ty)$.

If x and y satisfy (z_3), then we have

$$\begin{aligned} H(Tx, Ty) &\leq c[d(x, Ty) + d(y, Tx)] \\ &\leq c[d(x, Ty) + [d(y, x) + d(x, Ty) + H(Tx, Ty)]] \\ &= cd(x, y) + 2cd(x, Ty) + cH(Tx, Ty) \end{aligned}$$

and hence $H(Tx, Ty) \leq \frac{c}{1-c} d(x, y) + \frac{2c}{1-c} d(x, Ty)$.

Let

$$\theta = \max\{a, \frac{b}{1-b}, \frac{c}{1-c}\}.$$

Then we have $0 \leq \theta < 1$ and for all $x, y \in X$,

$$H(Tx, Ty) \leq \theta d(x, y) + 2\theta d(x, Ty).$$

Hence T is the multi-valued $(\theta, 2\theta)$-weak contraction.

The following example shows that a multi-valued weak contraction may not be a Zamfirescu mapping.

Example 3.3 Let $X = [0, 1]$ and $Tx = \{x\}$ for all $x \in X$. Then T is a multi-valued weak contraction and is not a Zamfirescu mapping.

Proof: Recall that, for all $x, y \in X$,

$$\begin{aligned} H(Tx, Ty) &= H(\{x\}, \{y\}) \\ &= d(x, y) \\ &= \theta d(x, y) + (1-\theta) d(x, y) \end{aligned}$$

for $\theta \in (0, 1)$. Then T is a $(\theta, 1-\theta)$-weak contraction for all $\theta \in (0, 1)$. Suppose that T is a multi-valued Zamfirescu mapping. Thus there exists $\theta = \max\{a, 2b, 2c\} \in (0, 1)$ such that for all $x, y \in X$,

$$\begin{aligned} |x - y| &= H(Tx, Ty) \\ &\leq \theta \max\{d(x, y), d(x, Tx), d(x, Ty), d(y, Ty), d(y, Tx)\} \\ &= \theta |x - y|. \end{aligned}$$

This is a contradiction.

Acknowledgment

The authors would like to thank the Thailand Research Fund for financial support.

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