Necessary and Sufficient Condition of Existence for the Quadrature Surfaces Free Boundary Problem

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Abstract

Performing the shape derivative (Sokolowski and Zolesio, 1992) and using the maximum principle, we show that the so-called Quadrature Surfaces free boundary problem

$$Q_S(f,k)\begin{cases} -\Delta u_\Omega = f \quad \text{in } \Omega \\ u_\Omega = 0 \text{ on } \partial\Omega \\ |\nabla u_\Omega| = k \text{ (constant) on } \partial\Omega. \end{cases}$$

has a solution which contains strictly the support of f if and only if

$$\int_C f(x)dx > k\int_{\partial C} d\sigma.$$

Where C is the convex hull of the support of f. We also give a necessary and sufficient condition of existence for the problem $Q_S(f,k)$ where the term source f is a uniform density supported by a segment.

Keywords: Dirichlet problem, Quadrature surfaces, Shape derivative, Shape optimization

1. Introduction

Assuming throughout that: $D \subset \mathbb{R}^N \quad (N \geq 2)$ is a bounded ball which contains all the domains we use. If ω is an open subset of D, let ν be the outward normal to $\partial\omega$ and let $|\partial\omega|$ (respectively $|\omega|$) be the perimeter (respectively the volume) of ω.

Let $k > 0$ and let f be a positive function belonging to $L^2(\mathbb{R}^N)$ and having a compact support K with nonempty interior. Denote by C the convex hull of K and consider the following free boundary problem. Find an open set $\Omega \subset D$ which contains strictly C and such that the following problem has a solution:

$$Q_S(f,k)\begin{cases} -\Delta u_\Omega = f \quad \text{in } \Omega \\ u_\Omega = 0 \text{ on } \partial\Omega \\ |\nabla u_\Omega| = k \text{ on } \partial\Omega. \end{cases}$$

Notice that since $u_\Omega = 0$ on $\partial\Omega$ then $|\nabla u_\Omega| = -\dfrac{\partial u_\Omega}{\partial \nu}$, where ν is the outward normal vector to $\partial\Omega$.

Imposing boundary conditions for both u_Ω and $|\nabla u_\Omega|$ on $\partial\Omega$ makes problem $Q_S(f,k)$ *overdetermined*, so that in general without any assumptions on data this problem has no solution.

The problem $Q_S(f,k)$ is called the quadrature surfaces free boundary problem and arises in many areas of physics (free streamlines, jets, Hele-show flows, electromagnetic shaping, gravitational problems etc.) It has been intensively studied from different points of view, by several authors. For more details about the methods used for solving this problem see the (Gustafsson and Shahgholian, Introduction, 1996). Using the maximum principle together with the compatibility condition of the Neumann problem, the authors gave sufficient condition of existence for problem $Q_S(f,k)$ (Barkatou and al., 2005).

Thegoal of this paper is to prove the following

Theorem 1.1 The problem $Q_S(f,k)$ has a solution if and only if

$$(NS) \qquad \int_C f(x)dx > k|\partial C|.$$

This theorem says that the inequality (NS) is a necessary and sufficient condition of existence for the quadrature surfaces free boundary problem.

To see that (NS) is a necessary condition of existence, let us suppose that Ω is a smooth solution of $Q_S(f,k)$. Integrating the equation (1) over Ω and using the fact that u_Ω vanishes on $\partial\Omega$, the Green formula gives:

$$\int_\Omega f(x)dx = \int_{\partial\Omega} -\frac{\partial u_\Omega}{\partial \nu}(s)d\sigma = k|\partial\Omega|.$$

Now, since the convex C is strictly contained in Ω, we obtain (NS).

To prove that (NS) is sufficient to get a solution of $Q_S(f,k)$, we proceed as follows.

By using the shape derivative (Sokolowski and Zolesio, 1992), the problem $Q_S(f,k)$ seems to be the Euler equation of the following optimization problem. Put

$$O_{\varepsilon,C} = \{\mathrm{int}(C) \subset \omega \subset D;\ \omega \in O_\varepsilon\},$$

where O_ε is the class of the domains which satisfy the ε-cone property (Chenais, 1975).

