Some New Families of Mean Graphs

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Abstract
We contribute some new results for mean labeling of graphs. It has been proved that the graphs obtained by the composition of paths $P_m$ and $P_n$ denoted by $P_{m}[P_{n}]$, the square of path $P_n$ and the middle graph of path $P_n$ admit mean labeling. We also investigate mean labeling for some cycle related graphs.

Keywords: Mean labeling, Mean graphs, Middle graphs

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1. Introduction
We begin with simple, finite, connected and undirected graph $G = (V(G), E(G))$ with $p$ vertices and $q$ edges. For standard terminology and notations we follow (Harary, F., 1972). We will provide brief summary of definitions and other information which are prerequisites for the present investigations.

Definition 1.1 The composition of two graphs $G_1$ and $G_2$ denoted by $G = G_1[G_2]$ has vertex set $V(G_1[G_2]) = V(G_1) \times V(G_2)$ and edge set $E(G_1[G_2]) = \{(u_1, v_1)(u_2, v_2) / u_1u_2 \in E(G_1) \text{ or } u_1 = u_2 \text{ and } v_1v_2 \in E(G_2)\}$

Definition 1.2 Duplication of a vertex $v_k$ by a new edge $e = v'_k v''_k$ in a graph $G$ produces a new graph $G_1$ such that $N(v'_k) \cap N(v''_k) = v_k$.

Definition 1.3 Duplication of an edge $e = v_i v_{i+1}$ by a vertex $v_k$ in a graph $G$ produces a new graph $G_1$ such that $N(v_k) = \{v_i, v_{i+1}\}$.

Definition 1.4 Square of a graph $G$ denoted by $G^2$ has the same vertex set as of $G$ and two vertices are adjacent in $G^2$ if they are at a distance of 1 or 2 apart in $G$.

Definition 1.5 The middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$ and in which two vertices are adjacent if and only if either they are adjacent edges of $G$ or one is a vertex of $G$ and the other is an edge incident on it.

Definition 1.6 If the vertices are assigned values subject to certain conditions then it is known as graph labeling.

For comprehensive survey on graph labeling we refer to (Gallian, J., 2009).

Definition 1.7 A function $f$ is called a mean labeling of graph $G$ if $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^*: E(G) \rightarrow \{1, 2, \ldots, q\}$ defined as

$f^*(e = uv) = \frac{f(u) + f(v)}{2};$ if $f(u) + f(v)$ is even

$= \frac{f(u) + f(v) + 1}{2};$ if $f(u) + f(v)$ is odd

is bijective. The graph which admits mean labeling is called a mean graph.

The mean labeling was introduced by (Somasundaram, S., 2003, p.29-35) and they proved the graphs $P_n$, $C_n$, $P_n \times P_m$, $P_m \times C_n$ etc. admit mean labeling. The same authors (Somasundaram, S., 2004, p.47-58) have discussed the mean labeling of subdivision of $K_{1,n}$ for $n \leq 4$ while in (Somasundaram, S., 2003, p.103-111) they proved that the wheel $W_n$ does not admit mean labeling for $n > 3$. In the present work seven new results corresponding to mean labeling and some new families of mean graphs are investigated.

Observations 1.8 In a mean graph $G$

(1) $v \in V(G)$ and $d(v) \geq 2$, having label 0, then edge label 1 can be produced only if $v$ is adjacent to the vertex having label either 1 or 2 and edge label 2 can be produced only if $v$ is adjacent to the vertex having label either 3 or 4 or the vertex with label 1 is adjacent to a vertex having label 3.
(2) The edge label \( q \) can be produced only when the vertices having labels \( q \) and \( q-1 \) are adjacent in \( G \).

2. Main Results

Theorem-2.1: The composition of paths \( P_m \) and \( P_2 \) denoted as \( P_m[P_2] \) admits mean labeling except for \( m = 2 \).

Proof: Let \( u_1, u_2, \ldots, u_m \) be the vertices of the path \( P_m \) and \( v_1, v_2 \) be the vertices of path \( P_2 \). The composition \( P_m[P_2] \) consists of \( 2m \) vertices, can be partitioned into two sets \( T_1=\{(u_i, v_1), i = 1, 2, \ldots, m\} \) and \( T_2=\{(u_i, v_2), i = 1, 2, \ldots, m\} \). To define \( f: V(G) \rightarrow \{0, 1, 2, \ldots, q\} \) following two cases are to be considered.

