

Some Exact Solutions of Plane Steady Hydromagnetic Flow with Variable Viscosity

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Abstract

The aim of the present paper is to find some exact solutions of steady two dimensional finitely conducting incompressible fluid flow under the presence of transverse magnetic field using transformation of variables. We have considered, vorticity distribution proportional to the stream function perturbed by a quadratic stream. The results are shown by various graphs.

Keywords: Steady flow, Navier stokes equations, Exact solution, Variable viscosity

Nomenclature

Latin Symbols

u, v, w	non-dimensional velocity component
H_1, H_2, H_3	non-dimensional components of the magnetic field vector H
H	non-dimensional transverse components of the magnetic field vector H
p	non-dimensional pressure
T	non-dimensional temperature
Re	Reynolds number
R_H	magnetic pressure number
R_σ	magnetic Reynolds number
J	generalized energy function
L, M, N, Z, H, G, P	functions
x, y	variables
K, U, m, n	real constants
$A_1(\theta), \dots, A_9(\theta)$	real constants dependent on the parameter θ , and $-\pi \leq \theta < \pi$
$B_1(\theta), \dots, B_9(\theta)$	real constants dependent on the parameter θ , and $-\pi \leq \theta < \pi$

Greek symbols

μ	non-dimensional viscosity of the fluid
ψ	stream function
ω	vorticity function
Ψ	functions
ξ, η, θ	variables

Subscripts

x, y, z, xx, yy differentiation with respect to Cartesian coordinates x and y .
 $\xi, \eta, \xi\xi, \eta\eta$ differentiation with respect to ξ and η .

superscripts

' dimensional quantities

1. Introduction

In order to describe the flow behavior of any fluid, we have to solve the Navier-Stokes equations arising in fluid flow. The importance of Navier-Stokes equations comes from their wide applicability for different kind of fluid flow, ranging from thin film to large scale atmospheric, even cosmic flows. However, Navier-Stokes equations are highly non-linear in nature and hence we face difficulties in solving them exactly. The full set of general solution of Navier-Stokes equations has not been found and is an open problem till the date. In order to overcome this difficulty one adopt transformations, inverse or semi- inverse method for the reformulation of equations in solvable form. Following the Martin's formulation (Martin, M.H., 1971), some researchers (Chandna, O.P., *et al*, 1982; Siddiqui, A.M., *et al*, 2008) have used hodograph transformation (Ames, W.F., *et al*, 1965) in order to linearized the system of governing equations and successfully got some exact solutions. Some authors (Chandna, O.P., *et al*, 1994; Hayat, T., *et al*, 1988) have used inverse method (Nemenyi, 1951) where some *a priori* condition is assumed about the flow variables and have found some exact solutions.

The above said solutions have been found for the flow of fluid with constant viscosity. But in many situations in the fluid flow, where the pressure and temperature gradients are high or in case of electrically conducting fluid flow where the magnetic field plays dominant role, the viscosity is no longer constant (Myers, *et al*, 2006; Sunil, *et al*, 2008; Kannan, *et al*, 2008). So In order to study fluid flows in such situations, we have to consider the viscosity of fluid as a variable, which gives rise the corresponding Navier-Stokes equations in more complicated form. The exact solution of Navier-Stokes equations for the fluid of variable viscosity are rare and very few work has been done in this aspect. As for the analytical solution is concern, Martin's approach, where system of equations are reformulated in curvilinear coordinates, was previously employed by Naeem in the compressible fluid of constant viscosity. This work was extended by Naeem and Nadeem (1996) for incompressible fluid of variable viscosity. Naeem (1994), utilizing one parameter group of transformation, transformed the governing equations of an incompressible fluid with variable viscosity into a system of ordinary differential equation and successfully got some exact solutions.

