On Anti Fuzzy Ideals in Left Almost Semigroups

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Abstract

In this paper we have studied the concept of Anti fuzzy ideals in Left Almost Semigroups (LA-semigroup in short). The equivalent statement for an LA-semigroup to be a commutative semigroup is proved. The set of all anti fuzzy left ideals, which are idempotents, forms a commutative monoid. Moreover it has been shown that the union of any family of Anti fuzzy left ideals of an LA-semigroup is an anti fuzzy left ideal of $F(S)$. The relation of anti fuzzy left(right) ideals, anti fuzzy interior ideals and anti fuzzy bi-ideals in LA-semigroups has been studied. Anti fuzzy points have been defined in an LA-semigroup and has been shown the representation of largest fuzzy left ideal generated by a fuzzy point.

Keywords: LA-semigroup, Anti fuzzy ideal, Anti fuzzy point

1. Introduction

An LA-semigroup (Kazim, M. A. 1972), is a groupoid S whose elements satisfy the left invertive law,

$$(ab)c = (cb)a \text{ for all } a, b \text{ and } c \text{ in } S. \tag{1}$$

An LA-semigroups is a midway between a groupoid and a commutative semigroup. P. Holgate (1992) has explored this useful non-associative structure and called it as simple invertive groupoid. It has wide applications in theory of flocks (Naseeruddin, N. 1970). In an LA-semigroups the medial law (Kazim, M. A. 1972),

$$(ab)(cd) = (ac)(bd) \text{ holds for all } a, b, c \text{ and } d \text{ in } S. \tag{2}$$

If there exists an element e in an LA-semigroups S such that $ex = x$ for all x in S then S is called an LA-semigroups with left identity e. It is to be noted that if LA-semigroups S contains a right identity then it becomes commutative monoid. If an LA-semigroups S contains left identity then paramedial law,

$$(ab)(cd) = (dc)(ba) \text{ holds for all } a, b, c \text{ and } d \text{ in } S. \tag{3}$$

Also $a(bc) = b(ac)$ holds for all a, b and c in an LA-semigroups with left identity. In an LA-semigroups S, an element $a \in S$ is called idempotent if $a^2 = a$.

Rosenfeld was the first who studied fuzzy sets in the structure of groups (Rosenfeld, A. 1971). Kuroki (1981) has studied the bi-ideals in semigroups. A fuzzy subset f of a set S is a function from S to a closed interval $[0, 1]$. The concept of a fuzzy set was introduced by Zadeh in 1965. Anti fuzzy bi-ideals have been studied in semigroups in (Shabir, M. 2009). Here in this paper we have generalized the concept of anti fuzzy ideals for LA-semigroups and proved some interesting results. Some preliminaries are given below.

Let $F(S)$ denote the collection of all fuzzy subsets of an LA-semigroup S. For subsets A, B of S, $AB = \{ab \in S : a \in A, b \in B\}$. A non-empty subset A of S is called left(right) ideal of S if $SA \subseteq A$ ($AS \subseteq A$). Further A is called two-sided ideal if it is both left and right ideal of S. A non-empty subset A of S is called an interior ideal of S if $(SA)S \subseteq A$. A non-empty LA-subsemigroup A of S is called bi-ideal of S if $(AS)A \subseteq A$. A non-empty subset A of S is called idempotent if $AA = A$.

Let f and g be two fuzzy subsets of an LA-semigroup S. The anti product $f * g$ is defined by

$$(f * g)(x) = \begin{cases} \bigwedge\limits_{x=yz} \{f(y) \vee g(z)\}; & \text{if } \exists\, y \text{ and } z \in S, \text{ such that } x = yz, \\ 1 & \text{otherwise.} \end{cases}$$

A fuzzy subset of S is called an anti fuzzy LA-subsemigroup of S if $f(ab) \leq f(a) \vee f(b)$ for all a and b in S, and is called an anti fuzzy left(right) ideal of S if $f(ab) \leq f(b)$ ($f(ab) \leq f(a)$) for all a and b in S. A fuzzy subset f of S is called an anti fuzzy two sided ideal(or an anti fuzzy ideal) of S if it is both anti fuzzy left and anti fuzzy right ideal of S. A fuzzy subset f of an LA-semigroup S is called a fuzzy bi-ideal of S if $f((xy)z) \leq f(x) \vee f(z)$ for all x, y and z of S.

