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On a Generalization of Hilbert's Inequality and Its Applications

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Abstract

In this paper, the author gives a new extension of Hilbert's inequality by introducing some parameters. Also its application has been taken into consideration.

Keywords: Hilbert's inequality, Hölder inequality, Homogeneous function

1. Introduction

If $a_n \geq 0,\ b_n \geq 0$, such that $0 < \sum\limits_{n=0}^{\infty} a_n^2 < \infty, 0 < \sum\limits_{n=0}^{\infty} b_n^2 < \infty$,then the famous Hilbert's inequality (Hardy, G. H.(1952)) is given by

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} < \pi\left(\sum_{n=0}^{\infty} a_n^2\right)^{\frac{1}{2}}\left(\sum_{n=0}^{\infty} b_n^2\right)^{\frac{1}{2}}, \tag{1}$$

where the constant factor is the best possible.

Recently, some improvements and extensions of (1) are given in papers (Gao, M. Z. (1998), Kuang, J. C. (2000), Yang, B. C. (1999, 2002), Yang, B. C. and Debnath, L. (1998, 1999)), We also have a classical extension of Hilbert's inequality with(p, q)-parameter form as follow:

If $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ a_n \geq 0,\ b_n \geq 0$, such that $0 < \sum\limits_{n=0}^{\infty} a_n^p < \infty, 0 < \sum\limits_{n=0}^{\infty} b_n^q < \infty$,then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{m+n+1} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{n=0}^{\infty} a_n^p\right)^{\frac{1}{p}}\left(\sum_{n=0}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{2}$$

where the constant factor $\pi/_{\sin(\pi/p)}$ is the best possible (Hardy, G. H. (1952)).

Ingham, A. E. (1936) has proved that if $a_n \geq 0, 0 < \sum\limits_{n=0}^{\infty} a_n^2 < \infty, \lambda > 0$, then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m a_n}{m+n+\lambda} \leq M(\lambda)\sum_{n=0}^{\infty} a_n^2, \tag{3}$$

where

$$M(\lambda) = \begin{cases} \pi/_{\sin\lambda\pi}, & 0 < \lambda < 1/2 \\ \pi, & \lambda \geq 1/2 \end{cases}. \tag{4}$$

Yang, B. C. (1999) proved that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(m+n+1)^t} < B\left(\frac{p+t-2}{p}, \frac{q+t-2}{q}\right)\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-t} a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-t} b_n^q\right\}^{\frac{1}{q}},$$

where $B(u, v)$ is beta function, $p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ a_n \geq 0,\ b_n \geq 0, 2 - \min\{p, q\} < t \leq 2$ and $0 < \sum\limits_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-t} a_n^p < \infty$, $0 < \sum\limits_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{1-t} b_n^q < \infty$.

Yang, B. C. (2005) proved that

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_m b_n}{(m+n+1)^t} < B\left(\frac{p+t-2}{p},\frac{q+t-2}{q}\right)\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{(p-1)(2-t)-1}a_n^p\right\}^{\frac{1}{p}}\left\{\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{(q-1)(2-t)-1}b_n^q\right\}^{\frac{1}{q}}. \tag{5}$$

Kuang, J. C.(2000) discussed the general series

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}K(m+\lambda,n+\lambda)a_m b_n, \tag{6}$$

they gave the following theorem:

Theorem A Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_n\geq 0$, $b_n\geq 0$, and $0<\sum\limits_{n=0}^{\infty}(n+\lambda)^{1-t}a_n^p<\infty$, $0<\sum\limits_{n=0}^{\infty}(n+\lambda)^{1-t}b_n^q<\infty$, $\frac{1}{2}\leq\lambda\leq\frac{1}{2}\min\{p,q\}$,and let $K(x,y)$be nonnegative and homogeneous function of degree $-t(t>0)$. If $K(1,y)$ has first four derivatives continuous on$(0,\infty)$, and $(-1)^n K^{(n)}(1,y)\geq 0$, for $n=0,1,2,3,4$,$K^{(m)}(1,y)y^{-\frac{2\lambda}{r}}\to 0, y\to\infty$, for $m=0,1$,$I(r,\lambda)=\int_0^{\infty}K(1,u)u^{-\frac{2\lambda}{r}}du<\infty, r=p,q$.then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}K(m+\lambda,n+\lambda)a_m b_n < \left\{\sum_{m=0}^{\infty}[I(q,\lambda)-\phi(q,m,t,\lambda)](m+\lambda)^{1-t}a_m^p\right\}^{\frac{1}{p}}\times\left\{\sum_{n=0}^{\infty}[I(p,\lambda)-\phi(p,n,t,\lambda)](n+\lambda)^{1-t}b_n^q\right\}^{\frac{1}{q}},$$