Find $\Omega \in O_{\varepsilon,C}$ such that

$$J(\Omega) = \mathrm{Min}\{J(\omega),\ \omega \in O_{\varepsilon,C}\},$$

and

$$J(\omega) = \int_\omega (|\nabla u_\omega(x)|^2 - 2f(x)u_\omega(x) + k^2)dx.$$

u_ω is the solution of the following Dirichlet problem.

$$P(f,\omega)\begin{cases} -\Delta u_\omega = f & \text{in } \omega \\ u_\omega = 0 \text{ on } \partial\omega. \end{cases}$$

We begin by proving the following propositions.

Proposition 1.2

1. There exists $\Omega \in O_{\varepsilon,C}$ such that
$$J(\Omega) = \mathrm{Min}\{J(\omega),\ \omega \in O_{\varepsilon,C}\}.$$
2. If Ω is of class C^2, then
$$(I)\begin{cases} |\nabla u_\Omega| \le k \text{ on } \partial\Omega \cap \partial C \\ |\nabla u_\Omega| = k \text{ on } \partial\Omega \setminus \partial C. \end{cases}$$

Now, put

$$M_C = \frac{1}{|\partial C|}\int_C f(x)dx,$$

$$F(\omega) = \int_\omega (|\nabla u_\omega(x)|^2 - 2f(x)u_\omega(x) + M_C^2)dx, \text{ and}$$

$$O_\Omega = \{\omega \subset \Omega,\ \omega \in O_{\varepsilon,C}\},$$

u_ω being the solution of the problem $P(f,\omega)$. We prove

Proposition 1.3

1. There exists $\Omega^* \in O_\Omega$ such that
$$F(\Omega) = \mathrm{Min}\{F(\omega),\ \omega \in O_\Omega\}.$$
2. If Ω^* is of class C^2, then
$$(II)\begin{cases} |\nabla u_{\Omega^*}| \le M_C \text{ on } \partial\Omega^* \cap \partial C \\ |\nabla u_{\Omega^*}| \ge M_C \text{ on } \partial\Omega^* \cap \partial\Omega \\ |\nabla u_{\Omega^*}| = M_C \text{ on } \partial\Omega^* \setminus (\partial C \cup \partial\Omega). \end{cases}$$

Next, we prove by contradiction that (NS) is sufficient to solve $Q_S(f,k)$. The contradiction is obtained according to (I), (II) and by applying the maximum principle to Ω and Ω^*.

The paper is ended by two sections. Section 5 is concerned by the special case of the uniform density supported by a segment for which we obtain a necessary and sufficient condition of existence. Section 6 contains some remarks.

2. Preliminaries

2.1 Definitions and lemmas

Definition 2.1

Let K_1 and K_2 be two compact subsets of D. We call a Hausdorff distance of K_1 and K_2 (or briefly $d_H(K_1, K_2)$), the following positive number:

$$d_H(K_1, K_2) = \max\left[\rho(K_1, K_2), \rho(K_2, K_1)\right],$$

where $\rho(K_i, K_j) = \max_{x\in K_i} d(x, K_j)$, $i, j = 1, 2$, and $d(x, K_j) = \min_{y\in K_j} |x - y|$.

Definition 2.2

Let ω_n be a sequence of open subsets of D and let ω be an open subset of D. Let K_n and K be their complements in $\bar{D}$. We say that the sequence ω_n converges in the Hausdorff sense, to ω (or briefly $\omega_n \xrightarrow{H} \omega$) if

$$\lim_{n\to+\infty} d_H(K_n, K) = 0.$$

Definition 2.3

Let $\{\omega_n, \omega\}$ be a sequence of open subsets of D. We say that the sequence ω_n converges in the compact sense, to ω (or briefly $\omega_n \xrightarrow{K} \omega$) if

- every compact subset of ω is included in ω_n, for *n large enough*, and
- every compact subset of $\bar{\omega}^c$ is included in $\overline{\omega}_n^c$, for *n large enough.*

Definition 2.4

Let $\{\omega_n, \omega\}$ be a sequence of open subsets of D. We say that the sequence ω_n converges in the sense of characteristic functions, to ω (or briefly $\omega_n \xrightarrow{L} \omega$) if χ_{ω_n} converges to χ_ω in $L^p_{\text{loc}}(\mathbb{R}^N)$, $p \neq \infty$, (χ_ω is the characteristic function of ω).