A fuzzy subset f of an LA-semigroup S is called an anti fuzzy interior ideal of S if $f((xa)y) \leq f(a)$ for all x, a and y of S. It can be easily seen that, if A is a non-empty subset of an LA-semigroup S, then A is an interior ideal of S if and only if C_{A^c} is an anti fuzzy interior ideal of S. A fuzzy subset f of S is called an anti fuzzy idempotent if $f * f = f$. For a subset A of S the anti characteristic function, C_{A^c} is defined by $C_{A^c} = \{.0$ if $x \in A$,1 if $x \notin A$.

Note that an LA-semigroup S can be considered as an anti fuzzy subset of itself (denoted by Θ) and we write $\Theta(x) = 0$, for all x in S. Let a be an arbitrary element of S, then for λ in $[0, 1)$ and for x in S we define anti fuzzy point a_λ of S as; $a_\lambda(x) = \{.\lambda$ if $x = a$,1 otherwise..

Proposition 1. *Let S be an LA-semigroup, then the set $(F(S), *)$ is an LA-semigroup.*

proof. Clearly $F(S)$ is closed. Let f, g and h be in $F(S)$. Let x be any element of S such that it is not expressible as product of two elements in S then we have, $((f * g) * h)(x) = 1 = ((h * g) * f)(x)$. Let the element x can be written as product of two elements in S then we have,

$$\begin{aligned} ((f * g) * h)(x) &= \bigwedge_{x=yz} \{(f * g)(y) \vee h(z)\} \\ &= \bigwedge_{x=yz} \left\{ \bigwedge_{y=pq} \{f(p) \vee g(q)\} \vee h(z) \right\} \\ &= \bigwedge_{x=(pq)z} \{f(p) \vee g(q) \vee h(z)\} \\ &= \bigwedge_{x=(zq)p} \{h(z) \vee g(q) \vee f(p)\} \\ &= \bigwedge_{x=wp} \left\{ \bigwedge_{w=zq} (h(z) \vee g(q) \vee f(p)) \right\} \\ &= \bigwedge_{x=wp} \{(h * g)(w) \vee f(p)\} \\ &= ((h * g) * f)(x). \end{aligned}$$

Hence $(F(S), *)$ is an LA-semigroup.

Corollary 1. *Let S be an LA-semigroup, then the medial law holds in $F(S)$.*

proof. Let $f, g, h,$ and k be arbitrary elements of $F(S)$. By successive use of left invertive law, $(f * g) * (h * k) = ((h * k) * g) * f = ((g * k) * h) * f = (f * h) * (g * k)$.

Theorem 1. *Let S be an LA-semigroup with left identity, then the following properties hold in $F(S)$;*

(i) $f * (g * h) = g * (f * h)$ *for all f, g and h in $F(S)$,*

(ii) $(f * g) * (h * k) = (k * h) * (g * f)$ *for all f, g, h and k in $F(S)$.*

proof. (i) Let x be an arbitrary element of S. If x is not expressible as a product of two elements in S, then $(f * (g * h))(x) =$

$1 = (g * (f * h))(x)$. Let there exists y and z in S such that $x = yz$, then

$$\begin{aligned}(f * (g * h))(x) &= \bigwedge_{x=yz} \{f(y) \vee (g * h)(z)\} \\ &= \bigwedge_{x=yz} \left\{ f(y) \vee \bigwedge_{z=pq} \{g(p) \vee h(q)\} \right\} \\ &= \bigwedge_{x=y(pq)} \{f(y) \vee g(p) \vee h(q)\} \\ &= \bigwedge_{x=p(yq)} \{g(p) \vee f(y) \vee h(q)\} \\ &= \bigwedge_{x=pw} \left\{ g(p) \vee \bigwedge_{w=yq} \{f(y) \vee h(q)\} \right\} \\ &= \bigwedge_{x=pw} \{g(p) \vee (f * h)(w)\} \\ &= (g * (f * h))(x).\end{aligned}$$