Moreover, exact solutions of the steady plane incompressible fluid flow with variable viscosity, employing transformation of variables (Naeem, *et al*, 2009) and von-mises variables (Naeem, *et al*, 2001) have been obtained. Recently Naeem and Jamil (2006), defined a one dimensional transformed variable $\xi = (x \cos \theta + y \sin \theta)$; $-\pi \leq \theta \leq \pi$, converted the governing equations into simple ordinary differential equations and have got a class of exact solutions to flow of fluid of variable viscosity for which the vorticity function is proportional to the stream function perturbed by a uniform stream parallel to X-axis. Further, Jamil and Khan (2006), using the same technique, extended this work by taking electrically conducting fluid of variable viscosity under the presence of transverse magnetic field and considering the vorticity distribution proportional to the stream function perturbed by a uniform stream, $U(x + y)$; where U is a real constant, inclined to the X-axis.

In the present analysis we have extended the work of Jamil and Khan (2006) to find some exact solutions of governing equations of the flow of electrically conducting fluid of variable viscosity under the presence of transverse magnetic field $\mathbf{H} = (0, 0, H)$ by considering the vorticity distribution proportional to the stream function perturbed by a generalized quadratic stream, $U(x + y) - (A(\theta)x^2 + B(\theta)y^2)$, where $A(\theta)$, $B(\theta)$ are parametric constants depending on the parameter θ .

2. Equations of Motion

The non-dimensional equation of steady plane flow of an incompressible electrically conducting fluid of variable viscosity under the presence of transverse magnetic field following Jamil and Khan (2006) are

$$u_x + v_y + w_z = 0, \quad (1)$$

$$\begin{aligned} [\mu u_x + \nu u_y + w u_z] = & -P_x + \frac{1}{Re} \left[(2\mu u_x)_x + (\mu(u_y + v_x))_y + (\mu(w_x + u_z))_z \right] \\ & + R_H [H_3(H_{1z} - H_{3x}) - H_2(H_{2x} - H_{1y})], \end{aligned} \quad (2)$$

$$\begin{aligned} [\mu v_x + \nu v_y + w v_z] = & -P_y + \frac{1}{Re} \left[(2\mu v_y)_y + (\mu(v_z + w_y))_z + (\mu(u_y + v_x))_x \right] \\ & + R_H [H_1(H_{2x} - H_{1y}) - H_3(H_{3y} - H_{2z})], \end{aligned} \quad (3)$$

$$\begin{aligned} [uw_x + vw_y + ww_z] = & -P_y + \frac{1}{R_e} \left[(2\mu w_z)_z + (\mu(w_x + u_z))_x + (\mu(v_z + w_y))_y \right] \\ & + R_H [H_2(H_{3y} - H_{2z}) - H_1(H_{1z} - H_{3x})], \end{aligned} \quad (4)$$

$$(vH_1 - uH_2)_x - (uH_3 - wH_1)_z = \frac{1}{R_\sigma} (H_{1xx} + H_{1yy} + H_{1zz}) \quad (5)$$

$$(wH_2 - vH_3)_z - (vH_1 - uH_2)_x = \frac{1}{R_\sigma} (H_{2xx} + H_{2yy} + H_{2zz}) \quad (6)$$

$$(uH_3 - wH_1)_x - (wH_2 - vH_3)_y = \frac{1}{R_\sigma} (H_{3xx} + H_{3yy} + H_{3zz}). \quad (7)$$

Since we have considered transverse plane flow i.e magnetic field is applied normal to the plane of flow, so we must have

$$\begin{aligned} (u, v, w) &= (u, v, 0) \\ (H_1, H_2, H_3) &= (0, 0, H) \end{aligned} \quad (8)$$

Now in view of equation (8) we have the two dimensional form of governing equations as

$$u_x + v_y = 0, \quad (9)$$

$$[uu_x + vu_y] = -P_x + \frac{1}{R_e} \left[(2\mu u_x)_x + (\mu(u_y + v_x))_y \right], \quad (10)$$

$$[uv_x + vv_y] = -P_y + \frac{1}{R_e} \left[(2\mu v_y)_y + (\mu(u_y + v_x))_x \right] \quad (11)$$

with magnetic diffusion equation as

$$uH_x + vH_y = \frac{1}{R_\sigma} [H_{xx} + H_{yy}]. \quad (12)$$

Where H is the transverse component of magnetic field and $P = p + R_H \frac{H^2}{2}$. Further for non-dimensionalisation we have used the scaling parameters L , U_0 , μ_0 and ρU_0^2 as reference length, velocity, viscosity and pressure. These are defined as

$$u = \frac{u'}{U_0}, \quad v = \frac{v'}{U_0}, \quad w = \frac{w'}{U_0}, \quad \mu = \frac{\mu'}{\mu_0}, \quad p = \frac{p'}{\rho U_0^2}, \quad R_e = \frac{\rho U_0 L}{\mu_0}. \quad (13)$$