where

$$\phi(r,m,t,\lambda)=\left(\frac{\lambda}{\lambda+m}\right)^{1-\frac{2\lambda}{r}}\left\{K(1,\frac{\lambda}{\lambda+m})\left[\frac{1}{1-{}^{2\lambda}\!/_{r}}-\frac{1}{2\lambda}\left(1+\frac{1}{3r}\right)\right]-\frac{1}{24\lambda(\lambda+m)}K'(1,\frac{\lambda}{\lambda+m})\right\}>0,$$

and $r=p,q$.

The major objective of this paper is to formulate a new inequality related to the general series (6). which is an extension of (1), (5).

2. Some lemmas

Lemma 2.1 Let $f(x)$ have its first four derivatives continuous on $[0,\infty)$, $(-1)^n f^{(n)}(x)>0 \quad (n=0,1,2,3,4)$, $\int_0^{\infty}f(x)dx<\infty$,$f(x),f'(x)\to 0(x\to\infty)$, then

$$\sum_{k=0}^{\infty}f(k)<\int_0^{\infty}f(x)dx+\frac{1}{2}f(0)-\frac{1}{12}f'(0). \tag{7}$$

The proof of Lemma 2.1 is given by Kuang, J. C.(2000).

Lemma 2.2 Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$,$\frac{1}{2}\leq\lambda\leq\frac{1}{2}\min\{p,q\}$, $2-\frac{\min(p,q)}{2\lambda}<t\leq 2$,$K(x,y)$be nonnegative and homogeneous function of degree $-t$. If $K(1,y)$ has first four derivatives continuous on$(0,\infty)$, and , for $n=0,1,2,3,4$,$K^{(m)}(1,y)y^{-\frac{2\lambda}{r}(2-t)}\to 0, y\to\infty$, for $m=0,1$.$I(r,\lambda)=\int_0^{\infty}K(1,u)u^{-\frac{2\lambda}{r}(2-t)}du<\infty, r=p,q$. Define the weight coefficient as:

$$\omega(r,m,t,\lambda)=\sum_{n=0}^{\infty}K(m+\lambda,n+\lambda)\left(\frac{m+\lambda}{n+\lambda}\right)^{\frac{2\lambda(2-t)}{r}}, \tag{8}$$

then

$$\omega(r,m,t,\lambda)<(m+\lambda)^{1-t}[I(r,\lambda)-\phi(r,m,t,\lambda)]. \tag{9}$$

Where

$$\phi(r,m,t,\lambda)=\left(\frac{\lambda}{\lambda+m}\right)^{1-\frac{2\lambda}{r}(2-t)}\left\{K(1,\frac{\lambda}{\lambda+m})\left[\frac{1}{1-{}^{2\lambda}\!/_{r}\,(2-t)}-\frac{1}{2\lambda}\left(1+\frac{2-t}{3r}\right)\right]-\frac{1}{24\lambda(\lambda+m)}K'(1,\frac{\lambda}{\lambda+m})\right\}>0.$$

Proof Let

$$f(x)=K(m+\lambda,x+\lambda)\left(\frac{x+\lambda}{m+\lambda}\right)^{-\frac{2\lambda}{r}(2-t)}=(m+\lambda)^{-t}K(1,\frac{x+\lambda}{m+\lambda})\left(\frac{x+\lambda}{m+\lambda}\right)^{-\frac{2\lambda}{r}(2-t)},$$