Thus, $(f * (g * h))(x) = (g * (f * h))(x)$. If z is not expressible as a product of two elements in S, then$(f * (g * h))(x) = 1 = (g * (f * h))(x)$. Hence, $(f * (g * h))(x) = (g * (f * h))(x)$ for all x in S.

(ii) If any element x of S is not expressible as product of two elements in S at any stage, then$((f * g) * (h * k))(x) = 1 = ((k * h) * (g * f))(x)$. Let there exists y, z in S such that $x = yz$, then

$$\begin{aligned}((f * g) * (h * k))(x) &= \bigwedge_{x=yz} \{(f * g)(y) \vee (h * k)(z)\} \\ &= \bigwedge_{x=yz} \left\{ \bigwedge_{y=pq} \{f(p) \vee g(q)\} \vee \bigwedge_{z=uv} \{h(u) \vee k(v)\} \right\} \\ &= \bigwedge_{x=(pq)(uv)} \{f(p) \vee g(q) \vee h(u) \vee k(v)\} \\ &= \bigwedge_{x=(vu)(qp)} \{k(v) \vee h(u) \vee g(q) \vee f(p)\} \\ &= \bigwedge_{x=mn} \left\{ \bigwedge_{m=vu} \{k(v) \vee h(u)\} \vee \bigwedge_{n=qp} \{g(q) \vee f(p)\} \right\} \\ &= \bigwedge_{x=mn} \{(k * h)(m) \vee (g * f)(n)\} \\ &= ((k * h) * (g * f))(x).\end{aligned}$$

Proposition 2. *An LA-semigroup S with $F(S) = (F(S))^2$ is commutative semigroup if and only if $(f * g) * h = f * (h * g)$ holds for all fuzzy subsets f, g and h of S.*

proof. It is simple.

Lemma 1. *A non-empty subset A of an LA-semigroup S is LA-subsemigroup if and only if C_{A^c} is an anti fuzzy LA-subsemigroup of S.*

proof. Let A be a non-empty subset of an LA-semigroup S and x and y be arbitrary elements of S. Let A be an LA-subsemigroup of S. Let x and y be in A then xy is also in A, so we have,

$$C_{A^c}(xy) = 0 = C_{A^c}(x) \vee C_{A^c}(y).$$

Now let x be in A and y in not in A then $C_{A^c}(x) = 0$ and $C_{A^c}(y) = 1$ and so we have,

$$C_{A^c}(xy) \leq 1 = C_{A^c}(x) \vee C_{A^c}(y).$$

Now let both x and y are not in A then $C_{A^c}(x) = 1$ and $C_{A^c}(y) = 1$ and so we have,

$$C_{A^c}(xy) \leq 1 = C_{A^c}(x) \vee C_{A^c}(y).$$

Thus for all x and y in S we have $C_{A^c}(xy) \leq C_{A^c}(x) \vee C_{A^c}(y)$, which implies that C_{A^c} is an anti fuzzy LA-subsemigroup of S.

Conversely, let C_{A^c} be an anti fuzzy LA-subsemigroup of S. If the elements x and y are in A then $C_{A^c}(x) = 0 = C_{A^c}(y)$. But $C_{A^c}(xy) \leq C_{A^c}(x) \vee C_{A^c}(y) = 0$, which implies that xy is in A. Hence A is an LA-subsemigroup of S.