Symbols in the above equations have there usual meaning and are listed in the Nomenclature. Equation (9) implies the existence of stream function ψ as

$$u = \psi_y, \quad v = -\psi_x. \quad (14)$$

Utilizing (14) in above equations we get

$$\psi_x \omega = -J_x + \frac{1}{R_e} \left[\mu (\psi_{yy} - \psi_{xx}) \right]_y, \quad (15)$$

$$\psi_y \omega = -J_y - \frac{4}{R_e} (\mu \psi_{xy})_y + \frac{1}{R_e} \left[\mu (\psi_{yy} - \psi_{xx}) \right]_x, \quad (16)$$

$$\psi_y H_x - \psi_x H_y = \frac{1}{R_e} (H_{xx} + H_{yy}), \quad (17)$$

$$\omega = -(\psi_{xx} + \psi_{yy}), \quad J = P + \frac{1}{2} (\psi_x^2 + \psi_y^2) - \frac{2\mu \psi_{xy}}{R_e}. \quad (18)$$

Now we let

$$\psi_{xx} + \psi_{yy} = K (\psi - Ux - Uy + Ax^2 + By^2). \quad (19)$$

Again from equation (18) and (19) we have

$$\omega = -K\Psi \quad (20)$$

where

$$\Psi = \psi - Ux - Uy + Ax^2 + By^2. \quad (21)$$

Now using (20) and (21), equations (15) and (16) becomes

$$L_x = UK\Psi - 2KAx\Psi + \frac{1}{R_e} \left[\mu \{ (\Psi_{yy} - \Psi_{xx}) + 2(A - B) \} \right]_y \quad (22)$$

$$L_y = UK\Psi - 2KBy\Psi - \frac{4}{R_e} (\Psi_{xy})_y + \frac{1}{R_e} \left[\mu \{ (\Psi_{yy} - \Psi_{xx}) + 2(A - B) \} \right]_y \quad (23)$$

where $L = J - \frac{1}{2}K\Psi^2$, now using the Integrability criteria $L_{xy} = L_{yx}$ we get

$$(M_{xx} - M_{yy}) - \frac{4}{R_e} (\mu\Psi_{xy})_{xy} - 2K(By\Psi_x - Ax\Psi_y) + KU(\Psi_x - \Psi_x) = 0 \quad (24)$$

where $M = \frac{\mu(\Psi_{yy} - \Psi_{xx} + 2(A - B))}{R_e}$.

(17) employing equation (21) becomes

$$(\Psi_y + U - 2By)H_x - (\Psi_x + U - 2Ax)H_y = \frac{1}{R_e} (H_{xx} + H_{yy}) \quad (25)$$

3. Solutions

In this section we find exact solutions of the governing equations. Using equations (19) and (21) we have

$$\nabla^2\Psi = K\Psi + 2(A + B). \quad (26)$$

Now we seek the solution of the above equation of the form

$$\Psi(x, y) = N(\xi) \quad (27)$$

where

$$\xi = (x \cos \theta + y \sin \theta), \quad -\pi \leq \theta \leq \pi. \quad (28)$$

Using (27) in equation (26) we have

$$N''(\xi) - KN(\xi) = 2(A + B) \quad (29)$$

Now for the solution of above equation we have two cases

Case (I): $K = -n^2, n \geq 0$

Case (II): $K = m^2, m \geq 0$

Now considering the case I we have the solution of (29) as

$$N(\xi) = A_1(\theta) \cos(n\xi + A_2(\theta)) + \frac{2(A + B)}{n^2} \quad (30)$$