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then

$$\begin{aligned}
f'(x) =&(m+\lambda)^{-t-1}K'(1,\frac{x+\lambda}{m+\lambda})\Big(\frac{x+\lambda}{m+\lambda}\Big)^{-\frac{2\lambda}{r}(2-t)}\\
&-\frac{2\lambda}{r}(2-t)(m+\lambda)^{-t-1}K(1,\frac{x+\lambda}{m+\lambda})\Big(\frac{x+\lambda}{m+\lambda}\Big)^{-\frac{2\lambda}{r}(2-t)-1},\\
f(0) =&(m+\lambda)^{-t}K(1,\frac{\lambda}{m+\lambda})\Big(\frac{\lambda}{m+\lambda}\Big)^{-\frac{2\lambda}{r}(2-t)},\\
f'(0) =&(m+\lambda)^{-t-1}K'(1,\frac{\lambda}{m+\lambda})\Big(\frac{\lambda}{m+\lambda}\Big)^{-\frac{2\lambda}{r}(2-t)}\\
&-\frac{2\lambda}{r}(2-t)(m+\lambda)^{-t-1}K(1,\frac{\lambda}{m+\lambda})\Big(\frac{\lambda}{m+\lambda}\Big)^{-\frac{2\lambda}{r}(2-t)-1}.
\end{aligned}$$

By lemma 2.1 we have

$$\begin{aligned}
\omega(r,m,t,\lambda) &= \sum_{k=0}^{\infty} f(k) < \int_0^{\infty} f(x)dx + \frac{1}{2}f(0) - \frac{1}{12}f'(0)\\
&= \int_0^{\infty}(m+\lambda)^{-t}K(1,\frac{x+\lambda}{m+\lambda})\Big(\frac{x+\lambda}{m+\lambda}\Big)^{-\frac{2\lambda}{r}(2-t)}dx + \frac{1}{2}(m+\lambda)^{-t}K(1,\frac{\lambda}{m+\lambda})\Big(\frac{\lambda}{m+\lambda}\Big)^{-\frac{2\lambda}{r}(2-t)}\\
&-\frac{1}{12}(m+\lambda)^{-t-1}K'(1,\frac{\lambda}{m+\lambda})\Big(\frac{\lambda}{m+\lambda}\Big)^{-\frac{2\lambda}{r}(2-t)} + \frac{\lambda}{6r}(2-t)(m+\lambda)^{-t-1}K(1,\frac{\lambda}{m+\lambda})\Big(\frac{\lambda}{m+\lambda}\Big)^{-\frac{2\lambda}{r}(2-t)-1},
\end{aligned}$$

$$\omega(r,m,t,\lambda) < (m+\lambda)^{1-t}\left[\int_0^{\infty}K(1,u)u^{-\frac{2\lambda}{r}(2-t)}du - \psi(r,m,t,\lambda)\right], \tag{10}$$

$$\begin{aligned}
\psi(r,m,t,\lambda) = &\int_0^{\frac{\lambda}{m+\lambda}}K(1,u)u^{-\frac{2\lambda}{r}(2-t)}du - \Big(\frac{\lambda}{m+\lambda}\Big)^{1-\frac{2\lambda}{r}(2-t)}\left(\frac{1}{2\lambda}+\frac{2-t}{6r\lambda}\right)K(1,\frac{\lambda}{m+\lambda})\\
&+\frac{1}{12\lambda^2}\Big(\frac{\lambda}{m+\lambda}\Big)^{2-\frac{2\lambda}{r}(2-t)}K'(1,\frac{\lambda}{m+\lambda}).
\end{aligned} \tag{11}$$