Lemma 2. *A non-empty subset A of an LA-semigroup S left(right, two-sided) ideal of S if and only if C_{A^c} is an anti fuzzy left(right, two-sided) ideal of S.*

proof. Let A be a non-empty subset of an LA-semigroup S and is an anti fuzzy left ideal. Let x and y be arbitrary elements of S such that both x and y are in A. Then since A is left ideal so xy is also in A. Thus we have,

$$C_{A^c}(xy) = 0 = C_{A^c}(y).$$

Now let x be in A and y in not in A then $C_{A^c}(x) = 0$ and $C_{A^c}(y) = 1$ and so we have,

$$C_{A^c}(xy) \leq 1 = C_{A^c}(y).$$

Now let both x and y are not in A then $C_{A^c}(x) = 1$ and $C_{A^c}(y) = 1$ and so we have,

$$C_{A^c}(xy) \leq 1 = C_{A^c}(y).$$

Thus for all x and y in S we have $C_{A^c}(xy) \leq C_{A^c}(y)$, which implies that C_{A^c} is an anti fuzzy left ideal of S.

Conversely, let C_{A^c} be an anti fuzzy left ideal of S. If the elements x and y are in A then $C_{A^c}(x) = 0 = C_{A^c}(y)$. But $C_{A^c}(xy) \leq C_{A^c}(y) = 0$, which implies that xy is in A. Hence A is left ideal of S.

Similarly we can prove the result for right and two sided ideal of S.

For non-empty subsets A and B of an LA-semigroup S, $C_A * C_B = C_{AB}$. It is easy to see that for every fuzzy subset f of an LA-semigroup S, we have $f \subseteq S$. The following lemmas have the same proof as in (Holgate, P. 1992).

Lemma 3. *Let f be a fuzzy subset of an LA-semigroup S, then the following properties hold.*

(i) f is an anti fuzzy LA-subsemigroup of S if and only if $f * f \supseteq f$.

(ii) f is an anti fuzzy left ideal of S if and only if $\Theta * f \supseteq f$.

(iii) f is an anti fuzzy right ideal of S if and only if $f * \Theta \supseteq f$.

(iv) f is an anti fuzzy ideal of S if and only if $\Theta * f \supseteq f$ and $f * \Theta \supseteq f$.

proof. It is same as in (Shabir, M. 2009).

Lemma 4. *Let S be an LA-semigroup. Then the following properties hold.*

(i) Let f and g be two anti fuzzy LA-subsemigroups of S. Then $f \cup g$ is also an anti fuzzy LA-subsemigroup of S.

(ii) The union of any family of anti fuzzy left(right, two-sided) ideal of S an anti fuzzy left(right, two-sided) ideal of S.

proof. *(i) Let* f and g be two anti fuzzy LA-subsemigroups of S and x, y be any two arbitrary elements of S, then $(f \cup g)(xy) = f(xy) \vee g(xy) \leq f(x) \vee f(y) \vee g(x) \vee g(y) = f(x) \vee g(x) \vee f(y) \vee g(y)$ thus we have $(f \cup g)(xy) \leq (f \cup g)(x) \vee (f \cup g)(y)$. Hence $f \cup g$ is an anti fuzzy LA-subsemigroup.

(ii) Let $\{f_i\}_{i \in I}$ be a family of anti fuzzy left ideals of S, for x and y in S we have

$$\begin{aligned}
\left(\bigcup_{i \in I} f_i\right)(xy) &= \bigvee_{i \in I} (f_i(xy)) \\
&\leq \bigvee_{i \in I} f_i(y) \\
&= \left(\bigcup_{i \in I} f_i\right)(y).
\end{aligned}$$

Hence $\bigcup_{i \in I} f_i$ is an anti fuzzy left ideal of S.

Lemma 5. *In an LA-semigroup with left identity $\Theta * \Theta = \Theta$.*

proof. Every x in S can be written as $x = ex$, where e is the left identity in S. So $(\Theta * \Theta)(x) = \bigwedge_{ex=yz} \{\Theta(y) \vee \Theta(z)\} \leq \{\Theta(e) \vee \Theta(x)\} = 0$. Hence $(\Theta * \Theta)(x) = 0 = \Theta(x)$ for all x in S.

