On the Elementary Solution of the Operator $\circledast_B^k$

Somboon Niyom

Department of Mathematics, Chiangmai University

Chiangmai, 50200, THAILAND

E-mail: sbniyom@hotmail.com

Amnuay Kananthai (Corresponding author)

Department of Mathematics, Chiangmai University

Chiangmai, 50200, THAILAND

E-mail: malamnka@science.cmu.ac.th

The research is financed by Office of the Higher Education Commission, Thailand(Sponsoring information)

Abstract

In this paper, we study the elementary solution of the operator $\circledast_B^k$ which is defined by

$$\circledast_B^k = \left[\left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right)^3 + \left(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\right)^3\right]^k,$$

where $p + q = n$ is the dimension of $\mathbb{R}_n^+ = \{(x = x_1, x_2, \ldots, x_n) : x_1 > 0, x_2 > 0, \ldots, x_n > 0\}$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i}\frac{\partial}{\partial x_i}$, $2v_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$, $x_i > 0$, $i = 1, 2, \ldots, n$ and k is a positive integer. After that, we apply such an elementary solution to solve the equation $\circledast_B^k u(x) = f(x)$, where f is a generalized function and u is an unknown function.

Keywords: Dirac delta distribution, Tempered distribution, Fourier-Bessel transform, Bessel operator

1. Introduction

I. M. Gelfand and G. E. Shilov (1964) have first introduced the elementary solution of the $n-$dimensional classical diamond operator. S. E. Trione has shown that the $n-$dimensional ultra-hyperbolic equation has $u(x) = R_{2k}(x)$ as unique elementary solution. Later, M. A. Tellez has proved that $R_{2k}(x)$ exists only for case p is odd with $p + q = n$. A. Kananthai has showed that the solution in the convolution form $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution of the $\diamondsuit^k u(x) = \delta$. Furthermore, M. Z. Sarikaya and H. Yildirim have introduced the Bessel diamond operator and have proved that the convolution solution $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is the unique elementary solution of the $\diamondsuit_B^k u(x) = \delta$, where $\diamondsuit_B^k$ is the Bessel diamond operator iterated k times with $x \in \mathbb{R}_n^+$,

$$\diamondsuit_B^k = \left[\left(B_{x_1} + B_{x_2} + \cdots + B_{x_p}\right)^2 - \left(B_{x_{p+1}} + \cdots + B_{x_{p+q}}\right)^2\right]^k, p + q = n. \tag{1}$$

The Bessel diamond operator can be expressed in the form $\diamondsuit_B = \Box_B \triangle_B = \triangle_B \Box_B$, where $\triangle_B$ is the Laplace-Bessel operator which is defined by

$$\triangle_B = B_{x_1} + B_{x_2} + \cdots + B_{x_n}, \tag{2}$$

and $\Box_B$ is the Bessel ultra-hyperbolic operator which is defined by

$$\Box_B = B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - B_{x_{p+2}} - \cdots - B_{x_{p+q}}. \tag{3}$$

In this paper, at first we study the elementary solution of the $\circledast_B^k$ operator, that is

$$\circledast_B^k G(x) = \delta, \tag{4}$$

where $G(x)$ is the elementary solution of such equation, δ is the Dirac delta distribution, k is nonnegative integer and the

$\circledast_B$ operator is defined by

$$\begin{aligned}
\circledast_B &= \left(\sum_{i=1}^{p} B_{x_i}\right)^3 + \left(\sum_{i=p+1}^{p+q} B_{x_i}\right)^3 \\
&= \left[\sum_{i=1}^{p} B_{x_i} + \sum_{i=p+1}^{p+q} B_{x_i}\right]\left[\left(\sum_{i=1}^{p} B_{x_i}\right)^2 - \sum_{i=1}^{p} B_{x_i}\sum_{i=p+1}^{p+q} B_{x_i} + \left(\sum_{i=p+1}^{p+q} B_{x_i}\right)^2\right] \\
&= \triangle_B\left[\triangle_B^2 - \frac{3}{4}\left(\triangle_B + \Box_B\right)\left(\triangle_B - \Box_B\right)\right] \\
&= \frac{3}{4}\triangle_B\Box_B^2 + \frac{1}{4}\triangle_B^3 \\
&= \frac{3}{4}\Diamond_B\Box_B + \frac{1}{4}\triangle_B^3. \qquad (5)
\end{aligned}$$

After that, we apply such an elementary solution to solve for the solution of the equation $\circledast_B^k G(x) = f(x)$, where $f(x)$ is a generalized function and $u(x)$ is an unknown function for $x \in \mathbb{R}_n^+$.