Now using (27), (28) and (30) in (34) with suitable choice of

$$A = -V \cot \theta, \quad B = V \tan \theta \quad (31)$$

where

$$\theta \in (-\pi, \pi) \setminus \left\{ -\frac{\pi}{2}, \frac{\pi}{2} \right\}$$

and V is some real constant, we have

$$\left[\left\{ \frac{n^2 A_1(\theta)}{R_e} \cos(n\xi + A_2(\theta)) - \frac{4V}{R_e} \cot 2\theta \right\} \mu \right]_{\xi\xi} \quad (32)$$

$$= nKA_1(\theta) [U(\cos \theta - \sin \theta) - 2V\xi] \sin(n\xi + A_2(\theta))$$

Thus using (28), (30) and (31) we the function $\Psi(x, y)$ as

$$\Psi(x, y) = A_1(\theta) \cos(n(x \cos \theta + y \sin \theta) + A_2(\theta)) - \frac{4V}{n^2} \cot 2\theta \quad (33)$$

again using (21) and (33) we have the stream function as under:

$$\psi(x, y) = A_1(\theta) \cos(n(x \cos \theta + y \sin \theta) + A_2(\theta)) - \frac{4V}{n^2} \cot 2\theta + U(x + y) + V(x^2 \cot \theta - y^2 \tan \theta) \quad (34)$$

Now introducing new variable

$$\eta = n\xi + A_2(\theta) \quad (35)$$

the equation (32) becomes

$$Z_{\eta\eta} = (A_3(\theta) + A_0\eta) \sin \eta \quad (36)$$

where

$$Z = (\cos \eta + A_4(\theta))\mu, \quad (37)$$

$$A_0 = -\frac{2KVR_e}{n^4}, A_3(\theta) = \frac{KR_e}{n^3} \left\{ U(\cos \theta - \sin \theta) + \frac{2A_2(\theta)V}{n} \right\} \text{ and } A_4(\theta) = -\frac{4V \cot 2\theta}{n^2 A_1(\theta)}.$$

Now on solving (36) and using (37) we get viscosity as

$$\mu = -\frac{1}{(\cos \eta + A_4(\theta))} \left[\{(A_3(\theta) + A_0\eta) \sin \eta + 2A_0 \cos \eta\} + A_5(\theta)\eta + A_6(\theta) \right] \quad (38)$$

where $A_5(\theta)$ and $A_6(\theta)$ are parametric constants. Using (17), (21), (28) and (35) we have

$$H_{\eta\eta} = (A_7(\theta) + H_0\eta) H_\eta \quad (39)$$

where $A_7(\theta) = \frac{R_e U(\cos \theta - \sin \theta)}{n}$ and $H_0 = -2\frac{R_e}{n}$. which gives the solution as

$$H = A_8(\theta) \int e^{\frac{1}{2H_0}(A_7(\theta) + H_0\eta)^2} d\eta + A_9(\theta) \quad (40)$$

where $A_8(\theta)$ and $A_9(\theta)$ are parametric constants. Finally using (28) and (35) we have viscosity and magnetic field in cartesian coordinates.

Case (II): $K = m^2, m \geq 0$

Considering this case, solving equation (29) and then using (27), we have the solution for stream function as

$$\Psi(x, y) = B_1(\theta)e^{m\xi} + B_2(\theta)e^{-m\xi} + \frac{2}{m^2}(A + B) \quad (41)$$

Now using (27) (28) and (41) in (24) with suitable choice of

$$A = -V \cot \theta, \quad B = V \tan \theta \quad (42)$$

where

$$\theta \in (-\pi, \pi) \setminus \left\{ -\frac{\pi}{2}, \frac{\pi}{2} \right\}$$

We have

$$S_{\xi\xi} = \{B_3(\theta)e^{m\xi} + B_4(\theta)e^{-m\xi}\} \xi + B_5(\theta)e^{m\xi} + B_6(\theta)e^{-m\xi}. \quad (43)$$

where

$$S = \mu \left\{ (B_1(\theta)e^{m\xi} + B_2(\theta)e^{-m\xi}) + \frac{4V}{m^2} \cot 2\theta \right\} \quad (44)$$

and

$$B_3(\theta) = -\frac{2KVR_e B_1(\theta)}{m}, \quad B_4(\theta) = \frac{2KVR_e B_2(\theta)}{m} \quad (45)$$

$$B_5(\theta) = \frac{B_1(\theta)R_e K U(\cos \theta - \sin \theta)}{m^3}, \quad B_6(\theta) = -\frac{B_2(\theta)R_e K U(\cos \theta - \sin \theta)}{m^3}$$

using (39) and (40) we have

$$\Psi(x, y) = B_1(\theta)e^{m\xi} + B_2(\theta)e^{-m\xi} - \frac{4V}{m^2} \cot 2\theta \quad (46)$$