Lemma 6. *In an LA-semigroup S with left identity, for every anti fuzzy left ideal f of S, we have* $(\Theta * f) = f$.

proof. It is sufficient to show that $\Theta * f \subseteq f$. Now for any x in S, $(\Theta * f)(x) = \bigwedge_{x=yz} \{\Theta(y) \vee f(z)\}$. Since $x = ex$, for all x in S, as e is left identity in S, so $\bigwedge_{ex=yz} \{\Theta(y) \vee f(z)\} \leq \Theta(e) \vee f(x) = f(x)$.

Proposition 3. Let S be an LA-semigroup with left identity and f and k are fuzzy left ideals in S then for any fuzzy subsets g and h of S, $f * g = h * k$ implies that $g * f = k * h$.

proof. Since f and h are fuzzy left ideals in S so by above lemma 4, $\Theta * f = f$ and $\Theta * h = h$. Now $g * f = (\Theta * g) * f = (f * g) * \Theta = (h * k) * \Theta = (\Theta * k) * h = k * h$.

The following corollary is direct consequence of the successive use of left invertive law in fuzzy LA-semigroup shown in proposition 1.

Theorem 2. *If S is an LA-semigroup then $Q = \{f \mid f \in \Theta,\ f * h = f \text{ where } h = h * h\}$ is a commutative subsemigroup with identity of S.*

proof. It is simple.

Let S be an LA-semigroup and a_λ be an anti fuzzy point in S. The largest anti fuzzy left ideal of S containing a_λ is called anti fuzzy left ideal of S generated by a_λ. Let us by $\langle a_\lambda \rangle_L$ we shall mean an anti fuzzy left ideal of S generated by a_λ.

Theorem 3. *Let S be an LA-semigroup with left identity e and a_λ be an anti fuzzy point in S, then $\langle a_\lambda \rangle_L = f$,*

$$where f(x) = \begin{cases} \lambda \in [0, 1) \; if \text{ there exists } b \in S \text{ such that } x = ba \\ 1 \qquad\qquad\qquad\qquad\qquad\qquad otherwise. \end{cases}$$

proof. Let x and y be arbitrary elements of S. Let $f(y) = 1$ then $f(xy) \leq 1 = f(y)$ and if $f(y) \neq 1$ then there exists an element b in S such that $y = ba$. Now, $f(xy) = f(x(ba)) = f((ex)(ba)) = f((ab)(xe)) = f(((xe)b)a) = \lambda \leq f(y)$, which implies that f is an anti fuzzy left ideal in S. Since $a = ea$, so $f(a) = f(ea) = \lambda$ and hence a_λ is in f. Let g be any other anti fuzzy left ideal of S containing a_λ, then $g(a) \leq \lambda$. Let $f(x) = 1$ for all x in S then $g(x) \leq 1 = f(x)$. On the other hand if $f(x) = \lambda$ then there exists b in S such that $x = ba$ and $g(x) = g(ba) \leq g(a) \leq \lambda = f(x)$, which implies that $g \subseteq f$. Hence f is the largest anti fuzzy left ideal generated by a in S.

Proposition 4. *Every idempotent anti fuzzy left ideal of an LA-semigroup S is an anti fuzzy ideal of S.*

proof. Let f be an anti fuzzy left ideal of S, which is idempotent. Consider, $f * \Theta = (f * f) * \Theta = (\Theta * f) * f \supseteq f * f = f$.

Theorem 4. Let f is an anti fuzzy idempotent in an LA-semigroup S with left identity, then $\Theta * f$ is an idempotent.

proof. Let f be an anti fuzzy idempotent in S, by lemma 3 and using medial law we have $(\Theta * f) * (\Theta * f) = (\Theta * \Theta) * (f * f) = (\Theta * f)$.