2. Preliminaries

The generalized shift operator, T_x^y has the following form (B.M. Levitan, 1951, p.102-143),

$$T_x^y = C_v^* \int_0^\pi \cdots \int_0^\pi \varphi(s_1, \ldots, s_n)\left(\prod_{i=1}^{n} \sin^{2v_i-1}\theta_i\right) d\theta_1 \cdots d\theta_n,$$

where $s_i^2 = x_i^2 + y_i^2 - 2x_iy_i\cos\theta_i$, $x, y \in \mathbb{R}_n^+$ and $C_v^* = \prod_{i=1}^n \frac{\Gamma(v_i+1)}{\Gamma(\frac{1}{2})\Gamma(v_i)}$. We remark that this shift operator is closely connected with the Bessel differential operator (B.M. Levitan, 1951, p.102-143),

$$\begin{aligned}
\frac{d^2\varphi}{dx_i^2} + \frac{2v_i}{x_i}\frac{d\varphi}{dx_i} &= \frac{d^2\varphi}{dy_i^2} + \frac{2v_i}{y_i}\frac{d\varphi}{dy_i}, \\
\varphi(x_i, 0) &= f(x), \\
\varphi_{y_i}(x_i, 0) &= 0,
\end{aligned}$$

where $x_i, y_i \in \mathbb{R}_n^+$ for $i = 1, 2, \ldots, n$. The convolution operator denoted by T_x^y is defined as follows

$$(f * \varphi)(x) = \int_{\mathbb{R}_n^+} f(y) T_x^y \varphi(x)\left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy. \qquad (6)$$

Convolution in (6) is known as a B-convolution. We note the following properties of the B-convolution and the generalized shift operator,

(a) $T_x^y \cdot 1 = 1$.

(b) $T_x^0 \cdot f(x) = f(x)$.

(c) If $f(x), g(x) \in C(\mathbb{R}_n^+)$, $g(x)$ is a bounded function for $x \in \mathbb{R}_n^+$ and

$$\int_{\mathbb{R}_n^+} |f(x)|\left(\prod_{i=1}^{n} x_i^{2v_i}\right) dx < \infty,$$

then

$$\int_{\mathbb{R}_n^+} T_x^y f(x) g(y)\left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy = \int_{\mathbb{R}_n^+} f(y) T_x^y g(x)\left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy.$$

(d) From (c), we have the following equality for $g(x) = 1$,

$$\int_{\mathbb{R}_n^+} T_x^y f(x)\left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy = \int_{\mathbb{R}_n^+} f(y)\left(\prod_{i=1}^{n} y_i^{2v_i}\right) dy.$$

(e) $(f * g)(x) = (g * f)(x)$.

The Fourier-Bessel transformation and its inverse transformation are defined as follows (H. Yildirim, 1995),

$$(F_B f)(x) = C_v \int_{\mathbb{R}_n^+} f(y) \left(\prod_{i=1}^{n} J_{v_i - \frac{1}{2}}(x_i y_i) y_i^{2v_i} \right) dy,$$

$$(F_B^{-1} f)(x) = (F_B f)(-x),\ C_v = \left(\prod_{i=1}^{n} 2^{v_i - \frac{1}{2}} \Gamma\left(v_i + \frac{1}{2}\right) \right)^{-1},$$

where $J_{v_i-\frac{1}{2}}(x_i y_i)$ is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. There are following equalities for Fourier-Bessel transformation (H. Yildirim, 1995),

$$F_B \delta(x) = 1 \text{ and } F_B(f * g)(x) = F_B f(x) \cdot F_B g(x).$$

Lemma 1. *There is a following equality for Fourier-Bessel transformation*

$$F_B(|x|^{-\alpha}) = 2^{n+2|v|-2\alpha} \Gamma\left(\frac{n+2|v|-\alpha}{2}\right) \left[\Gamma\left(\frac{\alpha}{2}\right)\right]^{-1} |x|^{\alpha-n-2|v|},$$

where $|v| = v_1 + v_2 + \cdots + v_n$.

