# On the Solution of the Black-Sholes Equation with Jump Process

A. Kananthai & C. Bunpog

Department of Mathematics, Chiang Mai University

Chiang Mai 50200, Thailand

&

Center of Excellence in Mathematics, Faculty of Science at Mahidol University

272 Rama 6 Road, Phayathai Campus, Ratchathewi, Bangkok, 10400, Thailand

E-mail: malamnka@science.cmu.ac.th

*The research is financed by Center of Excellence in Mathematics Postgraduate Education and Research Development Office Faculty of Science at Mahidol University for financial support.(Sponsoring information)*

**Abstract**

In this paper, we study the well known equation that is the Black-Scholes equation by considering its solution for the case of jump processes, particularly the jumps of prices of the stock models can be interpreted by the concepts of distribution theory.

**AMS Subject Classification:** 91B28

**Keywords:** The Black-Scholes equation, Jump process

## 1. Introduction

In the year 1973, F. Black and M. sholes has first introduced the well known equation that can be solved for the call option of the stocks. Such equation is named the Black-Scholes equation which is given by

$$\frac{\partial}{\partial t}u(s,t) + rs\frac{\partial}{\partial s}u(s,t) + \frac{\sigma^2 s^2}{2}\frac{\partial^2}{\partial s^2}u(s,t) - ru(s,t) = 0 \tag{1}$$

with the terminal condition

$$u(s_T, T) = (s_T - k)^+ \tag{2}$$

denotes $(s_T - k)^+ = max(s_T - k, 0)$, for $0 \leq t \leq T$ where $u(s,t)$ is the option price at time $t$, $r$ is the interest rate, $s = s(t)$ is the price of stock at time $t$, $s_T$ is the price of stock on the expiration date at time $T$, $k$ is the strike price and $\sigma$ is the volatility of stock. They obtain such solution

$$\begin{aligned} u(s,t) =& s\Phi\Big(\frac{\ln\left(\frac{s}{k}\right) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\Big) \\ &- ke^{-r(T-t)}\Phi\Big(\frac{\ln\left(\frac{s}{k}\right) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\Big) \end{aligned} \tag{3}$$

which is call the Black-Sholes Formula where $\Phi$ denote by

$$\Phi(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{z^2}{2}}dz,$$

(J.Michael, 2001). In this paper, we study such solution for the case of the jumps of $s$. Let $s = s(t)$ be the prices of the stock having the jumps of magnitude $a_1, a_2, a_3, \ldots, a_m$ at time $t = t_1, t_2, t_3, \ldots, t_m$ respectively for $0 < t_1 < t_2 < t_3, \ldots < t_m < T$. By using the jumps in the distributional sense (I.M, 1964, p22) and the method of option prices with the stock model. We let

$$x(t) = s(t) - \sum_{i=1}^{m} a_i H(t - t_i)$$

where

$$H(t - t_i) = \begin{cases} 1, & \text{for } t_i < t; \\ 0, & \text{for } t < t_i. \end{cases}$$

is a Heaviside function, and $x(t)$ is a continuous functions for all $t$. Thus

$$s(t) = x(t) + \sum_{i=1}^{m} a_i H(t - t_i).$$

If we put

$$u(s,t) = u\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i), t\Big) = v(x(t), t),$$

we obtain the Black-Sholes in (1) for the jump processes in the form

$$\frac{\partial}{\partial t} v(x(t),t) + r\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big)\frac{\partial}{\partial x} v(x(t),t) +$$
$$\frac{\sigma^2}{2}\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big)^2 \frac{\partial^2}{\partial x^2} v(x(t),t) - rv(x(t),t) = 0 \tag{4}$$

with the terminal condition

$$v(x(T),T) = \Big(x(T) + \sum_{i=1}^{m} a_i H(T-t_i) - k\Big)^+, \tag{5}$$

we obtain

$$v(x(t),t) = \Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big)$$
$$\Phi\Big[\frac{\ln\Big(\frac{x(t)+\sum_{i=1}^{m} a_i H(t-t_i)}{k}\Big) + \Big(r + \frac{\sigma^2}{2}\Big)(T-t)}{\sigma\sqrt{T-t}}\Big]$$
$$- ke^{-r(T-t)}\Phi\Big[\frac{\ln\Big(\frac{x(t)+\sum_{i=1}^{m} a_i H(t-t_i)}{k}\Big) + \Big(r - \frac{\sigma^2}{2}\Big)(T-t)}{\sigma\sqrt{T-t}}\Big] \tag{6}$$

as a solution of (3) for the case of jump processes. For the case of no jumps that is $a_i = 0$, equation (6) reduces to (1) and (5) reduces to (2).

