Fuzzy Anti-2-Normed Linear Space

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Abstract

In this paper, we study fuzzy anti-2-norm on a linear space and some results are introduced in fuzzy anti-2-norms on a linear space. We shall introduce the notions of convergent sequence, Cauchy sequence in fuzzy anti-2-normed linear space and also introduce the concept of compact subset and bounded subset in fuzzy anti-2-normed linear space.

Keywords: Fuzzy norm, Fuzzy anti-norm, Fuzzy 2-norm, Fuzzy anti-2-norm

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1. Introduction

Fuzzy set theory is a useful tool to describe situations in which the data are imprecise or vague. Fuzzy sets handle such situation by attributing a degree to which a certain object belongs to a set. The idea of fuzzy norm was initiated by Katsaras in [1984]. Felbin [1992] defined a fuzzy norm on a linear space whose associated fuzzy metric is of Kaleva and Seikkala type [1984]. Cheng and Mordeson [1994] introduced an idea of a fuzzy norm on a linear space whose associated metric is Kramosil and Michalek type [1975].

Bag and Samanta in [2003] gave a definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [1975]. They also studied some properties of the fuzzy norm in [2005] and [2008]. Bag and Samanta discussed the notions of convergent sequence and Cauchy sequence in fuzzy normed linear space in [2003]. They also made in [2008] a comparative study of the fuzzy norms defined by Katsaras [1984], Felbin [1992], and Bag and Samanta [2003].

In [2010] Iqbal H. Jebril and Samanta introduced fuzzy anti-norm on a linear space depending on the idea of fuzzy anti-norm was introduced by Bag and Samanta [2008] and investigated their important properties.

In this paper, we study fuzzy anti-2-norm on a linear space and some results are introduced in fuzzy anti-2-norms on a linear space. We shall introduce the notions of convergent sequence, Cauchy sequence in fuzzy anti-2-normed linear space and also introduce the concept of compact subset and bounded subset in fuzzy anti-2-normed linear space.

In [1992], Felbin introduced the concept of a fuzzy norm based on a Kaleva and Seikkala type [1984] of fuzzy metric using the notion of fuzzy number. Let X be a vector space over R(set of real numbers). Let $\| \bullet \| : X \longrightarrow R^*(I)$ be a mapping and let the mappings $L, U : [0,1] \times [0,1] \longrightarrow [0,1]$, be symmetric, non-decreasing in both arguments and satisfying $L(0,0) = 0$ and $U(1,1) = 1$. Write $[\|x\|]_\alpha = [\|x\|^1_\alpha R \|x\|^2_\alpha]$ for $x \in X$, $0 < \alpha \leq 1$ and suppose for all $x \in X$, $x \neq \underline{0}$ there exists $\alpha_0 \in (0,1]$ independent of x such that for all $\alpha \leq \alpha_0$,

(A) $\|x\|^2_\alpha < \infty$ (B) $inf \|x\|^1_\alpha > 0$

The quadruple $(X, \| \bullet \|, L, U)$ is called a Felbin-fuzzy normed linear space and $\| \bullet \|$ is a Felbin-fuzzy norm if

(i) $\|x\| = 0$ if and only if $x = \underline{0}$ (the null vector),

(ii) $\|rx\| = |r| \, \|x\|$, $x \in X$, $r \in R$,

(iii) For all $x, y \in X$, (a) Whenever $s \leq \|x\|^1_1$, $t \leq \|y\|^1_1$ and $s + t \leq \|x + y\|^1_1$,

$$\|x + y\|(s + t) \geq L(\|x\|(s), \|y\|(t)).$$

(b) Whenever $s \geq \|x\|^1_1$, $t \geq \|y\|^1_1$ and $s + t \geq \|x + y\|^1_1$,

$$\|x + y\|(s + t) \leq U(\|x\|(s), \|y\|(t)).$$

Definition 1.1. Let X be a vector space over R(set of real numbers). Let $\|\bullet, \bullet\| : X \times X \longrightarrow R^*(I)$ be a mapping and let the mappings $L, U : [0,1] \times [0,1] \longrightarrow [0,1]$, be symmetric, non-decreasing in both arguments and satisfying $L(0,0) = 0$ and $U(1,1) = 1$. Write $[\|x,z\|]_\alpha = [\|x,z\|^1_\alpha R \|x,z\|^2_\alpha]$ for $x, z \in X$, $0 < \alpha \leq 1$ and suppose for all $x, z \in X$, $x \neq \underline{0}$, $z \neq \underline{0}$, there exists $\alpha_0 \in (0,1]$ independent of x, z such that for all $\alpha \leq \alpha_0$,

(A) $\|x,z\|^2_\alpha < \infty$ (B) $inf \|x,z\|^1_\alpha > 0$

The quadruple $(X, \|\bullet, \bullet\|, L, U)$ is called a Felbin-fuzzy 2-normed linear space and $\|\bullet, \bullet\|$ is a Felbin-fuzzy 2-norm if

(i) $\|x, z\| = 0$ if and only if x, z are linearly dependent,

(ii) $\|rx, z\| = |r|\,\|x, z\|$, $x, z \in X$, $r \in R$

(iii) $\|x, z\|$ is invariant under any permutation of x, z,

(iv) For all $x, y, z \in X$, (a) Whenever $s \le \|x, z\|_1^1$, $t \le \|y, z\|_1^1$ and $s + t \le \|x + y, z\|_1^1$,

$\|x + y, z\|(s + t) \ge L\,(\|x, z\|(s), \|y, z\|(t))$.