Again using (46) and (21), we have the stream function as

$$\begin{aligned} \psi(x, y) = & B_1(\theta)e^{m\xi} + B_2(\theta)e^{-m\xi} - \frac{4V}{m^2} \cot 2\theta \\ & + U(x + y) + V(x^2 \cot \theta - y^2 \tan \theta). \end{aligned} \quad (47)$$

Now on solving equation (43) we have

$$\begin{aligned} S = & \frac{1}{m^2} (B_3(\theta)e^{m\xi} + B_4(\theta)e^{-m\xi}) \xi + B_7(\theta)e^{m\xi} + B_8(\theta)e^{-m\xi} \\ & + B_9(\theta)\xi + B_{10}(\theta) \end{aligned} \quad (48)$$

where $B_7(\theta) = \left(\frac{B_5(\theta)}{m^2} - 2\frac{B_3(\theta)}{m^3}\right)$, $B_8(\theta) = \left(\frac{B_6(\theta)}{m^2} + 2\frac{B_4(\theta)}{m^3}\right)$, $B_9(\theta)$ and $B_{10}(\theta)$ are arbitrary constants. Again using (17), (21), (28) and (35) we have

$$H_{\xi\xi} = (B_{11}(\theta) + B_0\xi) H_{\xi} \quad (49)$$

where $B_{11}(\theta) = R_{\sigma}(\cos \theta - \sin \theta)U$ and $B_0 = -2R_{\sigma}$.

Now solving equation (49) we have

$$H = B_{12}(\theta) \int e^{(B_{11}(\theta) + B_0\xi)^2} d\xi + B_{13}(\theta) \quad (50)$$

where $B_{12}(\theta)$ and $B_{13}(\theta)$ are arbitrary constants depending on parameter θ .

4. Result and Discussion

Stream lines patterns have been obtained for two different cases $k = -n^2$ and $k = m^2$. In the analysis we observe that when quadratic perturbation term, $Ax^2 + By^2$ dominates over uniform perturbation term $U(x + y)$ then stream lines are hyperbolic in nature and there exist stagnation point, Figure 1 and 2. If uniform perturbation term $U(x + y)$ dominates over quadratic term ($Ax^2 + By^2$) then stream lines obtained are of wavy nature including some closed graphs, Figure 1(a), (b), (c), (d), (e) and (f).

Moreover for the solution in the case (II) when $k = m^2$ where the exponential terms dominates over all other perturbed terms resulting no wavy solution and hence there are stream lines patterns of hyperbolic nature including some closed curves shown in Figure 2(g), (h), (i), (j) and (k).

At last we have shown some viscosity variation in three dimensional surface graphs shown in Figure 3(a), (b), (c), (d) which shows there are significant variation of viscosity in the flow field corresponding to the considered kinematical condition for the vorticity.

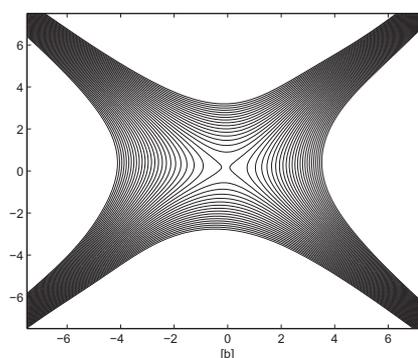
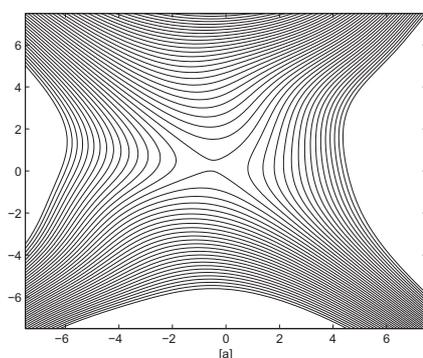
5. Concluding Remark