Definition 2.5 (Chenais, 1975)

We say that a domain ω satisfies the ε-cone property if for all $x \in \partial\omega$ there exist a direction vector $\xi \in \mathbb{R}^N$ such that the cone $C(y, \xi, \varepsilon) \subset \omega$ for all $y \in B(x, \varepsilon) \cap \overline{\omega}$. ε denotes both the angle and the hight of the cone.

Denoting by $\mathcal{O}_\varepsilon$ the class of domains which have the ε-cone property, we can have this lemma.

Lemma 2.6 (Chenais, 1975)

If $\omega_n \in \mathcal{O}_\varepsilon$, then there exist an open subset $\omega \subset D$ and a subsequence (again denoted by ω_n) such that (i) $\omega_n \xrightarrow{H} \omega$, (ii) $\bar{\omega}_n \xrightarrow{H} \bar{\omega}$, (iii) $\partial\omega_n \xrightarrow{H} \partial\omega$, (iv) χ_{ω_n} converges to χ_ω in $L^1(D)$, (v) $\omega \in \mathcal{O}_\varepsilon$ and (vi) u_{ω_n} converges strongly in $H^1_0(D)$ to u_ω (u_ω is the solution of $P(\omega)$).

Lemma 2.7 (Pironneau, 1984)

Let ω_n be a sequence of open and bounded subsets of D. There exist a subsequence (again denoted by ω_n) and some open subset ω of D such that

1. ω_n converges to ω in the Hausdorff sense, and
2. $|\partial\omega| \leq \liminf_{n\to+\infty} |\partial\omega_n|$.

2.2 Shape derivative

In this subsection, we use the standard tool of the domain derivative to write down the optimality conditions. Before doing this, recall the definition of the domain derivative (Sokolowski and Zolesio, 1992). Suppose that ω is of class C^2. Consider a deformation field $V \in C^2\left(\mathbb{R}^N; \mathbb{R}^N\right)$ and set $\omega_t = \{x + tV(x) : x \in \omega\}$, $t > 0$. The application $\mathrm{Id} + tV$ (a perturbation of the identity) is a Lipschitz diffeomorphism for t small enough and by definition, the derivative of J at ω in the direction V is

$$dJ(\omega, V) = \lim_{t\to0} \frac{J(\omega_t) - J(\omega)}{t}.$$

As the functional J depends on the domain ω through the solution of some Dirichlet problem, we need to define also the domain derivative u'_ω of u_ω:

$$u'_\omega = \lim_{t\to0} \frac{u_{\omega_t} - u_\omega}{t}.$$

Furthermore, u'_ω is the solution of the following problem (Sokolowski and Zolesio, 1992):

$$\begin{cases} -\Delta u'_\omega = 0 \text{ in } \omega\,, \\ u'_\omega = -\dfrac{\partial u_\omega}{\partial \nu} V \cdot \nu \quad \text{on } \partial\omega. \end{cases} \tag{1}$$

Now to compute the derivative of the functionals we consider in the sequel, recall the following (Sokolowski and Zolesio, 1992).

1. The shape derivative of the volume is
$$\int_{\partial\omega} V \cdot \nu \, d\sigma. \tag{2}$$

2. If $G(\omega) = \int_\omega |\nabla u_\omega|^2 dx$, by the Hadamard formula we get
$$dG(\omega, V) = \int_{\partial\omega} |\nabla u_\omega|^2 V \cdot \nu d\sigma. \tag{3}$$

Since the domain ω satisfies ε-cone property, the deformation domain ω_t satisfies the same property (for t sufficiently small).

3. Proofs of Propositions 1.2 and 1.3

Let u_D and u_ω respectively denote the solution of $P(f, D)$ and $P(f, \omega)$. The maximum principle implies:

$$0 \le u_\omega \le u_D.$$

By using the variational formulation of $P(f, \omega)$,

$$\int_\omega |\nabla u_\omega(x)|^2 dx = \int_\omega f(x) u_\omega(x) dx.$$

Then,

$$J(\omega) = \int_\omega (k^2 - f(x) u_\omega(x)) dx, \text{ and } F(\omega) = \int_\omega (M_C^2 - f(x) u_\omega(x)) dx.$$

Therefore,

$$J(\omega) \ge -\int_D f(x) u_D(x) dx, \text{ and } F(\omega) \ge -\int_D f(x) u_D(x) dx.$$

Hence, inf J and inf F exist.