(b) Whenever $s \ge \|x, z\|_1^1$, $t \ge \|y, z\|_1^1$ and $s + t \ge \|x + y, z\|_1^1$,

$\|x + y, z\|(s + t) \le U\,(\|x, z\|(s), \|y, z\|(t))$.

2. Fuzzy 2-Norms on a Linear Space

This section contains a few basic definitions and preliminary results which will be needed in the sequel.

Definition 2.1. [Somasundaram, 2009]. Let X be a real linear space of dimension greater than one and let $\|\bullet, \bullet\|$ be a real valued function on $X \times X$ satisfying the following conditions

$2N_1$: $\|x, y\| = 0$ if and only if x and y are linearly dependent

$2N_2$: $\|x, y\| = \|y, x\|$

$2N_3$: $\|\alpha x, y\| = |\alpha|\,\|x, y\|$, for every $\alpha \in R$

$2N_4$: $\|x, y + z\| \le \|x, y\| + \|x, z\|$

then the function $\|\bullet, \bullet\|$ is called a 2-norm on X and the pair $(X, \|\bullet, \bullet\|)$ is called a 2-normed linear space.

Definition 2.2. [Somasundaram, 2009]. Let X be a linear space over a field F. A fuzzy subset N of $X \times X \times R$ is called a fuzzy 2-norm on X if the following conditions are satisfied for all $x, y, z \in X$.

$(2 - N1)$: For all $t \in R$ with $t \le 0$, $N(x, y, t) = 0$,

$(2 - N2)$: For all $t \in R$ with $t > 0$, $N(x, y, t) = 1$ if and only if x, y are linearly dependent

$(2 - N3)$: $N(x, y, t)$ is invariant under any permutation of x, y

$(2 - N4)$: For all $t \in R$ with $t > 0$, $N(x, cy, t) = N(x, y, \frac{t}{|c|})$ if $c \ne 0$, $c \in F$

$(2 - N5)$: For all $s, t \in R$, $N(x, y + z, s + t) \ge \min\{N(x, y, s), N(x, z, t)\}$

$(2 - N6)$: $N(x, y, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t\to\infty} N(x, y, t) = 1$.

Then N is said to be a fuzzy 2-norm on a linear space X and the pair (X, N) is called a fuzzy 2-normed linear space (briefly F-2-NLS).

The following condition of fuzzy 2-norm N will be required later on.

$(2 - N7)$: For all $t \in R$ with $t > 0$, $N(x, y, t) > 0$, implies that x, y are linearly dependent.

Example 2.3. [Bag, 2003]. Let $(X, \|\bullet, \bullet\|)$ be a 2-normed linear space. Define

$$\begin{aligned} N(x, y, t) &= \frac{t}{t + \|x, y\|}, \ \text{ when } t > 0,\ t \in R,\ x, y \in X \\ &= 0, \ \text{ when } t \le 0,\ t \in R,\ x, y \in X. \end{aligned}$$

Then (X, N) is an F-2-NLS.

Example 2.4. [Bag, 2003]. Let $(X, \|\bullet, \bullet\|)$ be a 2-normed linear space. Define

$$\begin{aligned} N(x, y, t) &= 0, \ \text{ when } t \le \|x, y\|,\ t \in R,\ x, y \in X \\ &= 1, \ \text{ when } t > \|x, y\|,\ t \in R,\ x, y \in X. \end{aligned}$$

Then (X, N) is an F-2-NLS.

Theorem 2.5. Let (X, N) be a fuzzy 2-normed linear space. Define $\|x, y\|_\alpha = \inf\{t : N(x, y, t) \ge \alpha\}$; $\alpha \in (0, 1)$. Then $\{\|\bullet, \bullet\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of 2-norms on X. These 2-norms are called α-2-norms on X corresponding to fuzzy 2-norm on X.

Theorem 2.6. Let $\{\|\bullet, \bullet\|_\alpha : \alpha \in (0, 1)\}$ be an ascending family of 2-norms on linear space X. Define a function $N' : X \times X \times R \longrightarrow [0, 1]$ as

$$\begin{aligned} N'(x, y, t) &= \sup\{\alpha \in (0, 1) : \|x, y\|_\alpha \le t\}, \ \text{ when } (x, y, t) \ne 0, \\ &= 0, \ \text{ when } (x, y, t) = 0. \end{aligned}$$

Then N' is a fuzzy 2-norm on X

If the index set $(0, 1)$ of the family of crisp 2-norms $\{||\bullet, \bullet||_\alpha : \alpha \in (0, 1)\}$ of Theorem 2.6 is extended to $(0, 1]$ then a fuzzy 2-norm N is generated, satisfying an additional property that $N(x, y, t)$ attains the value 1 at some finite value t.

Theorem 2.7. Let $\{||\bullet, \bullet||_\alpha : \alpha \in (0, 1]\}$ be an ascending family of 2-norms on linear space X. Define a function $N' : X \times X \times R \longrightarrow [0, 1]$ as

$$\begin{aligned} N'(x, y, t) &= \sup\{\alpha \in (0, 1] : ||x, y||_\alpha \leq t\}, \text{ when } (x, y, t) \neq 0, \\ &= 0, \text{ when } (x, y, t) = 0. \end{aligned}$$

Then (a) N' is a fuzzy 2-norm on X

(b) For each $x, y \in X$, $\exists\, t = t(x, y) > 0$ such that $N'(x, y, s) = 1$, for all $s \geq t$.