In this paper we have taken the vorticity distribution proportional to the stream function perturbed by the more general quadratic stream and obtained the exact solution of equations of motion of a finitely conducting incompressible fluid of variable viscosity under the presence of transverse magnetic field. Transformation of variables has been used to find the solutions in terms of one dimensional variable, depending on the single parameter θ . Moreover, by means of graphs, we have presented stream line patterns and variation of viscosity corresponding to the various solutions. Further, we can get more exact solutions by taking different values of parameter θ , U and V . Moreover the result in all cases for $A = B = 0$, recovers the result of Jamil and Khan (2006). This guarantees the correctness of the mathematical calculations.

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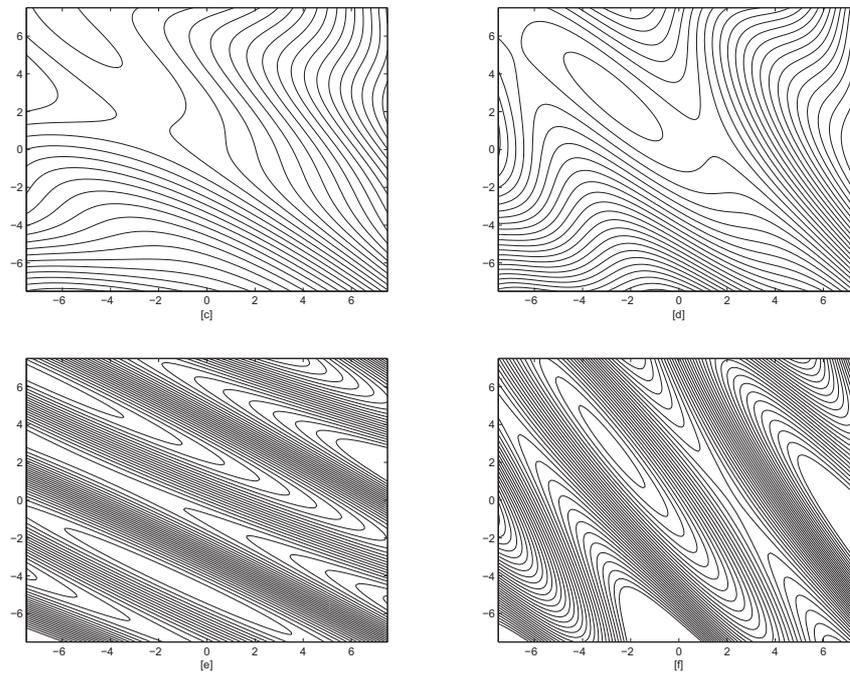


Figure 1.