Proposition 1.2

The first item is obtained by using item 1 of Lemma 2.7, and (iv) and (v) of Lemma 2.6. The continuity w.r.t. the domains for the Dirichlet problem $P(f, \omega)$ is obtained by (vi) of Lemma 2.6. For the item 2, using the same notations as in the subsection 2.2, to get $int(C)$ in $(\Omega)_t$ (for t small enough) the admissible directions V must satisfy

$$V \cdot \nu \ge 0 \quad \text{on } \partial\Omega \cap \partial C.$$

Notice that for $\partial\Omega \setminus \partial C$, each V is admissible. Now since u_Ω vanishes on $\partial\Omega$, (2) and (3) imply

$$dJ(\Omega, V) = \int_{\partial\Omega} (k^2 - |\nabla u_\Omega|^2) V \cdot \nu d\sigma.$$

And since $dJ(\Omega, V) \ge 0$ for each admissible direction V, according to what precedes we obtain (I).

Proposition 1.3

By replacing k by M_C, item 1 is obtained in the same way as in the previous proof. For the second item, on $\partial\Omega^* \setminus (\partial\Omega \cup \partial C)$, any direction V is admissible whereas V must satisfy

$$V \cdot \nu \ge 0 \quad \text{on } \partial\Omega^* \cap \partial C,$$

and

$$V \cdot \nu \le 0 \quad \text{on } \partial\Omega \cap \partial\Omega^*.$$

Then arguing as above, (2) and (3) imply (II).

4. Proof of Theorem 1.1

We would like to say that the minimum obtained in Proposition 1.2 is solution to the problem $Q_S(f,k)$. It is not so simple. In general, without any assumptions on f and k, $Q_S(f,k)$ does not have a solution : as we saw in the introduction, if Ω is a smooth solution of $Q_S(f,k)$ then

$$\int_\Omega f(x)dx = k|\partial\Omega|,$$

which shows that if f has a too small total mass or if k is too large, the perimeter of Ω will not be large enough so that Ω contains C. In such a case, the minimum that we found comes to intersect the convex C, i.e. $\partial\Omega$ and ∂C have a common part and so

$$|\nabla u_\Omega| \le k \text{ on } \partial\Omega \cap \partial C.$$

The condition (NS) will allow us to get

$$\partial\Omega \cap \partial C = \emptyset,$$

and then obtain

$$|\nabla u_\Omega| = k \text{ on } \partial\Omega.$$

Suppose by contradiction that $\partial\Omega \cap \partial C \neq \emptyset$. Since $\mathrm{int}(C) \subset \Omega^* \subset \Omega$, one of the following situations occurs.

1. $\partial\Omega \equiv \partial C$
2. $\partial\Omega \neq \partial C$ and $\partial\Omega^* \equiv \partial C$
3. $\partial\Omega \neq \partial C$ and $\partial\Omega^* \neq \partial C$
4. $\partial\Omega \neq \partial C$ and $\partial\Omega \equiv \partial\Omega^*$
5. $\partial\Omega \neq \partial C$ and $\partial\Omega \neq \partial\Omega^*$

The aim of the sequel is to prove that each of the five cases above contradicts the condition (NS).

Case 1. $\partial\Omega \equiv \partial C$
$\partial\Omega \equiv \partial C$ together with $\mathrm{int}(C) \subset \Omega^* \subset \Omega$ implies that $\mathrm{int}(C) \equiv \Omega \equiv \Omega^*$. Then by (I) and (II)

$$M_C = |\nabla u_{\Omega^*}| = |\nabla u_\Omega| \le k, \text{ on } \partial\Omega.$$

Case 2. $\partial\Omega \neq \partial C$ **and** $\partial\Omega^* \equiv \partial C$
$\partial\Omega^* \equiv \partial C$ together with $\mathrm{int}(C) \subset \Omega^*$ implies that $\mathrm{int}(C) \equiv \Omega^*$. Then by (I), (II) and the the maximum principle applied to Ω and Ω^* ($\Omega^* \subset \Omega$ and $\partial\Omega^* \neq \partial\Omega$)

$$M_C = |\nabla u_{\Omega^*}| < |\nabla u_\Omega| \le k, \text{ on } \partial\Omega \cap \partial\Omega^*.$$