**2. Preliminaries**

Let us consider the stock model

$$ds = \mu s dt + \sigma s dB \tag{7}$$

where $s = s(t)$ is the price of a stock at time $t$, $\mu$ is a drift and $\sigma$ is a volatility of the stock $s$ and $B$ is the Wiener process or Brownian motion. Suppose $s(t)$ has jumps of magnitude $a_1, a_2, a_3, \ldots, a_m$ at time $t = t_1, t_2, t_3, \ldots, t_m$ respectively. Let

$$x(t) = s(t) - \sum_{i=1}^{m} a_i H(t-t_i) \tag{8}$$

where

$$H(t-t_i) = \begin{cases} 1, & \text{for } t_i < t; \\ 0, & \text{for } t < t_i. \end{cases}$$

is a Heaviside function.

Now $x(t)$ is a continuous functions for all $t$ and the derivative $\frac{dx(t)}{dt} = \frac{ds(t)}{dt}$ except $t = t_1, t_2, t_3, \ldots, t_m$. Differentiable both sides of (8) and obtain

$$\frac{dx(t)}{dt} = \frac{ds(t)}{dt} - \sum_{i=1}^{m} a_i \delta(t-t_i) \tag{9}$$

where $\frac{ds(t)}{dt}$ is the derivative of $s(t)$ in the distributional sense.

Now from (7), (8) and (9) we obtain

$$dx(t) + \sum_{i=1}^{m} a_i \delta(t-t_i)dt = \mu\Big[x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big]dt$$
$$+ \sigma\Big[x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big]dB,$$

$$dx(t) = \mu x(t)dt + \sigma x(t)dB + \mu\sum_{i=1}^{m} a_i H(t-t_i)dt$$
$$+ \sigma\sum_{i=1}^{m} a_i H(t-t_i)dB - \sum_{i=1}^{m} a_i \delta(t-t_i)dt.$$

Let

$$J(t) = \mu \sum_{i=1}^{m} a_i H(t - t_i)dt + \sigma \sum_{i=1}^{m} a_i H(t - t_i)dB - \sum_{i=1}^{m} a_i \delta(t - t_i)dt. \tag{10}$$

Thus

$$dx(t) = \mu x(t)dt + \sigma x(t)dB + J(t). \tag{11}$$

Now from (1) we have $u(s, t)$ is the option price at time $t$. Let

$$\begin{aligned} u(s, t) &= u\Big(x(t) + \sum_{i=1}^{m} a_i H(t - t_i), t\Big) \\ &= v(x(t), t). \end{aligned}$$

By using Itô chain rule, we have

$$dv(x(t), t) = \frac{\partial v(x(t), t)}{\partial t} dt + \frac{\partial v(x(t), t)}{\partial x} dx(t) + \frac{1}{2}\frac{\partial^2 v(x(t), t)}{\partial x^2}(dx(t))^2.$$

By (11),

$$\begin{aligned} dv =& \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}[\mu x(t)dt + \sigma x(t)dB + J(t)] \\ &+ \frac{1}{2}\frac{\partial^2 v}{\partial x^2}[\mu x(t)dt + \sigma x(t)dB + J(t)]^2 \end{aligned}$$

$$\begin{aligned} dv =& \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}\Big[\mu x(t)dt + \sigma x(t)dB + J(t)\Big] \\ &+ \frac{1}{2}\frac{\partial^2 v}{\partial x^2}\Big[\mu^2 x^2(t)(dt)^2 + \sigma^2 x^2(t)(dB)^2 + J^2(t) \\ &+ 2\mu\sigma x^2(t)dtdB + 2\mu x(t)dtJ(t) + 2\sigma x(t)dBJ(t)\Big]. \end{aligned}$$

Since, we have $(dB)^2 \approx dt$, thus $dtdB \approx (dt)^{3/2}$. Now, $(dt)^2$ and $(dt)^{3/2}$ are not first order, so we discard the terms $(dt)^2$ and $dtdB$.