Integrating by parts twice gives

$$\begin{aligned}
&\int_0^{\frac{\lambda}{m+\lambda}}K(1,u)u^{-\frac{2\lambda}{r}(2-t)}du\\
=&\frac{1}{1-{}^{2\lambda}\!/_r\,(2-t)}\int_0^{\frac{\lambda}{m+\lambda}}K(1,u)du^{1-\frac{2\lambda}{r}(2-t)}\\
=&\frac{1}{1-{}^{2\lambda}\!/_r\,(2-t)}\left[K(1,\frac{\lambda}{m+\lambda})\Big(\frac{\lambda}{m+\lambda}\Big)^{1-\frac{2\lambda}{r}(2-t)} - \int_0^{\frac{\lambda}{m+\lambda}}K'(1,u)u^{1-\frac{2\lambda}{r}(2-t)}du\right]\\
=&\frac{1}{1-{}^{2\lambda}\!/_r\,(2-t)}K(1,\frac{\lambda}{m+\lambda})\Big(\frac{\lambda}{m+\lambda}\Big)^{1-\frac{2\lambda}{r}(2-t)}\\
&-\frac{1}{1-{}^{2\lambda}\!/_r\,(2-t)}\cdot\frac{1}{2-{}^{2\lambda}\!/_r\,(2-t)}\int_0^{\frac{\lambda}{m+\lambda}}K'(1,u)du^{2-\frac{2\lambda}{r}(2-t)}\\
=&\frac{1}{1-{}^{2\lambda}\!/_r\,(2-t)}K(1,\frac{\lambda}{m+\lambda})\Big(\frac{\lambda}{m+\lambda}\Big)^{1-\frac{2\lambda}{r}(2-t)}\\
&-\frac{1}{1-{}^{2\lambda}\!/_r\,(2-t)}\cdot\frac{1}{2-{}^{2\lambda}\!/_r\,(2-t)}K'(1,\frac{\lambda}{m+\lambda})\Big(\frac{\lambda}{m+\lambda}\Big)^{2-\frac{2\lambda}{r}(2-t)}\\
&+\frac{1}{1-{}^{2\lambda}\!/_r\,(2-t)}\cdot\frac{1}{2-{}^{2\lambda}\!/_r\,(2-t)}\int_0^{\frac{\lambda}{m+\lambda}}K''(1,u)u^{2-\frac{2\lambda}{r}(2-t)}du\\
>&\frac{1}{1-{}^{2\lambda}\!/_r\,(2-t)}K(1,\frac{\lambda}{m+\lambda})\Big(\frac{\lambda}{m+\lambda}\Big)^{1-\frac{2\lambda}{r}(2-t)}\\
&-\frac{1}{1-{}^{2\lambda}\!/_r\,(2-t)}\cdot\frac{1}{2-{}^{2\lambda}\!/_r\,(2-t)}K'(1,\frac{\lambda}{m+\lambda})\Big(\frac{\lambda}{m+\lambda}\Big)^{2-\frac{2\lambda}{r}(2-t)}.
\end{aligned}$$

From this and equality (11), we obtain

$$\begin{aligned}\psi(r,m,t,\lambda) >&\left[\frac{1}{1-{}^{2\lambda}\!/_{r}\,(2-t)}-\frac{1}{2\lambda}\left(1+\frac{2-t}{3r}\right)\right]K(1,\frac{\lambda}{m+\lambda})\left(\frac{\lambda}{m+\lambda}\right)^{1-\frac{2\lambda}{r}(2-t)}\\ &-\left[\frac{1}{1-{}^{2\lambda}\!/_{r}\,(2-t)}\cdot\frac{1}{2-{}^{2\lambda}\!/_{r}\,(2-t)}-\frac{1}{12\lambda^2}\right]K'(1,\frac{\lambda}{m+\lambda})\left(\frac{\lambda}{m+\lambda}\right)^{2-\frac{2\lambda}{r}(2-t)}.\end{aligned}\tag{12}$$

Since $\frac{1}{2}\le\lambda\le\frac{1}{2}\min\{p,q\}$, we have

$$2\lambda\ge 1>\frac{3r^2+r(2-t)}{3r^2+3r(2-t)+(2-t)^2}=\frac{r(3r+2-t)}{3r^2+3r(2-t)+(2-t)^2},$$

that is,

$$\frac{1}{2\lambda}<\frac{3r}{3r+2-t}+\frac{2-t}{r},$$

$$\frac{1}{2\lambda}-\frac{2-t}{r}<\frac{3r}{3r+2-t},$$

$$1-\frac{2\lambda(2-t)}{r}<\frac{6\lambda r}{3r+2-t},$$

So that

$$\frac{1}{2\lambda}\left(1+\frac{2-t}{3r}\right)\left(1-\frac{2\lambda}{r}(2-t)\right)<1.$$

This implies

$$\frac{1}{1-{}^{2\lambda}\!/_{r}\,(2-t)}-\frac{1}{2\lambda}\left(1+\frac{2-t}{3r}\right)>0.\tag{13}$$

Next note that $\frac{2}{r}(2-t)<\frac{1}{\lambda}\le 2$, which implies that

$$0<\left(\frac{1}{\lambda}-\frac{2(2-t)}{r}\right)\left(\frac{1}{\lambda}-\frac{2-t}{r}\right)\le 4.$$

