Clique Partitions of Glued Graphs

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Abstract

A glued graph at K_2-clone (K_3-clone) results from combining two vertex-disjoint graphs by identifying an edge (a triangle) of each original graph. The clique covering numbers of these desired glued graphs have been investigated recently. Analogously, we obtain bounds of the clique partition numbers of glued graphs at K_2-clones and K_3-clones in terms of the clique partition numbers of their original graphs. Moreover, we characterize glued graphs satisfying such bounds.

Keywords: Clique partitions, Clique coverings, Glued graphs

1. Introduction

Let G_1 and G_2 be any two graphs with disjoint vertex sets. Let H_1 and H_2 be nontrivial connected subgraphs of G_1 and G_2, respectively, such that $H_1 \cong H_2$ with an isomorphism f. We combine G_1 and G_2 by identifying H_1 and H_2 with respect to the isomorphism f. This resulting graph is called *glued graph of G_1 and G_2 at H_1 and H_2* with respect to f. We denote this glued graph by $G_1 \underset{H}{\oplus} G_2$ where H is the copy of H_1 and H_2 in the glued graph. We refer to H, H_1 and H_2 as the *clones* of the glued graph, G_1 and G_2, respectively, and refer to G_1 and G_2 as the *original graphs*. Thus the combined graph is also called the *glued graph of G_1 and G_2 at H-clone* The notation $G_1 \oplus G_2$ represents an arbitrary graph resulting from gluing graphs G_1 and G_2 at any isomorphic subgraph $H_1 \cong H_2$ with respect to any of their isomorphism. More details concerning glued graphs can be explored in Promsakon's thesis (Promsakon, 2006). Furthermore, planar glued graphs and perfection of glued graphs are also discussed recently (see (Uiyyasathian and Hemakul, 2006) and (Saduakdee and Uiyyasathian, 2009).

Cliques are complete subgraphs of a graph that are not necessarily maximal. An *n-clique* or a *clique of order n* of a graph G is a clique of G with n vertices. A *clique covering* of a graph G is a set of cliques of G, which together contain each edge of G at least once; if each edge is covered exactly once, then it is called *clique partition*. The *clique covering number* $cc(G)$ and *clique partition number* $cp(G)$ are the smallest cardinality among all clique coverings and clique partitions of G, respectively. A *minimum clique partition* of G is a clique partition of G with cardinality $cp(G)$. Since a clique partition of a graph G is also a clique covering of G, we always have $cc(G) \le cp(G)$.

$G - H$ denotes the graph derived from G obtained by deleting all edges of a subgraph H. In particular, if e is an edge of G, then $G - e$ is the graph G with the edge e deleted. The notation $e(G)$ stands for the number of edges in a graph G.

The question of calculating clique covering and clique partition numbers was raised by Orlin in 1977. DeBruijn and Erdős had already proved that partitioning a complete graph K_n into smaller cliques required at least n cliques (DeBruijn & Erdős, 1948). This yields the following theorem of Orlin which states that for $n \ge 3$, $cp(K_n - e) = n - 1$ where e is any edge of the complete graph K_n (Orlin, 1977). A recent paper by Pimpasalee et al. (Pimpasalee et al., 2008), investigates bounds of **clique covering numbers** of glued graphs at K_2-clones as follows:

$$cc(G_1) + cc(G_2) - 1 \le cc\left(G_1 \underset{K_2}{\oplus} G_2\right) \le cc(G_1) + cc(G_2).$$

They also give a characterization of glued graphs with the clique covering number of each possible value. This paper investigates analogously the **clique partition numbers** of glued graphs in terms of their original graphs.

Note first that a glued graph can possibly have its clique partition number less than the clique partition number of each original graph, see Example A.

Example A Let G_1 be a Hamiltonian graph on n vertices with a Hamiltonian path P as shown in the bold edges in Figure 1. and $G_2 = \overline{G_1} \cup P$. Then the resulting graph $G_1 \underset{P}{\oplus} G_2$ is a complete graph, so $cp(G_1 \underset{P}{\oplus} G_2) = 1$. Since neither G_1 nor G_2

is a complete graph, its clique partition number is more than 1. It is noticeable that the graph gluing of original graphs with any arbitrary large clique partition number could yield a resulting glued graph with clique partition number 1. This situation occurs because of the existence of new cliques in the glued graphs.