Proof. (H. Yildirim, 1995).

Lemma 2. *Given the equation* $\triangle_B^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$, *where* $\triangle_B^k$ *is the Laplace-Bessel operator iterated k-times defined by* (2). *Then* $u(x) = (-1)^k S_{2k}(x)$ *is an elementary solution of the* $\triangle_B^k$ *operator, where*

$$S_{2k}(x) = \frac{2^{n+2|v|-4k} \Gamma(\frac{n+2|v|-2k}{2})}{\prod_{i=1}^{n} 2^{v_i-\frac{1}{2}} \Gamma(v_i + \frac{1}{2}) \Gamma(k)} |x|^{2k-n-2|v|}. \tag{7}$$

Proof. (H. Yildirim, 1995).

Lemma 3. *Given the equation* $\Box_B^k u(x) = \delta(x)$ *for* $x \in \Gamma_+ = \{x \in \mathbb{R}_n^+ : x_1 > 0, x_2 > 0, \cdots, x_n > 0$ *and* $V > 0\}$, *where* $\Box_B^k$ *is the Bessel-ultra-hyperbolic operator iterated k-times defined by* (3). *Then* $u(x) = R_{2k}(x)$ *is an elementary solution of the* $\Box_B^k$ *operator, where*

$$R_{2k}(x) = \frac{V^{\frac{2k-n-2|v|}{2}}}{K_n(2k)} \tag{8}$$

for

$$V = x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$$

and

$$K_n(2k) = \frac{\pi^{\frac{n+2|v|-1}{2}} \Gamma(\frac{2+2k-n-2|v|}{2}) \Gamma(\frac{1-2k}{2}) \Gamma(2k)}{\Gamma(\frac{2+2k-p-2|v|}{2}) \Gamma(\frac{p-2k}{2})}$$

Proof. (H. Yildirim, M. Z. Sarikaya and S. Öztürk, 2004, p.375-387).

Lemma 4. *The functions* $S_{2k}(x)$ *and* $R_{2k}(x)$ *are homogeneous distributions of order* $(2k - n - 2|v|)$ *for* $Re(2k) < n + 2|v|$. *In particular, the B-convolution* $S_{2k}(x) * R_{2k}(x)$ *exists and is a tempered distribution.*

Proof. (H. Yildirim, M. Z. Sarikaya and S. Öztürk, 2004, p.375-387).

Lemma 5. *Given the equation* $\Diamond_B^k u(x) = \delta(x)$ *for* $x \in \mathbb{R}_n^+$, *where* $\Diamond_B^k$ *is the diamond Bessel operator iterated k-times defined by* (1). *Then* $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ *is an elementary solution of the* $\Diamond_B^k$ *operator.*

Proof. (H. Yildirim, M. Z. Sarikaya and S. Öztürk, 2004, p.375-387).

Lemma 6. *Let k and r be nonnegative integer.*

(a) *Let* $S_{2k}(x)$ *and* $S_{2r}(x)$ *be defined by* (7), *then* $S_{2k}(x) * S_{2r}(x) = S_{2k+2r}(x)$.

(b) *Let* $R_{2k}(x)$ *and* $R_{2r}(x)$ *be defined by* (8), *then* $R_{2k}(x) * R_{2r}(x) = R_{2k+2r}(x)$.

Proof. (M. Z. Sarikaya and H. Yildirim, 2009, p.18-22).

Lemma 7. *The convolution $S_{6k}(x) * R_{4k}(x)$ exists and is a tempered distribution where $S_{6k}(x) = S_{2k}(x) * S_{2k}(x) * S_{2k}(x)$ and $R_{4k}(x) = R_{2k}(x) * R_{2k}(x)$ such that $S_{2k}(x)$ and $R_{2k}(x)$ are defined by* (7) *and* (8)*, respectively.*

Proof. Since $S_{2k}(x) * R_{2k}(x)$ exists and is a tempered distribution, by W.F. Donoghue (1969, p.156-159), we obtain $S_{6k}(x) * R_{4k}(x)$ exists and is a tempered distribution. □

Lemma 8. *Let $S_6(x)$ with $k = 3$ and $R_4(x)$ with $k = 2$ be defined by* (7) *and* (8) *respectively. Then*

(a) $\Diamond_B \Box_B (S_6(x) * R_4(x)) = -S_4(x)$,

(b) $\triangle_B^3 (S_6(x) * R_4(x)) = -R_4(x)$.