Theorem 5. Let S be an LA-semigroup with left identity, then the collection of all anti fuzzy left ideals of S, which are idempotent forms a commutative monoid.

proof. Let $\hat{H}$ denote the collection of all anti fuzzy left ideals which are idempotent in S. Here, $\hat{H}$ is non-empty, since by lemma 3, $\Theta * \Theta = \Theta$ implies that S is in $\hat{H}$. Consider f, g in $\hat{H}$, then $(f * g) * (f * g) = (f * f) * (g * g) = f * g$, also by corollary 1, $\Theta * (f * g) = (\Theta * \Theta) * (f * g) = (\Theta * f) * (\Theta * g) \supseteq f * g$. Also for every f, g in $\hat{H}$ by use of theorem 1 and corollary 1, we get $f * g = (f * g) * (f * g) = (g * f) * (g * f) = (g * g) * (f * f) = g * f$, that is commutative law holds in $\hat{H}$. Now, for any f, g, and h in $\hat{H}$ we have $(f * g) * h = (h * g) * f = f * (h * g) = f * (g * h)$. Since every f in $\hat{H}$ is an anti left ideal, so lemma 4 implies that $\Theta * f = f$. Commutativity implies that $\Theta * f = f * \Theta = f$, which implies that S is identity in $\hat{H}$ and every f is an anti fuzzy ideal in $\hat{H}$.

Lemma 7. *Let S be an LA-semigroup with left identity e, then every fuzzy right ideal is an anti fuzzy ideal.*

proof. Let f be an anti fuzzy right ideal in S, so $f * \Theta \supseteq f$. By lemma 3 and proposition 1, $\Theta * f = (\Theta * \Theta) * f = (f * \Theta) * \Theta \supseteq f * \Theta \supseteq f$. So f is an anti fuzzy left ideal, and hence an anti fuzzy ideal in S.

Remark 1. *If f is a fuzzy right ideal of an LA-semigroup S with left identity then $f \cup (\Theta * f)$ and $f \cup (f * f)$ are fuzzy two-sided ideals of S.*

Lemma 8. *If f is an anti fuzzy left ideal of an LA-semigroup S with left identity then $f \cup (f * \Theta)$ and $f \cup (f * f)$ are anti fuzzy two-sided ideals of S.*

proof. Consider, $(f \cup (f * \Theta)) * \Theta = (f * \Theta) \cup ((f * \Theta) * \Theta) = (f * \Theta) \cup ((\Theta * \Theta) * f) = (f * \Theta) \cup (\Theta * f) = (f * \Theta) \cup f = f \cup (f * \Theta)$. Hence $f \cup (f * \Theta)$ is fuzzy right ideal of S, and by lemma 5, $f \cup (f * \Theta)$ is fuzzy two-sided ideal of S. Now $(f \cup (f * f)) * \Theta = (f * \Theta) \cup (f * f) * \Theta = (f * \Theta) \cup ((\Theta * f) * f) \supseteq (f * \Theta) \cup (f * f) = (f * f) \cup (\Theta * f) \supseteq (f * f) \cup f = f \cup (f * f)$, implies that $f \cup (f * f)$ is a fuzzy right ideal of S. By lemma 5, $f \cup (f * f)$ is a fuzzy left ideal of S.

Theorem 6. *A non-empty subset A of an LA-semigroup S is a bi-ideal of S if and only if C_{A^c} is an anti fuzzy bi-ideal of S.*

proof. Let A be a non-empty subset of LA-semigroup S and is a bi-ideal of S. Since A is an LA-subsemigroup of S, then C_{A^c} is an anti fuzzy LA-subsemigroup of S. Let x, y and z be arbitrary elements of S such that both x and z are in A. Then since A is a bi-ideal so $((xy)z)$ is also in A. Thus we have,

$$C_{A^c}((xy)z) = 0 \leq C_{A^c}(x) \vee C_{A^c}(z).$$

Now let x be not in A and z is in A then $C_{A^c}(x) = 1$ and $C_{A^c}(z) = 0$ and so we have,

$$C_{A^c}((xy)z) \leq 1 = C_{A^c}(x) \vee C_{A^c}(z).$$

Now let both x and z are not in A then $C_{A^c}(x) = 1$ and $C_{A^c}(z) = 1$ and so we have,

$$C_{A^c}((xy)z) \leq 1 = C_{A^c}(x) \vee C_{A^c}(z).$$

Thus for all x, y and z in S we have $C_{A^c}((xy)z) \leq C_{A^c}(x) \vee C_{A^c}(z)$, which implies that C_{A^c} is an anti fuzzy bi-ideal of S.