Consequently, we observed that if $G_1 \diamond G_2$ does not have a new clique, at least $cp(G_1)$ (and $cp(G_2)$) cliques are needed to partition the copy of G_1 (and G_2) in $G_1 \diamond G_2$. Hence,

$$cp(G_1 \diamond G_2) \geq \max\{cp(G_1), cp(G_2)\}.$$

Back to clique coverings of arbitrary glued graphs, since a set of union between minimum clique coverings of both original graphs is a clique covering of such glued graph (see Pimpasalee et al., 2008), $cc(G_1 \diamond G_2) \leq cc(G_1) + cc(G_2)$. However, the same situation is not true for clique partition numbers, see Example B.

Example B Claim that $cp(K_m \underset{K_2}{\diamond} K_n) = \min\{m, n\}$ for $m, n \geq 3$.

Let $m \geq n \geq 3$. Since $K_m \underset{K_2}{\diamond} K_n$ can be partitioned into the sets of an m-clique, an $(n-1)$-clique and $(n-2)$ copies of 2-cliques, $cp(K_m \underset{K_2}{\diamond} K_n) \leq n$. Orlin proved that $cp(K_n - e) = n - 1$ (Orlin, 1977) for $n \geq 3$ and e is an edge of K_n. Since $n \leq m$, we obtain that $cp(K_m \underset{K_2}{\diamond} K_n) > n - 1$. Thus, $cp(K_m \underset{K_2}{\diamond} K_n) = n$. Note that $cp(K_m) + cp(K_n) = 2$ while $cp(K_m \underset{K_2}{\diamond} K_n)$ can be arbitrary large. It is an evident that a graph gluing even at the smallest clone may use much more cliques to partition our resulting glued graphs than those of original graphs.

This leads us to consider a possible upper bound for the clique partition numbers of arbitrary glued graphs. For any glued graphs of G_1 and G_2 at a clone H, since the union of a minimum clique partition of G_1 (or G_2) and minimum clique partition of $G_2 - H$ (or $G_1 - H$) provides a clique partition of $G_1 \underset{H}{\diamond} G_2$, it yields that

$$cp(G_1 \underset{H}{\diamond} G_2) \leq \min_{\substack{i,j=1,2 \\ i \neq j}} \{cp(G_i) + cp(G_j - H)\}. \tag{1}$$

Our main results focus on the graph gluing at K_2-clone and K_3-clone. We give some characterizations of glued graphs with the clique partition numbers of some possible values.

2. Preliminaries

For convenience, throughout the rest of the paper, we refer K_n in the glued graph $G_1 \underset{K_n}{\diamond} G_2$ to be only the K_n-clone, not an arbitrary copy of K_n in our graphs. In a clique partition, if an edge e is covered by a 2-clique itself, e is also refered to as such a clique. Moreover, we will also refer $E(G)$ to be a set of 2-cliques.

Definition 1 Let $\mathscr{P}$ be a minimum clique partition of a glued graph $G_1 \underset{H}{\diamond} G_2$. We define
$\mathscr{P}[G_1] = \{C \in \mathscr{P} \mid C \text{ is a clique of } G_1\}$ and $\mathscr{P}[G_2] = \{C \in \mathscr{P} \mid C \text{ is a clique of } G_2\}$.

Note from the Definition 1 that $\mathscr{P}[G_1] \cup \mathscr{P}[G_2] \subseteq \mathscr{P}$. However $G_1 \underset{H}{\diamond} G_2$ contains a new clique C if and only if $C \notin \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. Any glued graph at K_n-clone does not have a new clique, so we have the following result:

Proposition 2 For a minimum clique partition $\mathscr{P}$ of $G_1 \underset{K_n}{\diamond} G_2$, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$.

The known results about an edge deletion or n-clique deletion on the clique partition number are concluded next. Theorems 3-7 help us to investigate bounds of clique partition numbers of glued graphs at K_2-clones and K_3-clones.