From (10), we have

$$\begin{aligned} J^2(t) =& \mu^2\Big(\sum_{i=1}^{m} a_i H(t - t_i)\Big)^2 (dt)^2 + \sigma^2\Big(\sum_{i=1}^{m} a_i H(t - t_i)\Big)^2 (dB)^2 \\ &+ \Big(\sum_{i=1}^{m} a_i \delta(t - t_i)\Big)^2 (dt)^2 + 2\mu\sigma\Big(\sum_{i=1}^{m} a_i H(t - t_i)\Big)^2 dtdB \\ &- 2\mu \sum_{i=1}^{m} a_i H(t - t_i) \sum_{i=1}^{m} a_i \delta(t - t_i)(dt)^2 \\ &- 2\sigma \sum_{i=1}^{m} a_i H(t - t_i) \sum_{i=1}^{m} a_i \delta(t - t_i)dtdB \end{aligned}$$

since $(dB)^2 \approx dt$ and the terms $(dt)^2$ and $dtdB$ are cancelled, Thus

$$J^2(t) = \sigma^2\Big(\sum_{i=1}^{m} a_i H(t - t_i)\Big)^2 d.t$$

Now consider the term $dtJ(t)$ and $dBJ(t)$, the same as before, $dtJ(t) = 0$ and

$$dBJ(t) = \sigma \sum_{i=1}^{m} a_i H(t - t_i)dt.$$

Thus

$$\begin{aligned} dv =& \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}\Big[\mu x(t)dt + \sigma x(t)dB + J(t)\Big] + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}\Big[\sigma^2 x^2(t)dt \\ &+ 2\sigma^2 x(t) \sum_{i=1}^{m} a_i H(t - t_i)dt + \sigma^2\Big(\sum_{i=1}^{m} a_i H(t - t_i)\Big)^2 dt\Big], \end{aligned}$$

or

$$dv = \frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial x}\Big[\mu x(t)dt + \sigma x(t)dB + J(t)\Big] + \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2}\Big[x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big]^2 dt. \tag{12}$$

Let $u(s,t) = \phi s + \psi p$ where $\phi$ is a number of shares of stock and $\psi$ is a number of bonds and $p$ is the value of a bond. Now, we have $du(s,t) = \phi ds + \psi dp$ and

$$\begin{aligned} ds &= dx(t) + \sum_{i=1}^{m} a_i \delta(t-t_i) \\ &= \mu\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big)dt + \sigma\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big)dB. \end{aligned}$$

Thus

$$\begin{aligned} du(s,t) &= du\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i), t\Big) \\ &= \phi\Big[\mu\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big)dt \\ &\quad + \sigma\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big)dB\Big] + \psi r p dt \end{aligned}$$

where $dp = rpdt$, $r$ is the interest rate. We set $u(s,t) = v(x(t),t)$ thus

$$dv = \phi\Big[\mu\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big)dt + \sigma\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big)dB\Big] + \psi r p dt \tag{13}$$

we equate the $dv$ from (12) and (13) and

$$\begin{aligned} &\mu x(t)dt + \sigma x(t)dB + J(t) \\ &= \mu\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big)dt + \sigma\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big)dB \end{aligned}$$

and choose $\phi = \frac{\partial v}{\partial x}$, we then obtain

$$\frac{\partial v}{\partial t}dt + \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2}\Big[x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big]^2 dt = \psi r p dt. \tag{14}$$

Now

$$\begin{aligned} u(s,t) &= u\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big) = v(x(t),t) \\ &= \phi s + \psi p = \frac{\partial v}{\partial x}\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big) + \psi p \end{aligned}$$

thus

$$\begin{aligned} \psi p &= v(x(t),t) - \frac{\partial v}{\partial x}\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big) \\ r\psi p dt &= \Big(rv(x(t),t) - r\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big)\frac{\partial v}{\partial x}\Big)dt. \end{aligned} \tag{15}$$

From (14) and (15)

$$\begin{aligned}\Big(\frac{\partial v}{\partial t}dt + \frac{\sigma^2}{2}\Big(x + \sum_{i=1}^{m} a_i H(t-t_i)\Big)^2 \frac{\partial^2 v}{\partial x^2}\Big)dt \\ = \Big(rv - r\Big(x + \sum_{i=1}^{m} a_i H(t-t_i)\Big)\frac{\partial v}{\partial x}\Big)dt\end{aligned}$$

thus

$$\begin{aligned}&\frac{\partial v}{\partial t} + r\Big(x + \sum_{i=1}^{m} a_i H(t-t_i)\Big)\frac{\partial v}{\partial x} \\ &+ \frac{\sigma^2}{2}\Big(x + \sum_{i=1}^{m} a_i H(t-t_i)\Big)^2 \frac{\partial^2 v}{\partial x^2} - rv = 0 \qquad (16)\end{aligned}$$