Case 1: \( m \) is even, \( m \neq 2 \)

We define the labeling as follows.
\[
\begin{align*}
    f(u_1, v_1) &= 0 \\
    f(u_1, v_2) &= 2 \\
    f(u_2, v_1) &= 4 \\
    f(u_2, v_2) &= 7 \\
    \text{For } 3 \leq i \leq (m-1) &
    \\
    f(u_i, v_1) &= f(u_{i-1}, v_1) + 5; \forall i. \\
    f(u_i, v_2) &= f(u_{i-1}, v_2) + 6; \text{ when } i \text{ is odd.} \\
    &= f(u_{i-1}, v_2) + 4; \text{ when } i \text{ is even.} \\
    \text{For } i = m; &
    \\
    f(u_i, v_1) &= f(u_{i-1}, v_1) + 6 \\
    f(u_i, v_2) &= f(u_{i-1}, v_2) + 3 \\
\end{align*}
\]

Case 2: \( m \) is odd.

We define the labeling as follows.
\[
\begin{align*}
    f(u_1, v_1) &= 0 \\
    f(u_1, v_2) &= 2 \\
    \text{For } 2 \leq i \leq m &
    \\
    f(u_i, v_1) &= f(u_{i-1}, v_1) + 6; \text{ when } i \text{ is odd.} \\
    &= f(u_{i-1}, v_1) + 4; \text{ when } i \text{ is even.} \\
    \text{For } 2 \leq i \leq (m-1) &
    \\
    f(u_i, v_2) &= f(u_{i-1}, v_2) + 6; \text{ when } i \text{ is even.} \\
    &= f(u_{i-1}, v_2) + 4; \text{ when } i \text{ is odd.} \\
    \text{For } i = m; &
    \\
    f(u_i, v_1) &= f(u_{i-1}, v_1) + 6 \\
    f(u_i, v_2) &= f(u_{i-1}, v_2) + 3 \\
\end{align*}
\]

In view of the above defined labeling pattern the graph under consideration admits mean labeling.

Case 3: \( m=2 \).

When \( m = 2 \) the resultant graph is \( K_4 \) and as proved by (Somasundaram S., 2004, p.47-58) the complete graph \( K_n \) does not admit mean labeling for \( n > 3 \).

Thus we conclude that the graph \( P_m[P_2] \) admits mean labeling except for \( m = 2 \).

Remark:2.2

It is obvious from the observations 1.8 that the composition \( P_2[P_m] \) does not admit mean labeling.

Illustration 2.3:

Consider the composition of \( P_2 \) and \( P_2 \). This is the case when \( n \) is odd. The mean labeling is as shown in Figure 1.

Theorem-2.4: The graph obtained by duplication of an arbitrary vertex by a new edge in cycle \( C_n \) admits mean labeling.

Proof: Let \( v_1, v_2, \ldots, v_n \) be the vertices of cycle \( C_n \) and let \( G \) be the graph obtained by duplicating an arbitrary vertex...
of \( C_n \) by a new edge. Without loss of generality let this vertex be \( v_1 \) and the edge be \( e = v_1'v_1'' \). To define \( f : V(G) \rightarrow \{0, 1, 2, \ldots, q\} \) the following two cases are to be considered.

**Case 1:** \( n \) is odd.

We define the labeling as follows.

\[
\begin{align*}
  f(v_1) &= 3 \\
  f(v_1') &= 0 \\
  f(v_1'') &= 2 \\
  f(v_i) &= 2(i + 1); \text{ For } 2 \leq i \leq \left( \frac{n+1}{2} \right) \\
  f(v_i) &= 2(n + 3 - i) - 1; \text{ For } \frac{n+3}{2} \leq i \leq n \\
\end{align*}
\]

**Case 2:** \( n \) is even.

We define the labeling as follows.

\[
\begin{align*}
  f(v_1) &= 3 \\
  f(v_1') &= 0 \\
  f(v_1'') &= 2 \\
  f(v_i) &= 2(i + 1); \text{ For } 2 \leq i \leq \left( \frac{n}{2} \right) \\
  f(v_i) &= 2(n + 3 - i) - 1; \text{ For } \frac{n+2}{2} \leq i \leq n \\
\end{align*}
\]

Above defined labeling pattern exhaust all the possibilities and in each case the graph under consideration admits mean labeling.