Example 2.8. Let $X = R^3$ be a linear space over R. Define $||\bullet, \bullet|| : X \times X \times R \longrightarrow [0, 1]$ by

$$||x, y|| = \max\{|x_1 y_2 - x_2 y_1|, |x_2 y_3 - x_3 y_2|, |x_3 y_1 - x_1 y_3|\},$$

where $(x_1, x_2, x_3) \in R^3$ and $(y_1, y_2, y_3) \in R^3$ then $(X, ||\bullet, \bullet||)$ is a 2-normed linear space.

Define $N : X \times X \times R \longrightarrow [0, 1]$ by

$$\begin{aligned} N(x, y, t) &= 1, \text{ if } t > ||x, y|| \\ &= 0.5, \text{ if } \frac{1}{2}||x, y|| < t \leq ||x, y|| \\ &= 0, \text{ if } t \leq \frac{1}{2}||x, y|| \end{aligned}$$

Then (X, N) is a fuzzy 2-normed linear space. Define $||x, y||_\alpha = \inf\{t : N(x, y, t) \geq \alpha\}$, $\alpha \in (0, 1)$. Then $\{||\bullet, \bullet||_\alpha : \alpha \in (0, 1)\}$ be an ascending family of 2-norms on a linear space X. The α-2-norms corresponding to fuzzy 2-norm on X are given by

$$\begin{aligned} ||x, y||_\alpha &= ||x, y|| \text{ if } 1 > \alpha > 0.5 \\ &= \frac{1}{2}||x, y|| \text{ if } 0 < \alpha \leq 0.5 \end{aligned}$$

If the index set $(0, 1)$ of the family of crisp 2-norms $\{||\bullet, \bullet||_\alpha : \alpha \in (0, 1)\}$ of Theorem 2.6 is extended to $(0, 1]$ then a fuzzy 2-norm N is generated, satisfying an additional property that $N(x, y, t)$ attains the value 1 at some finite value t.

Let $\{||\bullet, \bullet||_\alpha : \alpha \in (0, 1]\}$ be an ascending family of 2-norms on linear space X. Define a function

$$\begin{aligned} N' : X \times X \times R \longrightarrow [0, 1] \text{ as } N'(x, y, t) &= \sup\{\alpha \in (0, 1] : ||x, y||_\alpha \leq t\}, \text{ when } (x, y, t) \neq 0, \\ &= 0, \text{ when } (x, y, t) = 0. \end{aligned}$$

Then N' is a fuzzy 2-norm on X.

Definition 2.9. Let (X, N) be a fuzzy 2-normed linear space. Let $\{x_n\}$ be a sequence in X then $\{x_n\}$ is said to be convergent if there exists $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, y, t) = 1$, for all $t > 0$.

Definition 2.10. Let (X, N) be a fuzzy 2-normed linear space. Let $\{x_n\}$ be a sequence in X then $\{x_n\}$ is said to be a Cauchy sequence if $\lim_{n\to\infty} N(x_{n+p} - x_n, y, t) = 1$, for all $t > 0$ and $p = 1, 2, 3, \ldots$.

Definition 2.11. A subset B of a fuzzy 2-normed linear space (X, N) is said to be bounded if and only if there exists $t > 0$ and $0 < r < 1$ such that $N(x, y, t) > 1 - r$ for all $x, y \in B$.

Definition 2.12. A subset B of a fuzzy 2-normed linear space (X, N) is said to be compact if any sequence $\{x_n\}$ in B has a subsequence converging to an element of B.

3. Fuzzy Anti-2-Norms on a Linear Space

In this section, we introduce the notion of fuzzy anti-2-normed linear space and investigate their important properties.

Definition 3.1. Let U be a linear space over a real field F. A fuzzy subset N^* of $U \times U \times R$ such that for all $x, y, u \in U$

$(2 - N^*1)$: For all $t \in R$ with $t \leq 0$, $N^*(x, y, t) = 1$,

$(2 - N^*2)$: For all $t \in R$ with $t > 0$, $N^*(x, y, t) = 0$ if and only if x, y are linearly dependent

$(2 - N^*3)$: $N^*(x, y, t)$ is invariant under any permutation of x, y

$(2 - N^*4)$: For all $t \in R$ with $t > 0$, $N^*(x, cy, t) = N^*(x, y, \frac{t}{|c|})$ if $c \neq 0$, $c \in F$

$(2-N^*5)$: For all $s, t \in R$, $N^*(x, y+u, s+t) \leq \max\{N^*(x, y, s), N^*(x, u, t)\}$

$(2-N^*6)$: $N^*(x, y, t)$ is a non-increasing function of $t \in R$ and $\lim_{t\to\infty} N^*(x, y, t) = 0$.

Then N^* is said to be a fuzzy anti-2-norm on a linear space U and the pair (U, N^*) is called a fuzzy anti-2-normed linear space (briefly Fa-2-NLS).

The following condition of fuzzy anti-2-norm N^* will be required later on.

$(2-N^*7)$: For all $t \in R$ with $t > 0$, $N^*(x, y, t) < 1$, implies that x, y are linearly dependent.

Example 3.2. Let $(U, \|\bullet, \bullet\|)$ be a 2-normed linear space. Define

$$\begin{aligned} N^*(x, y, t) &= \frac{\|x, y\|}{t + \|x, y\|}, \ \text{when } t > 0,\ t \in R,\ x, y \in U \\ &= 1, \ \text{when } t \leq 0,\ t \in R,\ x, y \in U. \end{aligned}$$

Then (U, N^*) is an Fa-2-NLS.

Proof. Now we have to show that $N^*(x, y, t)$ is a fuzzy anti-2-norm in U.