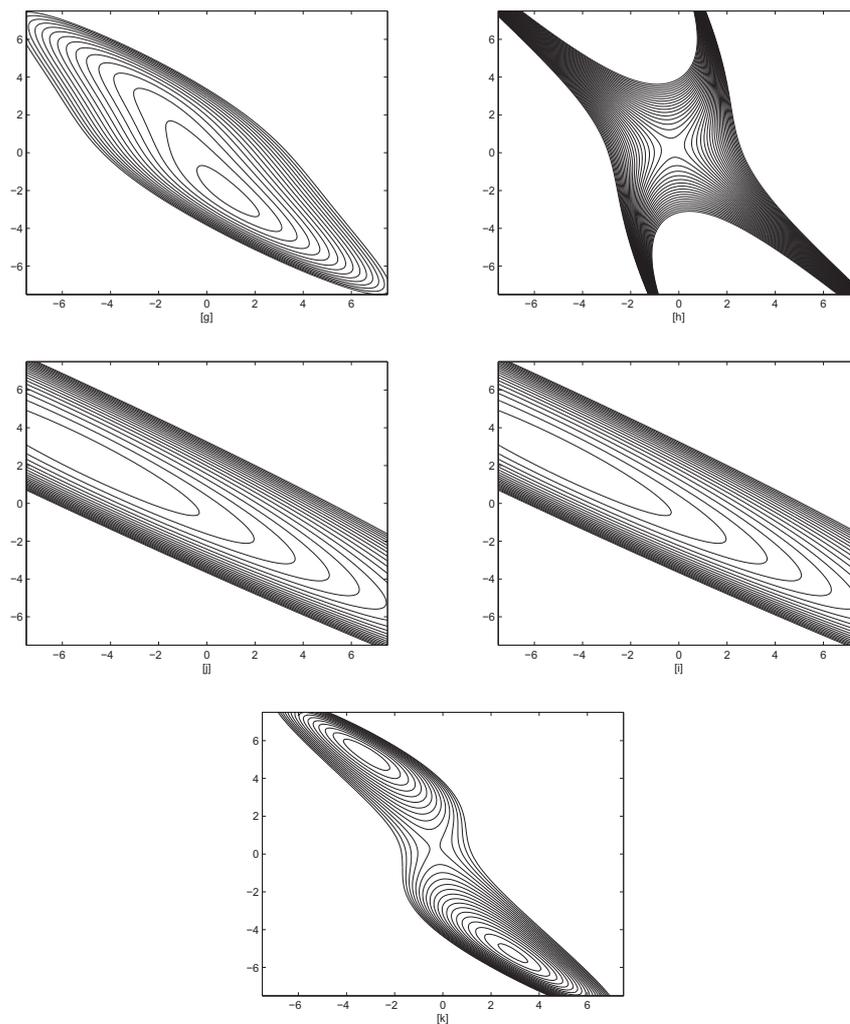


Figure 2.

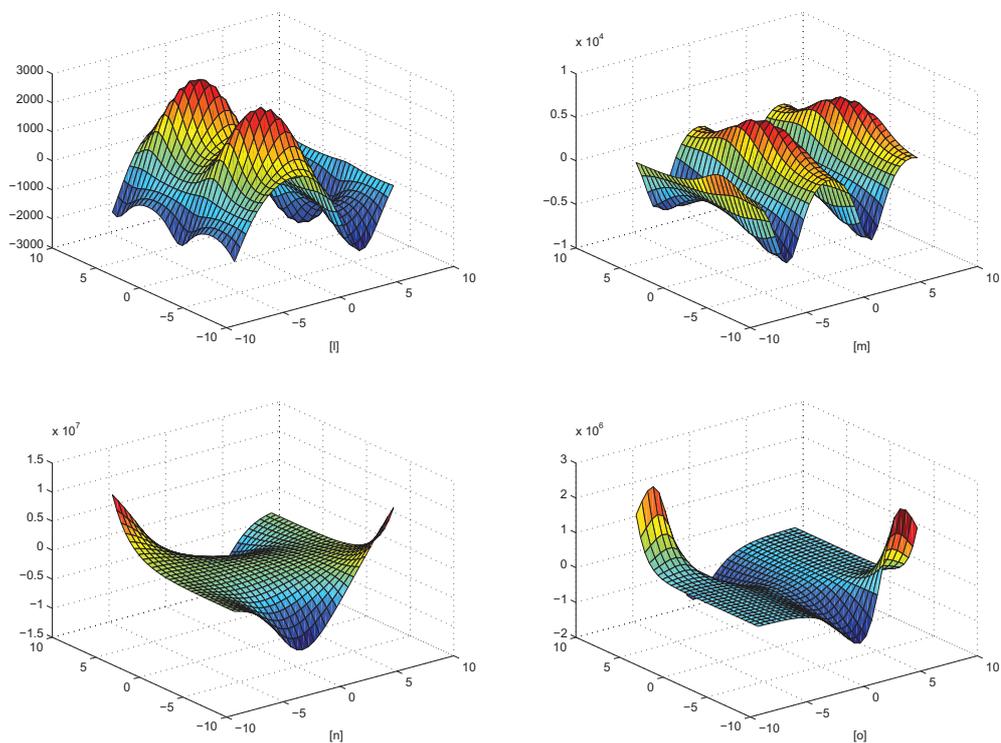


Figure 3.