Theorem 3 (Monson, 1996) *Let s be the order of the smallest clique containing the edge e among all of the minimum clique partitions of G. Then*

$$cp(G) - 1 \leq cp(G - e) \leq cp(G) + s - 2.$$

Theorem 4 (Pullman, 1981) *For $n \geq 4$, $n - 1 \leq cp(K_n - K_3) \leq 2n - 5$.*

Theorem 5 *For any graph G and clique C of G, $cp(G) - 1 \leq cp(G - C)$, and the equality holds if and only if there exists a minimum clique partition of G containing C.*

Proof. Let $\mathscr{P}$ and $\mathscr{P}'$ be minimum clique partitions of G and $G - C$, respectively. Then $\mathscr{P}' \cup \{C\}$ is a clique partition of G, so $cp(G) \leq |\mathscr{P}' \cup \{C\}|$ and hence $cp(G) - 1 \leq cp(G - C)$.

If every minimum clique partition of G does not contain C, then $\mathscr{P}' \cup \{C\}$ cannot be a minimum clique partition of G. So, $|\mathscr{P}' \cup \{C\}| > |\mathscr{P}|$. Hence, $cp(G - C) + 1 = |\mathscr{P}' \cup \{C\}| > |\mathscr{P}| = cp(G)$.

Suppose that $\mathscr{P}$ contains C. Then $\mathscr{P} \smallsetminus \{C\}$ is a clique partition of $G - C$. Thus, $cp(G - C) \leq |\mathscr{P}| - 1 = cp(G) - 1$. Therefore, the equality holds.

$\square$

Theorem 6 *Let G_1 and G_2 be graphs containing K_n as a subgraph. If there exists a minimum clique partition of G_1 or G_2 containing the K_n-clone, then* $cp(G_1 \underset{K_n}{\oplus} G_2) \leq cp(G_1) + cp(G_2) - 1$

Proof. It follows immediately from equation (1) and Theorem 5. $\square$

Lemma 12 *If there exists a minimum clique partition of $G_1 \underset{K_n}{\oplus} G_2$ containing the K_n-clone, then there exists a minimum clique partition of G_1 or G_2 containing the K_n-clone.*

Proof. Let $\mathscr{P}$ and $\mathscr{P}_1$ be minimum clique partitions of $G_1 \underset{K_n}{\oplus} G_2$ containing the K_n-clone and G_1, respectively. Since the glued graph at K_n-clone has no new cliques, $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$ are clique partitions of G_1 and G_2, respectively. Let $\mathscr{P}_1$ be a minimum clique partition of G_1. Note that $\mathscr{P} = (\mathscr{P}[G_1] \smallsetminus \{K_n\}) \cup (\mathscr{P}[G_2] \smallsetminus \{K_n\}) \cup \{K_n\}$. Now, suppose that all minimum clique partitions of G_1 and G_2 do not contain the K_n-clone. Then $\mathscr{P}_1$ does not contain the K_n-clone. Since $\mathscr{P}[G_1]$ is a clique partition of G_1 containing the K_n-clone, $|\mathscr{P}[G_1]| > |\mathscr{P}_1| = cp(G_1)$. Consequently, $|\mathscr{P}[G_1] \smallsetminus \{K_n\}| \geq cp(G_1)$. Thus,

$$|\mathscr{P}| = |\mathscr{P}[G_1] \smallsetminus \{K_n\}| + |\mathscr{P}[G_2] \smallsetminus \{K_n\}| + 1 \geq cp(G_1) + |\mathscr{P}[G_2] \smallsetminus \{K_n\}| + 1.$$

Observe that $\mathscr{P}_1 \cup (\mathscr{P}[G_2] \smallsetminus \{K_n\})$ is also a clique partition of $G_1 \underset{K_n}{\oplus} G_2$ and $|\mathscr{P}_1 \cup (\mathscr{P}[G_2] \smallsetminus \{K_n\})| = cp(G_1) + |\mathscr{P}[G_2] \smallsetminus \{K_n\}|$, this contradicts the minimality of $\mathscr{P}$. $\square$

Theorem 7 *If there exists a minimum clique partition of $G_1 \underset{K_n}{\oplus} G_2$ containing the K_n-clone, then*

$$cp(G_1 \underset{K_n}{\oplus} G_2) = cp(G_1) + cp(G_2) - 1.$$

Proof. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_n}{\oplus} G_2$ containing the K_n-clone. By Proposition 2, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. Since $\mathscr{P}$ contains the K_n-clone, $\mathscr{P}[G_1] \cap \mathscr{P}[G_2] = \{K_n\}$. Note that $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$ are clique partitions of G_1 and G_2, respectively, so $|\mathscr{P}[G_1]| \geq cp(G_1)$ and $|\mathscr{P}[G_2]| \geq cp(G_2)$. Thus,

$$cp(G_1 \underset{K_n}{\oplus} G_2) = |\mathscr{P}| = |\mathscr{P}[G_1] \cup \mathscr{P}[G_2]|$$
$$= |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| - |\mathscr{P}[G_1] \cap \mathscr{P}[G_2]|$$
$$\geq cp(G_1) + cp(G_2) - 1.$$