$(2-N^*1)$: For all $t \in R$ with $t \leq 0$, we have by definition $N^*(x, y, t) = 1$.

$(2-N^*2)$: For all $t \in R$ with $t > 0$,

$$N^*(x, y, t) = 0 \Leftrightarrow \frac{\|x, y\|}{t + \|x, y\|} = 0 \Leftrightarrow \|x, y\| = 0 \Leftrightarrow x, y \text{ are linearly dependent}$$

$(2-N^*3)$: As $\|x, y\|$ is invariant under any permutation of x, y, it follows that $N^*(x, y, t)$ is invariant under any permutation of x, y

$(2-N^*4)$: For all $t \in R$ with $t > 0$ and $c \neq 0$, $c \in F$, we get

$$N^*(x, cy, t) = \frac{\|x, cy\|}{t + \|x, cy\|} = \frac{|c|\ \|x, y\|}{t + |c|\ \|x, y\|} = \frac{\|x, y\|}{\frac{t}{|c|} + \|x, y\|} = N^*(x, y, \frac{t}{|c|}).$$

$(2-N^*5)$: For all $s, t \in R$ and $x, y, u \in U$. We have to show that $N^*(x, y+u, s+t) \leq \max\{N^*(x, y, s), N^*(x, u, t)\}$. If (a) $s+t < 0$ (b) $s = t = 0$ (c) $s+t > 0$; $s > 0, t < 0$; $s < 0, t > 0$, then in these cases the relation is obvious. If (d) $s > 0$, $t > 0$, $s+t > 0$. Then assume that

$$N^*(x, y, s) \leq N^*(x, u, t) \Rightarrow \frac{\|x, y\|}{s + \|x, y\|} \leq \frac{\|x, u\|}{t + \|x, u\|} \Rightarrow \|x, y\|(t + \|x, u\|) \leq \|x, u\|(s + \|x, y\|)$$

$$\Rightarrow t\|x, y\| \leq s\|x, u\| \tag{1}$$

$$\text{Now } \frac{\|x, y+u\|}{s+t+\|x, y+u\|} - \frac{\|x, u\|}{t + \|x, u\|} \leq \frac{\|x, y\| + \|x, u\|}{s+t+\|x, y\| + \|x, u\|} - \frac{\|x, u\|}{t + \|x, u\|} = \frac{t\|x, y\| - s\|x, u\|}{(s+t+\|x, y\| + \|x, u\|)(t + \|x, u\|)}.$$

By using equation (1), we get $\dfrac{\|x, y+u\|}{s+t+\|x, y+u\|} \leq \dfrac{\|x, u\|}{t + \|x, u\|}$. Simiarly $\dfrac{\|x, y+u\|}{s+t+\|x, y+u\|} \leq \dfrac{\|x, y\|}{s + \|x, y\|}$.

$$\text{Hence } N^*(x, y+u, s+t) \leq \max\{N^*(x, y, s), N^*(x, u, t)\}.$$

$(2-N^*6)$: If $t_1 < t_2 \leq 0$, then we have $N^*(x, y, t_1) = N^*(x, y, t_2) = 1$. If $0 < t_1 < t_2$ then

$$\frac{\|x, y\|}{t_1 + \|x, y\|} - \frac{\|x, y\|}{t_2 + \|x, y\|} = \frac{\|x, y\|(t_2 - t_1)}{(t_1 + \|x, y\|)(t_2 + \|x, y\|)} > 0 \Rightarrow N^*(x, y, t_1) \geq N^*(x, y, t_2).$$

Thus $N^*(x, y, t)$ is a non-increasing function of $t \in R$. Again

$$\lim_{t\to\infty} N^*(x, y, t) = \lim_{t\to\infty} \frac{\|x, y\|}{t + \|x, y\|} = 0, \text{ for all } x, y \in U. \text{ Hence } (U, N^*) \text{ is an Fa-2-NLS.}$$

Example 3.3. Let $(U, \|\bullet, \bullet\|)$ be a 2-normed linear space. Define $N^* : U \times U \times R \longrightarrow [0, 1]$ by

$$\begin{aligned} N^*(x, y, t) &= 0, \ \text{when } t > \|x, y\|,\ t \in R,\ x, y \in U \\ &= 1, \ \text{when } t \leq \|x, y\|,\ t \in R,\ x, y \in U. \end{aligned}$$

Then (U, N^*) is an Fa-2-NLS.

Proof. It can be easily verified that (U, N^*) is an Fa-2-NLS.

Remark 3.4. N^* is a fuzzy anti-2-norm on $U \Leftrightarrow 1 - N^*$ is a fuzzy 2-norm on U.

Lemma 3.5. Let (U, N^*) is an Fa-2-NLS. Then $N^*(x, y - u, t) = N^*(x, u - y, t)$ for all $x, y, u \in U$ and $t \in (0, \infty)$.

Proof. For $x, y, u \in U$ and $t \in (0, \infty)$, $N^*(x, y - u, t) = N^*(x, -(u - y), t) = N^*(x, u - y, \frac{t}{|-1|}) = N^*(x, u - y, t)$.

Definition 3.6. Let N^* be a fuzzy anti-2-norm on U satisfying $(2 - N^*7)$. Define

$\|x, y\|^*_\alpha = \inf\{t > 0 : N^*(x, y, t) < \alpha,\ \alpha \in (0, 1]\}$.