Case 3. $\partial\Omega \neq \partial C$ **and** $\partial\Omega^* \neq \partial C$
Applying the maximum principle to Ω and Ω^*, (I) and (II) give

$$M_C \le |\nabla u_{\Omega^*}| < |\nabla u_\Omega| \le k, \text{ on } \partial C \cap \partial\Omega \cap \partial\Omega^*.$$

Case 4. $\partial\Omega \neq \partial C$ **and** $\partial\Omega \equiv \partial\Omega^*$
$\partial\Omega^* \equiv \partial\Omega$ together with $\Omega^* \subset \Omega$ implies that $\Omega \equiv \Omega^*$. Then (I), and (II) imply

$$M_C \le |\nabla u_{\Omega^*}| = |\nabla u_\Omega| \le k, \text{ on } \partial\Omega \cap \partial C.$$

Case 5. $\partial\Omega \neq \partial C$ **and** $\partial\Omega \neq \partial\Omega^*$
(I), (II) and the the maximum principle applied to Ω and Ω^*

$$M_C \le |\nabla u_{\Omega^*}| < |\nabla u_\Omega| \le k, \text{ on } \partial C \cap \partial\Omega \cap \partial\Omega^*.$$

Remark 4.1

Barkatou and al. (2005) obtained, in the radial case, the following necessary and sufficient condition of existence for $Q_S(f,k)$.

$$\int_{B(0,R)} f(x)dx > NkV_N^{1/N}|B(0,R)|^{\frac{N-1}{N}},$$

where V_N is the volume of the unit ball in $\mathbb{R}^N$. Notice that this condition is exactly the condition (NS) obtained by Theorem 1.1.

5. The uniform density supported by a segment

Let $a > 0$ and put $C = [-1, 1] \times \{0\} \subset \mathbb{R}^2$. Consider the following free boundary problem. Find an open and bounded set $\Omega \subset \mathbb{R}^2$ which contains strictly C and such that the following free boundary problem has a solution:

$$Q_S(a,k)\begin{cases} -\Delta u_\Omega = a\delta_C \quad \text{in } \Omega \\ u_\Omega = 0 \text{ on } \partial\Omega \\ -\frac{\partial u_\Omega}{\partial \nu} = k \text{ on } \partial\Omega \end{cases}$$

Physically, this means that if the line segment in the complex plane is provided with a uniform density above a certain level, then there will exist a domain containing compactly the line segment such that the given measure on the line is equigravitational to the arc-length measure of the domain.

Theorem 5.1

The problem $Q_S(a, k)$ has a solution if and only if $a > 2k$.

We prove the theorem above in two propositions.

Proposition 5.2

Let (Ω_a, u_{Ω_a}) be the solution of the problem $Q_S(a, k)$. If Ω_a is lipschitz and $u_{\Omega_a} \in H^2(\Omega_a \setminus C)$ then $a > 2k$.

Proof

Let $\varepsilon \in [0, 1]$, put $V_\varepsilon = [-1-\varepsilon, 1+\varepsilon] \times [-\varepsilon, \varepsilon]$ and $\Omega_\varepsilon = \Omega_a \setminus V_\varepsilon$. u_{Ω_a} is harmonic on Ω_ε thus

$$0 = \int_{\Omega_\varepsilon} \Delta u_{\Omega_a} = \int_{\partial\Omega_a} \frac{\partial u_{\Omega_a}}{\partial \nu} + \int_{\partial V_\varepsilon} -\frac{\partial u_{\Omega_a}}{\partial \nu}.$$

Writing $u_{\Omega_a} = h - \frac{a}{2}|y|$ (where h is a harmonic function on Ω_a) and tending ε to 0, we obtain:

$$\lim_{\varepsilon \to 0} \int_{\partial V_\varepsilon} \frac{\partial u_{\Omega_a}}{\partial \nu} = -2a.$$

But $-\frac{\partial u_{\Omega_a}}{\partial \nu} = k$ on $\partial\Omega_a$, so

$$k|\partial\Omega_a| = 2a.$$

C is strictly contained in Ω_a thus $|\partial\Omega_a| > 4$ and consequently $a > 2k$.

Proposition 5.3

If $a > 2k$, then the problem $Q_S(a, k)$ has a solution.