Conversely, let C_{A^c} be an anti fuzzy bi-ideal of S. Since C_{A^c} is the anti fuzzy LA-subsemigroup of S so A is the LA-subsemigroup of S. If a belongs to $(AS)A$ that is $a = (xy)z$ for all x and z in A and y in S, then $C_{A^c}(x) = 0 = C_{A^c}(z)$. But $C_{A^c}((xy)z) \leq C_{A^c}(x) \vee C_{A^c}(z) = 0$, which implies that $((xy)z)$ is in A. Hence A is bi-ideal of S.

Lemma 9. *Let f be an anti fuzzy LA-subsemigroup of an LA-semigroup S. Then f is a fuzzy bi-ideal of S if and only if $(f * \Theta) * f \supseteq f$.*

proof. Let f be an anti fuzzy LA-subsemigroup of S, Let a be an arbitrary element of S. If there exist x and y in S such that $a = xy$, then

$$\begin{aligned} ((f * \Theta) * f)(x) &= \bigwedge_{a=xy} \{(f * \Theta)(x) \vee f(y)\} \\ &= \bigwedge_{a=xy} \left\{ \bigwedge_{x=pq} \{f(p) \vee \Theta(q)\} \vee f(y) \right\} \\ &= \bigwedge_{a=(pq)y} \{f(p) \vee f(y)\} \\ &\geq f((pq)y) = f(x) \end{aligned}$$

which implies $((f * \Theta) * f)(x) \geq f(x)$. If x is not expressible as a product of two elements then $((f * \Theta) * f)(x) = 1 \geq f(x)$. Hence $(f * \Theta) * f \supseteq f$.

Conversely, suppose that $(f * \Theta) * f \supseteq f$. Let x, y and z be arbitrary elements of S. Then

$$\begin{aligned} f((xy)z) &\leq ((f * \Theta) * f)((xy)z) \\ &= \bigwedge_{((xy)z)=ab} \{(f * \Theta)(a) \vee f(b)\} \\ &= \bigwedge_{((xy)z)=ab} \left\{ \bigwedge_{a=pq} \{f(p) \vee \Theta(q)\} \vee f(b) \right\} \\ &= \bigwedge_{((xy)z)=(pq)b} \{f(p) \vee f(b)\} \\ &\leq f(x) \vee f(z) \end{aligned}$$

Hence for all x, y and z in Θ, $f((xy)z) \leq f(x) \vee f(z)$ that is f is an anti fuzzy bi-ideal of S.

Lemma 10. *Let f and g be anti fuzzy right ideals of an LA-semigroup S with left identity. Then $f * g$ and $g * f$ are anti fuzzy bi-ideals of S.*

proof. By corollary 1, we have $(f * g) * (f * g) = (f * f) * (g * g) \supseteq f * g$. Hence $f * g$ is a fuzzy LA-subsemigroup of S. Now, by proposition 1, lemma 3 and corollary 1, we have $((f * g) * \Theta)) * (f * g) = ((f * g) * (\Theta * \Theta)) * (f * g) = ((f * \Theta) * (g * \Theta)) * (f * g) \supseteq (f * g) * (f * g) \supseteq f * g$. Similarly, $g * f$ is a bi-ideal of S.

Lemma 11. *Let f be an anti fuzzy LA-subsemigroup of an LA-semigroup S. Then f is an anti fuzzy interior ideal of S if and only if $(\Theta * f) * \Theta \supseteq f$.*

proof. Proof is similar to lemma 9.