Lemma 3.7. Let (U, N^*) be a Fa-2-NLS. For each $\alpha \in (0, 1]$ and $x, y, u \in U$. Then we have

(i) $\|x, y\|^*_{\alpha_1} \geq \|x, y\|^*_{\alpha_2}$ for $0 < \alpha_1 < \alpha_2 \leq 1$,

(ii) $\|x, cy\|^*_\alpha = |c|\, \|x, y\|^*_\alpha$ for any scalar c,

(iii) $\|x, y + u\|^*_\alpha \leq \|x, y\|^*_\alpha + \|x, u\|^*_\alpha$.

Proof. (i) For $0 < \alpha_1 < \alpha_2 \leq 1$, we note that

$\inf\{t > 0 : N^*(x, y, t) < \alpha_1\} \geq \inf\{t > 0 : N^*(x, y, t) < \alpha_2\} \Rightarrow \|x, y\|^*_{\alpha_1} \geq \|x, y\|^*_{\alpha_2}$.

(ii) For any scalar c and for all $\alpha \in (0, 1]$,

$$\begin{aligned}\|x, cy\|^*_\alpha &= \inf\{t > 0 : N^*(x, cy, t) < \alpha, \alpha \in (0, 1]\} = \inf\{t > 0 : N^*(x, y, \tfrac{t}{|c|}) < \alpha, \alpha \in (0, 1]\} \\ &= |c|\ \inf\{t > 0 : N^*(x, y, t) < \alpha, \alpha \in (0, 1]\} = |c|\, \|x, y\|^*_\alpha.\end{aligned}$$

(iii) For any $\alpha \in (0, 1]$,

$$\begin{aligned}\|x, y\|^*_\alpha + \|x, u\|^*_\alpha &= \inf\{t > 0 : N^*(x, y, t) < \alpha\} + \inf\{s > 0 : N^*(x, u, s) < \alpha\} \\ &\geq \inf\{s + t > 0 : N^*(x, y, t) < \alpha,\ N^*(x, u, s) < \alpha\} = \|x, y + u\|^*_\alpha.\end{aligned}$$

Theorem 3.8. Let (U, N^*) be a Fa-2-NLS. Then $\{\|\bullet, \bullet\|^*_\alpha : \alpha \in (0, 1]\}$ is a decreasing family of 2-norms on a linear space U.

Proof. By lemma 3.7 it can be easily verified.

Theorem 3.9. Let $\{\|\bullet, \bullet\|^*_\alpha : \alpha \in (0, 1]\}$ be a decreasing family of 2-norms on a linear space U. Now define a function $N^*_1 : U \times U \times R \longrightarrow [0, 1]$ as

$$\begin{aligned}N^*_1(x, y, t) &= \inf\{\alpha \in (0, 1] : \|x, y\|^*_\alpha \leq t\}, \text{ when } (x, y, t) \neq 0, \\ &= 1, \text{ when } (x, y, t) = 0.\end{aligned}$$

Then (a) N^*_1 is a fuzzy anti-2-norm on U.

(b) For each $x, y \in U$, $\exists\, r = r(x, y) > 0$ such that $N^*_1(x, y, t) = 1$.

Proof. (a) Now we have to show that N^*_1 is a fuzzy anti-2-norm on U.

$(2 - N^*1)$: (i) For all $t \in R$ with $t < 0$, $\{\alpha \in (0, 1] : \|x, y\|^*_\alpha \leq t\} = \Phi$, $\forall\ x, y \in U$, we have $N^*_1(x, y, t) = \inf\{\alpha \in (0, 1] : \|x, y\|^*_\alpha \leq t\} = 1$.

(ii) For $t = 0$ and $x \neq \underline{0}$, $y \neq \underline{0}$, $\{\alpha \in (0, 1] : \|x, y\|^*_\alpha \leq t\} = \Phi$, $\forall\ x, y \in U$, we have $N^*_1(x, y, t) = 1$.

(iii) For $t = 0$ and $x \neq \underline{0}$, $y \neq \underline{0}$, then from the definition $N^*_1(x, y, t) = 1$.

Thus for all $t \in R$ with $t \leq 0$, $N^*_1(x, y, t) = 1$, $\forall\ x, y \in U$.

$(2 - N^*2)$: For all $t \in R$ with $t > 0$, $N^*_1(x, y, t) = 0$. Choose any $\varepsilon \in (0, 1)$. Then for any $t > 0$, $\exists\ \alpha_1 \in (\varepsilon, 1]$ such that $\|x, y\|^*_{\alpha_1} \leq t$ and hence $\|x, y\|^*_\varepsilon \leq t$. Since $t > 0$ is arbitrary, this implies that $\|x, y\|^*_\varepsilon = 0$ then x, y are linearly dependent. If x, y are linearly dependent then for $t > 0$, $N^*_1(x, y, t) = \inf\{\alpha \in (0, 1] : \|x, y\|^*_\alpha \leq t\} = 0$. Thus for all $t \in R$ with $t > 0$, $N^*_1(x, y, t) = 0$ if and only if x, y are linearly dependent.

$(2 - N^*3)$: As $\|x, y\|^*_\alpha$ is invariant under any permutation of x, y, it follows that $N^*_1(x, y, t)$ is invariant under any permutation of x, y.

$(2 - N^*4)$: For all $t \in R$ with $t > 0$, and $c \neq 0$, $c \in F$, we have $N^*_1(x, cy, t) = \inf\{\alpha \in (0, 1] : \|x, cy\|^*_\alpha \leq t\} = \inf\{\alpha \in (0, 1] : |c|\, \|x, y\|^*_\alpha \leq t\} = \inf\{\alpha \in (0, 1] : \|x, y\|^*_\alpha \leq \frac{t}{|c|}\} = N^*_1(x, y, \frac{t}{|c|})\ \forall\ x, y \in U$.