Proof

Let B be the unit ball of $\mathbb{R}^2$. Put

$$O_{\varepsilon,B} = \{B \subset \omega \subset D,\ \omega \in O_\varepsilon\}, \text{ and}$$

$$J_a(\omega) = \int_\omega (|\nabla u_\omega(x)|^2 - 2au_\omega(x) + k^2)dx.$$

u_ω is the solution of the Dirichlet problem $P(a\delta_C, \omega)$. Arguing as in Section 3, we prove the existence of $\Omega_a \in O_{\varepsilon,B}$ which minimizes the functional J_a on $O_{\varepsilon,B}$. Then, supposing Ω_a of class C^2, we obtain the following optimality conditions:

$$(I_a)\begin{cases} |\nabla u_{\Omega_a}| \le k \text{ on } \partial\Omega_a \cap \partial C \\ |\nabla u_{\Omega_a}| = k \text{ on } \partial\Omega_a \setminus \partial C. \end{cases}$$

Now, put

$$O_{\Omega_a} = \{\omega \subset \Omega_a,\ \omega \in O_{\varepsilon,B}\}, \text{ and}$$

$$F_a(\omega) = \int_\omega (|\nabla u_\omega(x)|^2 - 2au_\omega(x) + (\frac{a}{2})^2)dx.$$

As above, there exists $\Omega_a^* \in O_{\Omega_a}$ which minimizes the functional F_a on O_{Ω_a}. If Ω_a^* is of class C^2, then:

$$(II_a)\begin{cases} |\nabla u_{\Omega_a^*}| \le \frac{a}{2} \text{ on } \partial\Omega_a^* \cap \partial C \\ |\nabla u_{\Omega_a^*}| \ge \frac{a}{2} \text{ on } \partial\Omega_a^* \cap \partial\Omega_a \\ |\nabla u_{\Omega_a^*}| = \frac{a}{2} \text{ on } \partial\Omega_a^* \setminus (\partial C \cup \partial\Omega_a). \end{cases}$$

Suppose by contradiction that $\partial\Omega_a \cap \partial B \neq \emptyset$. Replacing respectively in the proof of Theorem 1.1, C, Ω, Ω^*, and M_C by B, Ω_a, Ω_a^* and $\frac{a}{2}$, we obtain the desired contradiction.

Remark 5.4

Barkatou and Khatmi (2008) proved that if $a > 3.92k$ then $Q_S(a,k)$ has a solution while Shahgholian and Gustafsson (1996) showed that if $a \geq 24\pi k$, then the problem $Q_S(a,k)$ admits a solution.

6. Final remarks

Remark 6.1

For the p-Laplacian, the continuity with respect to the domain is a consequence of the γ_p-convergence (Bucur and Trebeschi, 1998). So using Hopf's comparison principle and considering, for $p > 1$, and $p \neq 2$, the functional

$$J_p(\omega) = \int_\omega (|\nabla u_\omega(x)|^p - pf(x)u_\omega(x) + k^p)dx,$$

one obtains the same necessary and sufficient condition of existence as in Theorem 1.1.

Remark 6.1

Gustafsson and Shahgholian (1996) showed the existence of a minimizer u to the functional

$$J(v) = \int_{\mathbb{R}^N} (|\nabla v|^2 - 2fv + k^2\chi_{\{}v > 0\})dx,$$

over all $0 \leq v \in H^1(\mathbb{R}^N)$. They prove that (Ω_u, u) $(\Omega_u = \{u > 0\})$ is a solution to $Q_S(f,k)$ but the overdetermined condition is given in a weak sense:

$$\lim_{\varepsilon \searrow 0} (|\nabla u|^2 - k^2)\eta \cdot v d\mathcal{H}^{N-1} = 0,$$

for every $\eta \in C_0^\infty(\mathbb{R}^N; \mathbb{R}^N)$. Then, they relate their minimization problem to quadrature domain $Q_D(f,k)$ and show that

$$\mathrm{Supp} f \subset \Omega_u \Leftrightarrow \Omega_u \in Q_D(f,k).$$

They conclude their paper by giving (Gustafsson and Shahgholian, 1996, Theorem 4.7) the following sufficient condition: If $\mathrm{Supp} f \subset B_R$ and if $\int_{B_R} f(x)dx > (\frac{6^N N}{3R}|B_R|)k$, then $\Omega_u \in Q_D(f,k)$ with $B_{3R} \subset \Omega_u$ (B_R being some ball of radius R).

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