Proposition 5. *Let S be an LA-semigroup. Then for any anti fuzzy left ideal, which is idempotent, in S, the following properties hold.*

(i) f is an anti fuzzy bi-ideal.

(ii) f is an anti fuzzy interior ideal.

proof. *(i)* Since a fuzzy subset f of S is an anti fuzzy left ideal so $f * f \supseteq f$. By corollary 1, we get $(f * \Theta) * f = (f * \Theta) * (f * f) = (f * f) * (\Theta * f) \supseteq f * f = f$.

(ii) Consider, $(\Theta * f) * \Theta \supseteq f * \Theta = (f * f) * \Theta = (\Theta * f) * f \supseteq f * f = f$, which implies that f is an anti fuzzy interior ideal of S.

Lemma 12. *Every fuzzy subset f of an LA-semigroup S with left identity is an anti fuzzy right ideal if and only if it is an anti fuzzy interior ideal.*

proof. Let every fuzzy subset f of S is an anti fuzzy right ideal. For x, a and y of S, consider $f((xa)y) \leq f(xa) = f((ex)a) = f((ax)e) \leq f(ax) \leq f(a)$, which implies that f is an interior ideal. Conversely, for any x and y in S we have, $f(xy) = f((ex)y) \leq f(x)$.

Lemma 13. *Let f be an anti fuzzy left ideal in an LA-semigroup S with left identity, then f being anti fuzzy interior ideal is an anti fuzzy bi-ideal of S.*

proof. Since f is an anti fuzzy left ideal in S, so $f(xy) \leq f(y)$ for all x and y in S. As e is left identity in S. So, $f(xy) = f((ex)y) \leq f(x)$, which implies that $f(xy) \leq f(x) \vee f(y)$ for all x and y in S. Thus f is an anti fuzzy LA-subsemigroup of S. For any x, y and z in S, we get $f((xy)z) = f((x(ey))z) = f((e(xy))z) \leq f(xy) = f((ex)y) \leq f(x)$. Also $f((xy)z) = f((zy)x) = f((z(ey))x) = f((e(zy))x) \leq f(zy) = f((ez)y) \leq f(z)$. Hence $f((xy)z) \leq f(x) \vee f(z)$ for all x, y and z in S.

Proposition 6. *Let f is a fuzzy subset of an LA-semigroup S with left identity. If f is an anti fuzzy left(right, two-sided) ideal in S then $f * f$ is an anti fuzzy ideal in S.*

proof. Let f be an anti fuzzy left ideal in an LA-semigroup S, then by lemma 1, we have $S * f \supseteq f$. By use of lemma 3 and corollary 1, $\Theta*(f * f) = (\Theta * \Theta)*(f * f) = (\Theta * f)*(\Theta * f) \supseteq f*f$. Also by proposition 1, $(f * f)*\Theta = (\Theta * f)*f \supseteq f*f$. If f is an anti fuzzy right ideal in S then by lemma 5, f is an anti fuzzy left ideal of S.

Corollary 2. *Let f is an anti fuzzy subset of an LA-semigroup S with left identity. If f is an anti fuzzy left ideal in S then $f * f$ is an anti fuzzy bi-ideal and an anti fuzzy interior ideal of S.*

proof. By proposition 6, $f * f$ is an anti fuzzy ideal in S. Now by lemmas 11 and 12, $f * f$ is an anti fuzzy interior and an anti fuzzy bi-ideal of S.

Theorem 7. *In an LA-semigroup S, every anti fuzzy ideal is an anti fuzzy bi-ideal and an anti fuzzy interior ideal of S.*

proof. Let f be an anti fuzzy ideal of an LA-semigroup S. Clearly f is an LA-subsemigroup of S by lemma 1, since $f * f \supseteq f$. Consider, $(f * \Theta) * f \supseteq f * f \supseteq f$, which by lemma 7, shows that f is a fuzzy bi-ideal of S. Now, consider $(\Theta * f) * \Theta \supseteq f * \Theta \supseteq f$, which by lemma 10 shows that f is a fuzzy interior ideal of S.

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