$(2 - N^*5)$: We have to show that $\forall\ s, t \in R$ and $\forall\ x, y, u \in U$, $N^*_1(x, y + u, s + t) \leq \max\{N^*_1(x, y, s), N^*_1(x, u, t)\}$.

Suppose that $\forall\ s, t \in R$ and $\forall\ x, y, u \in U$, $N^*_1(x, y + u, s + t) > \max\{N^*_1(x, y, s), N^*_1(x, u, t)\}$. Choose k such that $N^*_1(x, y + u, s + t) > k > \max\{N^*_1(x, y, s), N^*_1(x, u, t)\}$.

Now $N_1^*(x, y+u, s+t) > k \Rightarrow \inf\{\alpha \in (0,1] : ||x, y+u||_\alpha^* \leq s+t\} > k \Rightarrow ||x, y+u||_k^* \leq s+t \Rightarrow ||x, y||_k^* + ||x, u||_k^* > s+t$. Again $k > \max\{N_1^*(x, y, s), N_1^*(x, u, t)\} \Rightarrow k > N_1^*(x, y, s)$ and $k > N_1^*(x, u, t) \Rightarrow ||x, y||_k^* \leq s$ and $||x, u||_k^* \leq t \Rightarrow ||x, y||_k^* + ||x, u||_k^* \leq s+t$. Thus $s+t < ||x, y||_k^* + ||x, u||_k^* \leq s+t$, which is a contradiction. Hence $N_1^*(x, y+u, s+t) \leq \max\{N_1^*(x, y, s), N_1^*(x, u, t)\}$.

$(2-N^*6)$: Let $x, y \in U$, $\alpha \in (0, 1)$. Now $t > ||x, y||_\alpha^* \Rightarrow N_1^*(x, y, t) = \inf\{\beta \in (0, 1] : ||x, y||_\beta^* \leq t\} \leq \alpha$. So $\lim_{t\to\infty} N_1^*(x, y, t) = 0$. Next we verify that $N_1^*(x, y, t)$ is a non-increasing function of $t \in R$. If $t_1 < t_2 \leq 0$, then $N_1^*(x, y, t_1) = N_1^*(x, y, t_2) = 1$, $\forall\, x, y \in U$. If $0 < t_1 < t_2$ then $\{\alpha \in (0, 1] : ||x, y||_\alpha^* \leq t_1\} \subseteq \{\alpha \in (0, 1] : ||x, y||_\alpha^* \leq t_2\} \Rightarrow \inf\{\alpha \in (0, 1] : ||x, y||_\alpha^* \leq t_1\} \geq \inf\{\alpha \in (0, 1] : ||x, y||_\alpha^* \leq t_2\} \Rightarrow N_1^*(x, y, t_1) \geq N_1^*(x, y, t_2)$. Thus $N_1^*(x, y, t)$ is a non-increasing function of $t \in R$ and N_1^* is a fuzzy anti-2-norm on U.

(b) For each $x \neq \underline{0}$, $y \neq \underline{0}$, $||x, y||_\alpha^* > 0$. Thus $\exists\, r = r(x, y) > 0$ such that $||x, y||_\alpha^* \geq r(x, y) > 0 \Rightarrow ||x, y||_\alpha^* > r(x, y)$, $\forall\, \alpha \in (0, 1] \Rightarrow \inf\{\alpha \in (0, 1] : ||x, y||_\alpha^* \leq t\} = 1 \Rightarrow N_1^*(x, y, t) = 1$.

Definition 3.10. Let (U, N^*) be a Fa-2-NLS. A sequence $\{x_n\}$ in U is said to be convergent to $x \in U$ if given $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in N$ such that $N^*(x_n - x, y, t) < r$, for all $n \geq n_0$.

Theorem 3.11. In a Fa-2-NLS (U, N^*), a sequence $\{x_n\}$ converges to $x \in U$ if and only $\lim_{n\to\infty} N^*(x_n - x, y, t) = 0$, $\forall\, t > 0$.

Proof. Fix $t > 0$. Suppose $\{x_n\}$ converges to $x \in U$. Then for a given r, $0 < r < 1$ there exists an integer $n_0 \in N$ such that $N^*(x_n - x, y, t) < r$, for all $n \geq n_0$, and hence $N^*(x_n - x, y, t) \to 0$, as $n \to \infty$. Conversely, if for each $t > 0$, $N^*(x_n - x, y, t) \to 0$, as $n \to \infty$, then for every r, $0 < r < 1$, there exists an integer n_0 such that $N^*(x_n - x, y, t) < r$, for all $n \geq n_0$. Hence $\{x_n\}$ converges to x in U.

Definition 3.12. Let (U, N^*) be a Fa-2-NLS. A sequence $\{x_n\}$ in U is said to be a Cauchy sequence if given $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in N$ such that $N^*(x_{n+p} - x_n, y, t) < r$, for all $n \geq n_0$, $p = 1, 2, 3, \ldots$.

Theorem 3.13. In a Fa-2-NLS (U, N^*), a sequence $\{x_n\}$ is a Cauchy sequence in U if and only if $\lim_{n\to\infty} N^*(x_{n+p} - x_n, y, t) = 0$, for $p = 1, 2, 3, \ldots$ and $t > 0$.