**Illustration 2.5:**

Consider the graph obtained by duplicating the vertex \( v_1 \) by an edge \( v_1'v_1'' \) in \( C_8 \). This is the case when \( n \) is even. The corresponding mean labeling is as shown in Figure 2.

**Theorem-2.6:** Duplication of an arbitrary edge by a new vertex in cycle \( C_n \) produces a mean graph.

**Proof:** Let \( v_1, v_2, \ldots, v_n \) be the vertices of cycle \( C_n \). Let \( G \) be the graph obtained by duplication of an arbitrary edge in \( C_n \) by a new vertex. Without loss of generality let this edge be \( e = v_1v_2 \) and the vertex be \( v' \). To define \( f : V(G) \rightarrow \{0, 1, 2, \ldots, q\} \) the following two cases are to be considered.

**Case 1:** \( n \) is odd.

We define the labeling as follows.

\[
\begin{align*}
  f(v') &= 0 \\
  f(v_1) &= 4 \\
  f(v_2) &= 2 \\
  f(v_i) &= 2i; \text{ For } 3 \leq i \leq \left( \frac{n+1}{2} \right) \\
  f(v_i) &= 2(n + 2 - i) + 1; \text{ For } \frac{n+3}{2} \leq i \leq n \\
\end{align*}
\]

**Case 2:** \( n \) is even.

We define the labeling as follows.

\[
\begin{align*}
  f(v') &= 0 \\
  f(v_1) &= 4 \\
  f(v_2) &= 2 \\
  f(v_i) &= 2i; \text{ For } 3 \leq i \leq \left( \frac{n+2}{2} \right) \\
  f(v_i) &= 2(n + 2 - i) + 1; \text{ For } \frac{n+4}{2} \leq i \leq n \\
\end{align*}
\]

In view of the above defined labeling pattern the graph under consideration admits mean labeling.

**Illustration 2.7:** Consider the graph obtained by duplicating an arbitrary edge \( v_1v_2 \) by a vertex \( v' \) in cycle \( C_9 \). This is the case when \( n \) is odd. Labeling pattern is shown in Figure 3.

**Theorem-2.8:** \( P_n^2 \) is a mean graph.
Proof: Let $v_1, v_2, ..., v_n$ be the vertices of path $P_n$. Define $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ as follows.

$f(v_i) = 2(i - 1)$; For $i \leq (n - 1)$

$f(v_i) = 2i - 3$; For $i = n$

Above defined labeling pattern provides mean labeling for $P_n^2$. i.e. $P_n^2$ is a mean graph.

Illustration 2.9:
The graph $P_n^2$ and the corresponding mean labeling is shown in Figure 4.

Theorem-2.10: Crown (cycle with pendant edge attached at each vertex) is a mean graph.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of cycle $C_n$. Let $G_1$ be the crown with $2n$ vertices $v_1, v_2, ..., v_n, u_1, u_2, ..., u_n$ and $2n$ edges ($n$ edges of cycle $C_n$ together with $n$ pendant edges).

Case 1: $n$ is odd.

we can label the graph as follows.

$f(v_1) = 0$

$f(u_1) = 1$

$f(v_i) = 2i - 1$; For $2 \leq i \leq \frac{n + 1}{2}$

$f(v_i) = 2i$; For $\frac{n + 1}{2} \leq i \leq n$

$f(u_i) = 2i - 2$; For $2 \leq i < \frac{n + 1}{2}$

$f(u_i) = 2i$; For $i = \frac{n + 1}{2}$

$f(u_i) = 2i - 1$; For $\frac{n + 1}{2} \leq i \leq n$

Case 2: $n$ is even.

we can label the graph as follows.

$f(v_1) = 0$

$f(u_1) = 1$

For $2 \leq i \leq \frac{n}{2}$

$f(v_i) = 2i - 1$

$f(u_i) = 2i - 2$

For $\frac{n + 2}{2} \leq i \leq n$

$f(v_i) = 2i$

$f(u_i) = 2i - 1$

The above defined function $f$ provides mean labeling for crown. i.e. crown is a mean graph.

Illustration 2.11 Consider the cycle $C_{12}$ with 12 pendant vertices attached. This is the case when $n$ is even. The mean labeling pattern is as shown in Figure 5.

Theorem-2.12: Tadpoles $T(n, k)$ (The graph obtained by identifying a vertex of cycle $C_n$ to an end vertex of path $P_k$) admits mean labeling.