That is

$$\frac{1}{1-{}^{2\lambda}\!/_{r}\,(2-t)}\cdot\frac{1}{2-{}^{2\lambda}\!/_{r}\,(2-t)}\ge\frac{1}{8\lambda^2}.$$

$$\frac{1}{1-{}^{2\lambda}\!/_{r}\,(2-t)}\cdot\frac{1}{2-{}^{2\lambda}\!/_{r}\,(2-t)}-\frac{1}{12\lambda^2}\ge\frac{1}{8\lambda^2}-\frac{1}{12\lambda^2}=\frac{1}{24\lambda^2}.\tag{14}$$

By hypotheses $K'\left(1,\frac{\lambda}{m+\lambda}\right)\le 0$, this and inequalities (12), (13), imply that

$$\begin{aligned}\psi(r,m,t,\lambda) >&\left(\frac{\lambda}{m+\lambda}\right)^{1-\frac{2\lambda}{r}(2-t)}\left\{\left[\frac{1}{1-{}^{2\lambda}\!/_{r}\,(2-t)}-\frac{1}{2\lambda}\left(1+\frac{2-t}{3r}\right)\right]K(1,\frac{\lambda}{m+\lambda})-\frac{1}{24\lambda}\cdot\frac{1}{m+\lambda}K'(1,\frac{\lambda}{m+\lambda})\right\}\\ =&\phi(r,m,t,\lambda)>0.\end{aligned}$$

From (10), the lemma 2.2 is proved.

3. Main results

Theorem 3.1 Let $p>1$, $\frac{1}{p}+\frac{1}{q}=1$, $a_n\ge 0$, $b_n\ge 0$, $\frac{1}{2}\le\lambda\le\frac{1}{2}\min\{p,q\}$, $2-\frac{\min(p,q)}{2\lambda}<t\le 2$, and $0<\sum\limits_{m=0}^{\infty}(m+\lambda)^{2\lambda(p-2)(2-t)+1-t}a_m^p<\infty$, $0<\sum\limits_{n=0}^{\infty}(n+\lambda)^{2\lambda(q-2)(2-t)+1-t}b_n^q<\infty$,$K(x,y)$ be nonnegative and homogeneous function of degree $-t$. If $K(1,y)$ has first four derivatives continuous on$(0,\infty)$, and $(-1)^nK^{(n)}(1,y)\ge 0$, for $n=0,1,2,3,4$; $K^{(m)}(1,y)y^{-\frac{2\lambda}{r}(2-t)}\to 0, y\to\infty$,for $m=0,1$,

$$I(r,\lambda)=\int_0^\infty K(1,u)u^{-\frac{2\lambda}{r}(2-t)}du<\infty, r=p,q.\tag{15}$$

then

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}K(m+\lambda,n+\lambda)a_m b_n\\ <&\left\{\sum_{m=0}^{\infty}[I(p,\lambda)-\phi(p,m,t,\lambda)](m+\lambda)^{2\lambda(p-2)(2-t)+1-t}a_m^p\right\}^{\frac{1}{p}}\\ &\times\left\{\sum_{n=0}^{\infty}[I(q,\lambda)-\phi(q,n,t,\lambda)](n+\lambda)^{2\lambda(q-2)(2-t)+1-t}b_n^q\right\}^{\frac{1}{q}},\end{aligned}\tag{16}$$

where

$$\begin{aligned}&\phi(r,m,t,\lambda)\\ =&\left(\frac{\lambda}{\lambda+m}\right)^{1-\frac{2\lambda}{r}(2-t)}\left\{K(1,\frac{\lambda}{\lambda+m})\left[\frac{1}{1-2\lambda/r\,(2-t)}-\frac{1}{2\lambda}\left(1+\frac{2-t}{3r}\right)\right]-\frac{1}{24\lambda(\lambda+m)}K'(1,\frac{\lambda}{\lambda+m})\right\}>0,\end{aligned}\tag{17}$$

and $r=p,q$.