Proof. (a) We obtain

$$\begin{aligned}\Diamond_B \Box_B (S_6(x) * R_4(x)) &= \Diamond_B \Box_B \left((-1)^2 S_6(x) * R_4(x)\right) \\ &= \Diamond_B ((-1)S_2(x) * R_2(x)) * \Box_B (R_2(x)) * (-1)S_4(x) \\ &= \delta(x) * \delta(x) * (-1)S_4(x) \\ &= -S_4(x).\end{aligned}$$

(b) We get

$$\begin{aligned}\triangle_B^3 (S_6(x) * R_4(x)) &= \triangle_B^3 \left((-1)^4 (S_6(x) * R_4(x)\right) \\ &= \triangle_B^3 \left((-1)^3 S_{2(3)}(x)\right) * (-1)R_4(x) \\ &= \delta(x) * (-1)R_4(x) \\ &= -R_4(x).\end{aligned}$$

□

3. Main results

Theorem 1. *Given the equation*

$$\circledast_B^k G(x) = \delta(x), \tag{9}$$

then $G(x) = S_{6k}(x) * R_{4k}(x) * \left(C^{*k}(x)\right)^{*-1}$ is a Green function for the $\circledast_B^k$ operator iterated k-times where $\circledast_B$ is defined by (5)*, δ is the Direc delta distribution, $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_n^+$, k is a nonnegative integer and*

$$C(x) = -\left[\frac{3}{4} S_4(x) + \frac{1}{4} R_4(x)\right], \tag{10}$$

$C^{*k}(x)$ denotes the convolution of $C(x)$ itself k-times, $\left(C^{*k}(x)\right)^{*-1}$ denotes the inverse of $C^{*k}(x)$ in the convolution algebra. Moreover $C^{*k}(x)$ is a tempered distribution.

Proof. Since $\circledast_B = \frac{3}{4} \Diamond_B \Box_B + \frac{1}{4} \triangle_B^3$, by (9) we obtain

$$\left[\frac{3}{4} \Diamond_B \Box_B + \frac{1}{4} \triangle_B^3\right] \left[\frac{3}{4} \Diamond_B \Box_B + \frac{1}{4} \triangle_B^3\right]^{k-1} G(x) = \delta(x).$$

By Lemma 7 with $k = 1$, $S_6(x) * R_4(x)$ exists and is a tempered distribution. Convolving both side of the above equation by $S_6(x) * R_4(x)$, we have

$$\left[\frac{3}{4} \Diamond_B \Box_B + \frac{1}{4} \triangle_B^3\right] (S_6(x) * R_4(x)) * \left[\frac{3}{4} \Diamond_B \Box_B + \frac{1}{4} \triangle_B^3\right]^{k-1} G(x) = (S_6(x) * R_4(x)) * \delta(x).$$

By Lemma 8, we obtain

$$C(x) * \left[\frac{3}{4} \Diamond_B \Box_B + \frac{1}{4} \triangle_B^3\right]^{k-1} G(x) = S_6(x) * R_4(x).$$

Keeping on convolving both sides of the above equation by $S_6(x) * R_4(x)$ up to $k - 1$ times, we have

$$C^{*k}(x) * G(x) = (S_6(x) * R_4(x))^{*k}.$$

where the symbol $*k$ denotes the convolution of itself k-times. By M.A. Tellez (1994), we get