Proof: Let $v_1, v_2, ..., v_n$ be the vertices of cycle $C_n$ and $u_1, u_2, ..., u_k$ be the vertices of the path $P_k$. Let $G_1$ be the resultant graph obtained by identifying a vertex of cycle $C_n$ to an end vertex of the path $P_k$. To define $f: V(G) \rightarrow \{0, 1, 2, \ldots, q\}$ the following two cases are to be considered.

Case 1: $n$ is odd.

Identify one of the end vertices of $P_k$ to a vertex of $C_n$ in such a way that $v_{\frac{n+1}{2}} = u_1$. We can label the graph as follows.

$f(v_i) = 2(i - 1)$; For $1 \leq i \leq \frac{n+1}{2}$

$f(v_i) = 2(n - i) + 3$; For $\frac{n+1}{2} \leq i \leq n$

$f(u_1) = f(v_{\frac{n+1}{2}})$

$f(u_i) = n + i - 1, i = 2, 3, \ldots, k$.

Case 2: $n$ is even.


Identify one of the end vertices of \( P_k \) to a vertex of \( C_n \) in such a way that \( v_{\frac{n+2}{2}} = u_1 \). We label the graph as follows.

\[
\begin{align*}
    f(v_i) &= 2(i-1); \text{For } 1 \leq i \leq \frac{n+2}{2} \\
    f(v_i) &= 2(n-i) + 3; \text{For } \frac{n+4}{2} \leq i \leq n \\
    f(u_1) &= f(v_{\frac{n+2}{2}}) \\
    f(u_i) &= n + i - 1; \text{For } 2, 3, \ldots, k
\end{align*}
\]

In both the cases the above described function \( f \) provides mean labeling for the graph under consideration.

**Illustration 2.13** In Figure 6 the graph \( T(9,4) \) and its mean labeling is shown.

**Theorem 2.14:** The middle graph of a path \( P_n \) denoted by \( M(P_n) \) admits mean labeling.

**Proof:** Let \( v_1, v_2, \ldots, v_n \) be the vertices and \( e_1, e_2, \ldots, e_{n-1} \) be the edges of path \( P_n \). The middle graph of a path \( P_n \) denoted by \( M(P_n) \) is a graph with \( V[M(P_n)] = V(P_n) \cup E(P_n) \) and two vertices are adjacent in \( M(P_n) \) if and only if they are adjacent in \( P_n \) or one is a vertex and other is an incident edge to that vertex in \( P_n \). \( M(P_n) \) consists of two types of vertices \{\( v_1, v_2, \ldots, v_n \)\} and \{\( e_1, e_2, \ldots, e_{n-1} \)\}. To define \( f: V(M(P_n)) \to \{0, 1, 2, \ldots, q\} \) as follows.

\[
\begin{align*}
    f(e_1) &= 1 \\
    f(e_2) &= 4 \\
    f(e_3) &= 8 \\
    f(e_i) &= 3i - 1; \text{For } 4 \leq i \leq (n-1) \\
    f(v_1) &= 0 \\
    f(v_i) &= 2i - 1; \text{For } 2 \leq i \leq 4 \\
    f(v_i) &= 3i - 5; \text{For } 5 \leq i \leq n
\end{align*}
\]

The above defined function \( f \) provides mean labeling for \( M(P_n) \). i.e. \( M(P_n) \) is a mean graph.

**Illustration 2.15** The mean labeling for \( M(P_6) \) is as shown in Figure 7.

3. Concluding Remarks

We contribute here seven new results to the theory of mean graphs. This work is an effort to relate some graph operations with mean labeling. We also introduce two concepts namely duplication of a vertex by an edge as well as duplication of an edge by a vertex. We apply these operations to cycle \( C_n \) and prove that the resultant graphs are mean graphs. In addition to this we prove that the middle graph of a path \( P_n \) admits mean labeling.

Future Scope:

It is possible to investigate similar results for other graph families and in the context of different labeling techniques.

References


Figure 1. The mean labeling of graph $P_7[P_2]$

Figure 2. Duplication of a vertex by an edge in $C_8$ and its mean labeling

Figure 3. Duplication of an edge by a vertex in $C_9$ and its mean labeling
Figure 4. The mean labeling of $P^2_9$

Figure 5. The mean labeling of crown

Figure 6. The mean labeling of $\mathcal{T}(9, 4)$
Figure 7. The mean labeling of $M(P_6)$.