By Lemma 12, without loss of generality, G_1 has a minimum clique partition, $\mathscr{P}_1$, containing the K_n-clone. Let $\mathscr{P}_2$ be a minimum clique partition of G_2. Then $(\mathscr{P}_1 \smallsetminus \{K_n\}) \cup \mathscr{P}_2$ is a clique partition of $G_1 \underset{K_n}{\oplus} G_2$. Therefore,

$$cp(G_1 \underset{K_n}{\oplus} G_2) \leq |(\mathscr{P}_1 \smallsetminus \{K_n\}) \cup \mathscr{P}_2| = |(\mathscr{P}_1 \smallsetminus \{K_n\})| + |\mathscr{P}_2| = cp(G_1) + cp(G_2) - 1.$$

Hence, $cp(G_1 \underset{K_n}{\oplus} G_2) = cp(G_1) + cp(G_2) - 1$. $\square$

3. Clique Partitions of Glued Graphs at K_2-Clones

Theorem 9 shows bounds of clique covering numbers of any glued graphs at K_2-clones. An analogous statement for clique partition numbers is studied in Theorem 10. Notice that the lower bound has the same style while the upper bound does not. Example B illustrates that $cp(G_1) + cp(G_2)$ is not an upper bound for the clique partition numbers of $G_1 \underset{K_2}{\oplus} G_2$.

Remark 8 *Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_2}{\oplus} G_2$.*
(i) $\mathscr{P}[G_1] \cap \mathscr{P}[G_2] \subseteq \{K_2\}$.
(ii) If the K_2-clone is contained in $\mathscr{P}$, then $\mathscr{P}[G_1] \cap \mathscr{P}[G_2] = \{K_2\}$, and, $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$ are clique partitions of G_1 and G_2, respectively.
(iii) If the K_2-clone is not contained in $\mathscr{P}$, then $\mathscr{P}[G_1] \cap \mathscr{P}[G_2] = \emptyset$, furthermore, if the clone $K_2 \in \mathscr{P}[G_i] \smallsetminus \mathscr{P}[G_j]$ for some $i \neq j \in \{1, 2\}$, then $\mathscr{P}[G_i]$ and $\mathscr{P}[G_j]$ are clique partitions of G_i and $G_j - K_2$, respectively.

Theorem 9 (Pimpasalee et al, 2008) *For any graphs G_1 and G_2,*

$$cc(G_1) + cc(G_2) - 1 \leq cc(G_1 \underset{K_2}{\oplus} G_2) \leq cc(G_1) + cc(G_2). \qquad (2)$$

Theorem 10 *For any graphs G_1 and G_2,*

$$cp(G_1) + cp(G_2) - 1 \le cp(G_1 \underset{K_2}{\diamond} G_2) \le cp(G_1) + cp(G_2) + s - 2 \tag{3}$$

where s is the order of the smallest clique containing the K_2-clone among all of the minimum clique partitions of G_1 and G_2.

Proof. Without loss of generality, we may assume that G_2 has a minimum clique partition containing clique of order s containing the K_2-clone. By equation (1) and Theorem 3, $cp(G_1 \underset{K_2}{\diamond} G_2) \le cp(G_1) + cp(G_2 - K_2) \le cp(G_1) + cp(G_2) + s - 2$.