Proof By Hölder inequality, we have

$$\begin{aligned}&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}K(m+\lambda,n+\lambda)a_m b_n\\ =&\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}[K(m+\lambda,n+\lambda)]^{\frac{1}{p}}\left[\frac{(m+\lambda)^{\frac{1}{q^2}}}{(n+\lambda)^{\frac{1}{p^2}}}\right]^{2\lambda(2-t)}a_m\cdot[K(m+\lambda,n+\lambda)]^{\frac{1}{q}}\left[\frac{(n+\lambda)^{\frac{1}{p^2}}}{(m+\lambda)^{\frac{1}{q^2}}}\right]^{2\lambda(2-t)}b_n\\ \le&\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}K(m+\lambda,n+\lambda)\left[\frac{(m+\lambda)^{\frac{p}{q^2}}}{(n+\lambda)^{\frac{1}{p}}}\right]^{2\lambda(2-t)}a_m^p\right\}^{\frac{1}{p}}\times\left\{\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}K(m+\lambda,n+\lambda)\left[\frac{(n+\lambda)^{\frac{q}{p^2}}}{(m+\lambda)^{\frac{1}{q}}}\right]^{2\lambda(2-t)}b_n^q\right\}^{\frac{1}{q}}\\ <&\left\{\sum_{m=0}^{\infty}\omega(p,m,t,\lambda)(m+\lambda)^{2\lambda(p-2)(2-t)}a_m^p\right\}^{\frac{1}{p}}\times\left\{\sum_{n=0}^{\infty}\omega(q,n,t,\lambda)(n+\lambda)^{2\lambda(q-2)(2-t)}b_n^q\right\}^{\frac{1}{q}},\end{aligned}\tag{18}$$

where

$$\omega(r,m,t,\lambda)=\sum_{n=0}^{\infty}K(m+\lambda,n+\lambda)\left(\frac{m+\lambda}{n+\lambda}\right)^{\frac{2\lambda(2-t)}{r}},\tag{19}$$

thus, by (9), we have (16). Theorem 3.1 is proved.

Let $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ \frac{1}{2}\le\lambda\le\frac{1}{2}\min\{p,q\},2-\frac{\min(p,q)}{2\lambda}<t\le 2, t_0=2-\frac{\min(p,q)}{2\lambda}$. Setting $g_r(t)=t+\frac{2\lambda}{r}(2-t)-1,\ r=p,q$. then $g'_r(t)=1-\frac{2\lambda}{r}\ge 0$, $g_r(t)$is nondecreasing on $[t_0,2]$, $t_0<t\le 2$,$g_r(t)\ge g_r(t_0)$. If $p\le q$, $g_p(t_0)=2-\frac{p}{2\lambda}+\frac{p}{p}-1\ge 2-p\ge 0$; $g_q(t_0)=2-\frac{p}{2\lambda}+\frac{p}{q}-1\ge 2-p+p-2\ge 0$, $g_r(t)\ge 0$. When $p>q$, $g_r(t)\ge 0$ is valid.

We take $K(x,y)$ as $K(x,y)=(x+y)^{-t}$, then

$$I(r,\lambda)=\int_0^{\infty}(1+u)^{-t}u^{-\frac{2\lambda}{r}(2-t)}du=B\left(1-\frac{2\lambda}{r}(2-t),\ t+\frac{2\lambda}{r}(2-t)-1\right)$$

,

$$\begin{aligned}\phi(r,m,t,\lambda)=&\left(\frac{\lambda}{\lambda+m}\right)^{1-\frac{2\lambda}{r}(2-t)}\left\{\left(\frac{m+\lambda}{m+2\lambda}\right)^{t}\left[\frac{1}{1-2\lambda/r\,(2-t)}-\frac{1}{2\lambda}\left(1+\frac{2-t}{3r}\right)\right]+\frac{t(m+\lambda)^t}{24\lambda(m+2\lambda)^{t+1}}\right\}\\ >&\left(\frac{\lambda}{\lambda+m}\right)^{1-\frac{2\lambda}{r}(2-t)}\frac{1}{2^t}\left[\frac{1}{1-2\lambda/r\,(2-t)}-\frac{1}{2\lambda}\left(1+\frac{2-t}{3r}\right)\right]\\ =&h(r,m,t,\lambda).\end{aligned}$$

Theorem 3.2 Let $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ a_n\ge 0,\ b_n\ge 0,\ \frac{1}{2}\le\lambda\le\frac{1}{2}\min\{p,q\},\ 2-\frac{\min(p,q)}{2\lambda}<t\le 2$, and $0<$