Equation (16) is the Black-Sholes P.D.E with jump processes.
Now Put

$$R = x + \sum_{i=1}^{m} a_i H(t-t_i),$$

thus

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial R}\cdot\frac{\partial R}{\partial x} = \frac{\partial v}{\partial R}$$

and

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 v}{\partial R^2}\cdot\frac{\partial R}{\partial x} = \frac{\partial v^2}{\partial R^2}$$

thus (16) becomes

$$\frac{\partial v(R,t)}{\partial t} + rR\frac{\partial v(R,t)}{\partial R} + \frac{\sigma^2}{2}R^2\frac{\partial^2 v(R,t)}{\partial R^2} - rv(R,t) = 0 \qquad (17)$$

which is the Black-Scholes equation with $R = x(t) + \sum_{i=1}^{m} a_i H(t-t_i)$ and $0 \le t \le T$. From (2), we have the terminal condition $u(s,T) = (s_T - k)^+$, $k$ is the strike price. Since $s_T = x(T) + \sum_{i=1}^{m} a_i H(T-t_i)$, thus

$$v(R,T) = (R_T - k)^+ \qquad (18)$$

$R_T = x(T) + \sum_{i=1}^{m} a_i H(T-t_i)$ is the price of stock at time $t = T$.

In this work, we study the equation (17) with the terminal condition (18). So, we can say that the equation (1) with the terminal condition (2) is the Black-Sholes equation with no jump. If we have jump processes, we obtain (16). If we put $R = x(t) + \sum_{i=1}^{m} a_i H(t-t_i)$, we obtain (17) with the terminal condition (18). We next study the Black-Sholes formula with jump process. Recall that, for the call option price today is

$$v(s,T) = s_0 N(d_+(T,s_0)) - ke^{-rT}N(d_-(T,s_0))$$

which is called the Black-Sholes formula where $s_0 = s(0)$, $s_T = s_0\exp\Big[(r - \frac{\sigma}{2})T + \sigma B(T)\Big]$, for $t = T$, $0 \le t \le T$, $B(T)$ is the Brownian motion at time $t = T$, $r$ is the interest rate, $\sigma$ is the volatility of the stock $s$, $d_\pm(T,s_0) = \frac{1}{\sigma\sqrt{T}}\Big[\ln\Big(\frac{s_0}{k}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)T\Big]$, $k$ is the strike price and $N$ is the cumulative standard normal distribution function

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-z^2/2}dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-z^2/2}dz.$$

For the jump processes, we have $s(t) = x(t) + \sum_{i=1}^{m} a_i H(t-t_i)$.
Now, for $t = T$, $s(T) = s_T = x(T) + \sum_{i=1}^{m} a_i H(T-t_i)$
and $t = 0$, $s(0) = x(0)$, since $\sum_{i=1}^{m} a_i H(-t_i) = 0$ for $t_i > 0$
or $s_0 = x_0 = x(0)$. Thus for the call option to day, we obtain

$$v\Big(x(T) + \sum_{i=1}^{m} a_i H(T-t_i), T\Big) = x_0 N(d_+(T,x_0)) - ke^{-rT}N(d_-(T,x_0)).$$

Thus, we see that the call options to day for the jump and no jump are the same and can be computed from the same Black-Sholes formula.

We next study the solutions of the Black-sholes equation given by (17) with the given terminal condition (18). From the Black-Sholes formula we try the solution

$$v(R,t) = RN(d_+(T-t,R)) - ke^{-r(T-t)}N(d_-(T-t,R)) \tag{19}$$

where

$$d_\pm = \frac{1}{\sigma\sqrt{T-t}}\left[\ln\left(\frac{R}{k}\right) + \left(r \pm \frac{\sigma^2}{2}\right)(T-t)\right], \tag{20}$$

$0 \le t \le T$. We show that $v(R,t)$ satisfies (17).
At first we can verify that

$$ke^{-r(T-t)}N'(d_-) = RN'(d_+) \tag{21}$$

see [1, p.193], where $R = x(t) + \sum_{i=1}^{m} a_i H(t-t_i)$, $N'(d_\pm)$ is the derivative of $N(d_\pm)$. From (19), (20)and (21), we can compute

$$\frac{\partial v(R,t)}{\partial R} = N(d_+), \quad \frac{\partial^2 v(R,t)}{\partial R^2} = \frac{1}{\sigma R\sqrt{T-t}}N'(d_+)$$

and also

$$\frac{\partial v(R,t)}{\partial R} = rke^{-r(T-t)}N(d_-) - \frac{\sigma R}{2\sqrt{T-t}}N'(d_+) \qquad \text{see[1, p.159]}$$

substitute into (17),

$$\begin{aligned} &rke^{-r(T-t)}N(d_-) - \frac{\sigma R}{2\sqrt{T-t}}N'(d_+) + rRN(d_+) \\ &+ \frac{\sigma^2}{2}R^2 \cdot \frac{1}{\sigma R\sqrt{T-t}}N'(d_+) - rRN(d_+) - rke^{-r(T-t)}N(d_-) = 0 \end{aligned}$$

thus lefthand side are cancelled to be zero. That implies (17) holds. It follows that $v(R,t)$ given by (19) is the solution of (17).