Proof. Fix $t > 0$. Suppose $\{x_n\}$ is a Cauchy sequence in U. Then for a given r, $0 < r < 1$ and $p = 1, 2, 3, \ldots$ there exists an integer $n_0 \in N$ such that $N^*(x_{n+p} - x_n, y, t) < r$, for all $n \geq n_0$, and hence $N^*(x_{n+p} - x_n, y, t) \to 0$, as $n \to \infty$. Conversely, if for each $t > 0$, and $p = 1, 2, 3, \ldots$, $N^*(x_{n+p} - x_n, y, t) \to 0$, as $n \to \infty$, then for every r, $0 < r < 1$, there exists an integer n_0 such that $N^*(x_{n+p} - x_n, y, t) < r$, for all $n \geq n_0$. Hence $\{x_n\}$ is a Cauchy sequence in U.

Theorem 3.14. If a sequence $\{x_n\}$ in a Fa-2-NLS (U, N^*) is convergent then its limit is unique.

Proof. Let $\{x_n\}$ converges to x and z. Also let $s, t \in R^+$ then $\lim_{n\to\infty} N^*(x_n - x, y, t) = 0$ and $\lim_{n\to\infty} N^*(x_n - z, y, s) = 0$. Now

$$\begin{aligned} N^*(x - z, y, t + s) &= N^*(x - x_n + x_n - z, t + s) \leq \max\{N^*(x - x_n, y, t), N^*(x_n - z, y, s)\} \\ &= \max\{N^*(x_n - x, y, t), N^*(x_n - z, y, s)\}. \end{aligned}$$

Taking limit, we have $N^*(x - z, y, t + s) \leq \max\{\lim_{n\to\infty} N^*(x_n - x, y, t), \lim_{n\to\infty} N^*(x_n - z, y, s)\}$
$\Rightarrow N^*(x - z, y, t + s) = 0$, $\forall\, s, t \in R^+ \Rightarrow x - z = \underline{0} \Rightarrow x = z$.

Theorem 3.15. In a Fa-2-NLS (U, N^*), every subsequence of a convergent sequence converges to the limit of a sequence.

Proof. The proof is obvious.

Theorem 3.16. Let L be a linear space, N^* be a fuzzy anti-2-norm on L and $\widehat{N} = (1 - N^*)$ be a fuzzy 2-norm on L. Then

(a) $\{x_n\}$ is a convergent sequence in (L, N^*) if and only if $\{x_n\}$ is a convergent sequence in $(L, \widehat{N})$.

(b) $\{x_n\}$ is a Cauchy sequence in (L, N^*) if and only if $\{x_n\}$ is a Cauchy sequence in $(L, \widehat{N})$.

Proof. (a) Let $\{x_n\}$ be a convergent sequence in $(L, N^*) \Leftrightarrow \lim_{n\to\infty} N^*(x_n - x, y, t) = 0$, for all $t > 0 \Leftrightarrow \lim_{n\to\infty} \widehat{N}(x_n - x, y, t) = 1$ for all $t > 0 \Leftrightarrow \{x_n\}$ is a convergent sequence in $(L, \widehat{N})$.

(b) Let $\{x_n\}$ be a Cauchy sequence in $(L, N^*) \Leftrightarrow \lim_{n\to\infty} N^*(x_{n+p} - x_n, y, t) = 0$, $p = 1, 2, 3, \ldots$, for all $t > 0 \Leftrightarrow \lim_{n\to\infty} \widehat{N}(x_{n+p} - x_n, y, t) = 1$, $p = 1, 2, 3, \ldots$, for all $t > 0 \Leftrightarrow \{x_n\}$ is a Cauchy sequence in $(L, \widehat{N})$.

Theorem 3.17. In a Fa-2-NLS (U, N^*), every convergent sequence is a Cauchy sequence.

Proof. Let $\{x_n\}$ be a convergent sequence in a Fa-2-NLS (U, N^*) then $\lim_{n\to\infty} N^*(x_n - x, y, t) = 0$, for all $t > 0$. Let $s, t \in R^+$ and $p = 1, 2, 3, \ldots$, we have

$$\begin{aligned} N^*(x_{n+p} - x_n, y, s + t) &= N^*(x_{n+p} - x + x - x_n, s + t) \leq \max\{N^*(x_{n+p} - x, y, s), N^*(x - x_n, y, t)\} \\ &= \max\{N^*(x_{n+p} - x, y, s), N^*(x_n - x, y, t)\}. \end{aligned}$$

Taking limit, we have $\lim_{n\to\infty} N^*(x_{n+p} - x_n, y, s + t) \leq \max\{\lim_{n\to\infty} N^*(x_{n+p} - x, y, s), \lim_{n\to\infty} N^*(x_n - x, y, t)\} = 0$

$\Rightarrow \lim_{n\to\infty} N^*(x_{n+p} - x_n, y, s+t) = 0,\ \forall\ s,t \in R^+$ and $p = 1,2,3,\ldots$ Thus $\{x_n\}$ is a Cauchy sequence in Fa-2-NLS (U, N^*).

The converse of the above theorem is not necessarily true. This is justified by the following example.

Example 3.18. Let $(X, \|\bullet,\bullet\|)$ be a 2-normed linear space and $N^* : X \times X \times R \longrightarrow [0,1]$. Define

$$\begin{aligned} N^*(x,y,t) &= \frac{\|x,y\|}{t + \|x,y\|},\ \text{ when } t > 0,\ t \in R,\ x,y \in X \\ &= 1,\ \text{ when } t \leq 0,\ t \in R,\ x,y \in X. \end{aligned}$$

Then (X, N^*) is an Fa-2-NLS. Let $\{x_n\}$ be a sequence in X, then

(a) $\{x_n\}$ is a Cauchy sequence in $(X, \|\bullet,\bullet\|)$ if and only if $\{x_n\}$ is a Cauchy sequence in (X, N^*) .