$$(S_6(x) * R_4(x))^{*k} = S_{6k}(x) * R_{4k}(x).$$

Therefore,

$$C^{*k}(x) * G(x) = S_{6k}(x) * R_{4k}(x). \tag{11}$$

Since $S_4(x)$ and $R_4(x)$ are lies in S' where S' is a space of tempered distribution, $C(x) \in S'$. By W.F. Donoghue (1996, p. 152), we obtain $C^{*k}(x) \in S'$. Since $S_{6k}(x) * R_{4k}(x) \in S'$, choose $S' \subset D'_R$ where D'_R is the right-side distribution which is a subspace of D' of distribution. Thus $S_{6k}(x) * R_{4k}(x) \in D'_R$, it follows that $S_{6k}(x) * R_{4k}(x)$ is an element of the convolution algebra. By A.H. Zemanian (1964, p. 150-151) the equation (11) has an unique solution

$$G(x) = S_{6k}(x) * R_{4k}(x) * \left(C^{*k}(x)\right)^{*-1}$$

where $\left(C^{*k}(x)\right)^{*-1}$ is an inverse of $C^{*k}(x)$ in the convolution algebra, $G(x)$ is called the elementary solution of the $\circledast_B^k$ operator. Since $S_{6k}(x) * R_{4k}(x)$ and $\left(C^{*k}(x)\right)^{*-1}$ are tempered distribution, by W.F. Donoghue (1996, p. 152), we obtain $S_{6k}(x) * R_{4k}(x) * \left(C^{*k}(x)\right)^{*-1}$ is a tempered distribution. It follows that $G(x)$ is a tempered distribution. □

Theorem 2. *Given the equation*

$$\circledast_B^k u(x) = f(x) \tag{12}$$

where f is a given generalized function and $u(x)$ is an unknown function, we obtain

$$u(x) = G(x) * f(x)$$

is an unique solution of (12) *where $G(x)$ is an elementary solution for the operator $\circledast_B^k$.*

Proof. Convolving both sides of the equation (12) by the Green function $G(x)$ of the $\circledast_B^k$ operator in Theorem 1, we obtain

$$G(x) * \circledast_B^k u(x) = G(x) * f(x)$$

or

$$\circledast_B^k G(x) * u(x) = G(x) * f(x).$$

Applying Theorem 1, we have

$$\delta(x) * u(x) = G(x) * f(x)$$

or

$$u(x) = G(x) * f(x).$$

Since $G(x)$ is an unique, $u(x)$ is an unique solution of the equation (12). □

Acknowledgment

This work was granted by Office of the Higher Education Commission. Mr.Somboon Niyom was supported by CHE Ph.D. scholarship and Graduate School, Chiangmai University, Thailand.

References

A. H. Zemanian. (1964). *Distribution Theory and Transform Analysis*. Mc-Graw Hill, New York.

A. Kananthai. (1997). On the solution of the n-dimensional Diamond operator. *Applied Mathematics and Computation*, 88, p. 27-37.

B. M. Levitan. (1951). *Expansion in Fourier series and integrals with Bessel functions (N.S.)*. Uspeki Mat. Nauka, 2 (42), p. 102-143 (in Russian).

H. Yildirim. (1995). *Riesz Potentials Generated by a Gemeralized Shift Operator*. Ph. D. Thesis, Ankara University .

H. Yildirim, M. Z. Sarikaya and S. Öztürk. (2004). The solution of the n-dimensional Bessel diamond operator and the Fourier-Bessel transform of their convolution. *Proc. Indian Acad. Sci. (Math. Sci.)*, 114 (4), p. 375-387.

I. M. Gelfand and G. E. Shilov. (1964). *Generalized Function*. Vol. I, Academic Press, New York.

M. A. Tellez. (1995-96). The convolution product of $W_\alpha(u,m) * W_\beta(u,m)$. *Mathematicae Notae*, 38.

M. A. Tellez. (1994). The distribution Hankel transform of Marcel Rieszs ultra-hyperbolic kernel. *Studies in Applied Mathematics*, 93.

M. Z. Sarikaya and H. Yildirim. (2009). *On the B-convolutions of the Bessel diamond kernel of Riesz*. Appl. Math. Comp. 208, p. 18-22.

S. E. Trione. (1987). *On Marcel Rieszs ultra-hyperbolic kernel*. Trabajos de Math, 116.

W. F. Donoghue. (1969). *Distributions and Fourier transforms*. Academic Press, New York.

W.Satsanit, A.Kananthai.(2010). On the operator $\circledast^k$ Relate to Heat Equation. *Journal of Mathematics Research*, Vol.2, No.2, p.20-27.