## 3. Main results

The work in the preliminaries section, starting form (7)-(21) leading to the main theorem.

**Theorem 1.** *Given the Black-Scholes equation*

$$\frac{\partial v(R,t)}{\partial t} + rR\frac{\partial v(R,t)}{\partial R} + \frac{\sigma^2}{2}R^2\frac{\partial^2 v(R,t)}{\partial R^2} - rv(R,t) = 0 \tag{22}$$

*with the terminal condition*

$$v(R,T) = (R_T - k)^+ \tag{23}$$

*where $R = x(t) + \sum_{i=1}^{m} a_i H(t-t_i)$ and $R_T = x(T) + \sum_{i=1}^{m} a_i H(T-t_i)$ is the price of stock at time $t = T$ for the jump processes, $r$ is the interest rate and $\sigma$ is the volatility of the stock. Then we obtain the unique solution of* (22) *satisfies* (23) *which is given by*

$$v(R,t) = RN(d_+(T-t,R)) - ke^{-r(T-t)}N(d_-(T-t,R)), \tag{24}$$

*and*

$$R = x(t) + \sum_{i=1}^{m} a_i H(t-t_i), \tag{25}$$

*thus, for the jumps case, we obtain*

$$\begin{aligned} &v\Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i), t\Big) \\ &\qquad = \Big(x(t) + \sum_{i=1}^{m} a_i H(t-t_i)\Big)N(d_+) - ke^{-r(T-t)}N(d_-) \end{aligned} \tag{26}$$

*as the solution of the Black-Sholes equation with jump processes.*

**Proof.**

Now, we have

$$H(t-t_i) = \begin{cases} 1, & \text{for } t_i < t; \\ 0, & \text{for } t < t_i. \end{cases} \qquad i = 1, 2, \ldots, m$$

Thus for $t_i < t$ in (25) we obtain the solution

$$v = \Big(x(t) + \sum_{i=1}^{m} a_i\Big)N(d_+) - ke^{-r(T-t)}N(d_-)$$

where $\sum_{i=1}^{m} a_i$ is the sum of magnitude of the jumps of the prices of stock for $m$ times of jumps at time $t_1, t_2, t_3, \ldots, t_m$.

We can also show that $v(R, t)$ a unique solution of (22) which satisfies (23). Consider

$$N(d_\pm(T-t, R)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_\pm(T-t,R)} e^{-y^2/2} dy$$

where $d_\pm(T-t, R) = \frac{1}{\sigma\sqrt{T-t}}\Big[\ln\Big(\frac{R}{k}\Big) + \Big(r \pm \frac{\sigma^2}{2}\Big)(T-t)\Big]$.
Thus $\lim\limits_{t\to T} d_\pm(T-t, R) = \infty$ for $R > k$
and $\lim\limits_{t\to T} d_\pm(T-t, R) = -\infty$ for $0 < R < k$ see[1,p.193].
Thus $\lim\limits_{t\to T} N(d_\pm(T-t, R)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-y^2/2} dy = \frac{\sqrt{2\pi}}{\sqrt{2\pi}} = 1$ for $R > k$
and $\lim\limits_{t\to T} N(d_\pm(T-t, R)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{-\infty} e^{-y^2/2} dy = 0$ for $0 < R < k$.
Thus, from (24) we obtain

$$v(R, t) = \begin{cases} R_T - k, & \text{for } R > k; \\ 0, & \text{for } 0 < R < k. \end{cases}$$

where $R_T$ is the price of stock at $t = T$. Thus $v(R, t) = (R_T - k)^+$.

It follows that $v(R, t)$ satisfies the terminal condition (23). That implies $v(R, t)$ is the unique solution of (22). □

**Acknowledgment**

The author would like to thank Center of Excellence in Mathematics Postgraduate Education and Research Development Office Faculty of Science at Mahidol University for financial support.

**References**

Black, F and Scholes, M. (1973). *The pricing of options and corporale liabities*. J.Polit Econ, 81,637-659.

I.M Gelfund and G.E. shilov. (1964). *Generalized Functions*. Academic press,1.

Steven E. Shreve. (2004). *Stochastic Calculus for Finance II*. Spring Finance, Springer Science, Business Media, Inc.

J. Michael Steele. (2001). *Stochastic Calculus and Financial Applicaton*, Springer-Verlog New York, Inc.