(b) $\{x_n\}$ is a convergent sequence in $(X, \|\bullet,\bullet\|)$ if and only if $\{x_n\}$ is a convergent sequence in (X, N^*) .

Proof. (a) Let $\{x_n\}$ be a Cauchy sequence in $(X, \|\bullet,\bullet\|) \Leftrightarrow \lim_{n\to\infty} \|x_{n+p} - x_n, y\| = 0$, for all $p = 1,2,3,\ldots$

$$\Leftrightarrow \lim_{n\to\infty} N^*(x_{n+p} - x_n, y, t) = \lim_{n\to\infty} \frac{\|x_{n+p} - x_n, y\|}{t + \|x_{n+p} - x_n, y\|} = 0,\ \text{for all } t > 0,\ p = 1,2,3,\ldots$$

$$\Leftrightarrow \lim_{n\to\infty} N^*(x_{n+p} - x_n, y, t) = 0,\ \text{ for all } t > 0,\ p = 1,2,3,\cdots \Leftrightarrow \{x_n\} \text{ is a Cauchy sequence in } (X, N^*).$$

(b) Let $\{x_n\}$ be a convergent sequence in $(X, \|\bullet,\bullet\|) \Leftrightarrow \lim_{n\to\infty} \|x_n - x, y\| = 0$

$$\begin{aligned} \Leftrightarrow \lim_{n\to\infty} N^*(x_n - x, y, t) &= \lim_{n\to\infty} \frac{\|x_n - x, y\|}{t + \|x_n - x, y\|} = 0,\ \text{for all } t > 0 \Leftrightarrow \lim_{n\to\infty} N^*(x_n - x, y, t) = 0 \\ &\Leftrightarrow \{x_n\} \text{ is a convergent sequence in } (X, N^*). \end{aligned}$$

Remark 3.19. If there exist a 2-normed linear space $(X, \|\bullet,\bullet\|_0)$ which is not complete, then the fuzzy anti-2-norm induced by such a crisp 2-norm $\|\bullet,\bullet\|_0$ on a incomplete linear space X, is an incomplete Fa-2-NLS.

Definition 3.20. Let (U, N^*) be a Fa-2-NLS. A subset B of U is said to be closed if for any sequence $\{x_n\}$ in B converges to $x \in B$, that is $\lim_{n\to\infty} N^*(x_n - x, y, t) = 0,\ \forall\ t > 0$ implies that $x \in B$.

Definition 3.21. Let (U, N^*) be a Fa-2-NLS. A subset W of U is said to be the closure of $B \subset W$ if for any $w \in W$, there exists a sequence $\{x_n\}$ in B such that $\lim_{n\to\infty} N^*(x_n - x, y, t) = 0,\ \forall\ t \in R^+$, we denote the set W by $\overline{B}$.

Definition 3.22. A subset B of a Fa-2-NLS (U, N^*) is said to be bounded if and only if there exists $t > 0$ and $0 < r < 1$ such that $N^*(x,y,t) < r,\ \forall\ x,y \in B$.

Definition 3.23. A subset B of a Fa-2-NLS (U, N^*) is said to be compact if any sequence $\{x_n\}$ in B has a subsequence converging to an element of B.

Theorem 3.24. Let (U, N^*) be a Fa-2-NLS then every Cauchy sequence in (U, N^*) is bounded.

Proof. Let $\{x_n\}$ be a Cauchy sequence in a Fa-2-NLS (U, N^*). Then $\lim_{n\to\infty} N^*(x_{n+p} - x_n, y, t) = 0$, for $p = 1,2,3,\ldots$, and $t > 0$. Choose a fixed α_0, $0 < \alpha_0 < 1$. Then we have $\lim_{n\to\infty} N^*(x_n - x_{n+p}, y, t) = 0 < 1 - \alpha_0,\ \forall\ t > 0,\ p = 1,2,3,\ldots \Rightarrow$ For $t' > 0,\ \exists\ n'_0 = n'_0(t')$ such that $\lim_{n\to\infty} N^*(x_n - x_{n+p}, y, t') < 1 - \alpha_0,\ \forall\ n \geq n'_0,\ t > 0,\ p = 1,2,3,\ldots.$ Since $\lim_{n\to\infty} N^*(x,y,t) = 0$, we have for each x_i, $\exists\ t'_i > 0$ such that $N^*(x_i, y, t) < 1 - \alpha_0,\ \forall\ t > t'_i,\ i = 1,2,3,\ldots.$ Let $t'_0 = t' + \max\{t'_1, t'_2, \ldots, t'_{n_0}\}$. Then $N^*(x_n, y, t'_0) \leq N^*(x_n, y, t' + t'_{n_0}) = N^*(x_n - x_{n'_0} + x_{n'_0}, y, t' + t'_{n_0}) \leq \max\{N^*(x_n - x_{n'_0}, y, t'), N^*(x_{n'_0}, y, t'_{n_0})\} = (1 - \alpha_0),\ \forall\ n \geq n'_0$. i.e., $N^*(x_n, y, t'_0) \leq (1 - \alpha_0),\ \forall\ n \geq n'_0$. Therefore $\{x_n\}$ is bounded in (U, N^*).

Conclusion

One can introduce the notions of convergent sequence, Cauchy sequence in fuzzy anti-n-normed linear space and also introduce the concept of compact subset and bounded subset in fuzzy anti-n-normed linear space.

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