$\sum_{m=0}^{\infty}(m+\lambda)^{2\lambda(p-2)(2-t)+1-t}a_m^p<\infty,\ 0<\sum_{n=0}^{\infty}(n+\lambda)^{2\lambda(q-2)(2-t)+1-t}b_n^q<\infty$, then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_mb_n}{(m+n+2\lambda)^t}$$
$$<\left\{\sum_{m=0}^{\infty}\left[B\left(1-\frac{2\lambda}{p}(2-t),\ t+\frac{2\lambda}{p}(2-t)-1\right)-h(p,m,t,\lambda)\right](m+\lambda)^{2\lambda(p-2)(2-t)+1-t}a_m^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=0}^{\infty}\left[B\left(1-\frac{2\lambda}{q}(2-t),\ t+\frac{2\lambda}{q}(2-t)-1\right)-h(q,m,t,\lambda)\right](n+\lambda)^{2\lambda(q-2)(2-t)+1-t}b_n^q\right\}^{\frac{1}{q}},$$

where

$$h(r,m,t,\lambda)=\left(\frac{\lambda}{\lambda+m}\right)^{1-\frac{2\lambda}{r}(2-t)}\frac{1}{2^t}\left[\frac{1}{1-{}^{2\lambda}\!/_{r}\,(2-t)}-\frac{1}{2\lambda}\left(1+\frac{2-t}{3r}\right)\right].$$

when $\lambda={}^1\!/_2$, we have

Theorem 3.3 Let $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ a_n\geq 0,\ b_n\geq 0, 2-\min(p,q)<t\leq 2$, and $0<\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)^{(p-2)(2-t)+1-t}a_m^p<\infty, 0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{(q-2)(2-t)+1-t}b_n^q<\infty$, then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_mb_n}{(m+n+1)^t}$$
$$<\left\{\sum_{m=0}^{\infty}\left[B\left(\frac{p+t-2}{p},\ \frac{q+t-2}{q}\right)-\frac{1}{(2m+1)^{\frac{p+t-2}{p}}}\frac{1}{2^t}\left(\frac{p}{p+t-2}-1-\frac{2-t}{3p}\right)\right](m+\frac{1}{2})^{(p-2)(2-t)+1-t}a_m^p\right\}^{\frac{1}{p}} \tag{20}$$
$$\times\left\{\sum_{n=0}^{\infty}\left[B\left(\frac{p+t-2}{p},\ \frac{q+t-2}{q}\right)-\frac{1}{(2n+1)^{\frac{q+t-2}{q}}}\frac{1}{2^t}\left(\frac{q}{q+t-2}-1-\frac{2-t}{3q}\right)\right](n+\frac{1}{2})^{(q-2)(2-t)+1-t}b_n^q\right\}^{\frac{1}{q}}.$$

Evidently, inequality (20) is an improvement of (5).

By taking $t=1$ in Theorem 3.3, we obtain

Theorem 3.4 Let $p>1,\ \frac{1}{p}+\frac{1}{q}=1,\ a_n\geq 0,\ b_n\geq 0$, and $0<\sum_{m=0}^{\infty}\left(m+\frac{1}{2}\right)^{p-2}a_m^p<\infty, 0<\sum_{n=0}^{\infty}\left(n+\frac{1}{2}\right)^{q-2}b_n^q<\infty$, then

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_mb_n}{m+n+1}$$
$$<\left\{\sum_{m=0}^{\infty}\left[\frac{\pi}{\sin(\pi/p)}-\frac{1}{2(2m+1)^{\frac{1}{q}}}\left(q+\frac{1}{3q}-\frac{4}{3}\right)\right](m+\frac{1}{2})^{p-2}a_m^p\right\}^{\frac{1}{p}}$$
$$\times\left\{\sum_{n=0}^{\infty}\left[\frac{\pi}{\sin(\pi/p)}-\frac{1}{2(2m+1)^{\frac{1}{p}}}\left(p+\frac{1}{3p}-\frac{4}{3}\right)\right](n+\frac{1}{2})^{q-2}a_n^q\right\}^{\frac{1}{q}}.$$

By taking $p=q=2$ in Theorem 3.4, we have

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{a_mb_n}{m+n+1}<\left\{\sum_{m=0}^{\infty}\left[\pi-\frac{5}{12(2m+1)^{\frac{1}{2}}}\right]a_m^2\right\}^{\frac{1}{2}}\left\{\sum_{n=0}^{\infty}\left[\pi-\frac{5}{12(2n+1)^{\frac{1}{2}}}\right]a_n^2\right\}^{\frac{1}{2}}.$$

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