Now, let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_2}{\diamond} G_2$. If $K_2 \in \mathscr{P}$, then $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) - 1$. Suppose that $K_2 \notin \mathscr{P}$. Then $\mathscr{P}$ is partitioned into $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$. We may assume that $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2] \cup \{K_2\}$ are clique partitions of G_1 and G_2, respectively. Therefore,

$$cp(G_1 \underset{K_2}{\diamond} G_2) = |\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2] \cup \{K_2\}| - 1 \ge cp(G_1) + cp(G_2) - 1.$$

$\square$

Recall from Example B that $cp(K_m \underset{K_2}{\diamond} K_n) = \min\{m, n\}$. Note that for the glued graph $K_m \underset{K_2}{\diamond} K_n$, s defined in Theorem 10 is $\min\{m, n\}$. Thus, $cp(K_m) + cp(K_2) + s - 2 = 1 + 1 + \min\{m, n\} - 2 = cp(K_m \underset{K_2}{\diamond} K_n)$. Therefore, the upper bound in equation (3) is sharp.

A characterization of $G_1 \underset{K_2}{\diamond} G_2$ whose clique covering number satisfying the lower bound in equation (2) is investigated, see Pimpasalee et al. (Pimpasalee et al., 2008). Next, we reveal these analogous statements for glued graphs at K_2-clones with clique partition numbers satisfying the lower bound in equation (3).

Theorem 11 *For any graphs G_1 and G_2, the following statements are equivalent:*

(i) $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) - 1$,

(ii) G_1 *or* G_2 *has a minimum clique partition containing the K_2-clone, and*

(iii) $cp(G_1 - K_2) = cp(G_1) - 1$ *or* $cp(G_2 - K_2) = cp(G_2) - 1$.

Proof. $(ii) \Rightarrow (i)$ This follows directly from Theorems 6 and 10.

$(i) \Rightarrow (iii)$ Assume that $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) - 1$. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_2}{\diamond} G_2$. If $K_2 \in \mathscr{P}$, by Lemma 12, there exists a minimum clique partition of G_1 or G_2 containing the K_2-clone . Applying Theorem 5 yields $cp(G_1 - K_2) = cp(G_1) - 1$ or $cp(G_2 - K_2) = cp(G_2) - 1$. Assume that $K_2 \notin \mathscr{P}$. Then $|\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]|$. We may assume that $\mathscr{P}[G_2]$ is a clique partition of G_2, and then $\mathscr{P}[G_1]$ is a clique partition of $G_1 - K_2$. Thus $cp(G_1) + cp(G_2) - 1 = cp(G_1 \underset{K_2}{\diamond} G_2) = |\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| \ge cp(G_1 - K_2) + cp(G_2)$, so $cp(G_1) - 1 \ge cp(G_1 - K_2)$. Again apply Theorem 3 to have $cp(G_1) - 1 = cp(G_1 - K_2)$.

$(iii) \Rightarrow (ii)$ It follows immediately from Theorems 5. $\square$

Corollary 13 *If $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2)$, then $cp(G_1) \le cp(G_1 - K_2)$ and $cp(G_2) \le cp(G_2 - K_2)$.*

Proof. It follows immediately from Theorems 3 and 11. $\square$

Corollary 14 follows directly from Lemma 12 and Theorem 11 .

Corollary 14 *If there exists a minimum clique partition of $G_1 \underset{K_2}{\diamond} G_2$ containing the K_2-clone, then*

(i) $cp(G_1 \underset{K_2}{\diamond} G_2) = cp(G_1) + cp(G_2) - 1$, *or*

(ii) $cp(G_i - K_2) = cp(G_i) - 1$ *for some $i \in \{1, 2\}$.*

For any graph G, the statement in Theorem 3, namely $cp(G - e) \ge cp(G) - 1$, can be rewritten by $cp(G - e) = cp(G) + t$ for some $t \ge -1$. Next, we consider the clique partition number of a glued graph $G_1 \underset{K_2}{\diamond} G_2$ satisfying $cp(G_i - K_2) = cp(G_i) + t_i$ where $i \in \{1, 2\}$. The special case when $t_i = -1$ for some $i \in \{1, 2\}$ has been already examined in Theorem 11.

Theorem 12 *Let $G_1 \underset{K_2}{\oplus} G_2$ be a glued graph at K_2-clone. If $cp(G_1 - K_2) = cp(G_1) + t_1$ and $cp(G_2 - K_2) = cp(G_2) + t_2$ for some integers t_1, t_2, then $cp(G_1 \underset{K_2}{\oplus} G_2) = cp(G_1) + cp(G_2) + t$ where $t = \min\{t_1, t_2\}$.*

Proof. First, by Theorem 3 we note that $t_1, t_2 \geq -1$. If $t_i = -1$ for some $i \in \{1, 2\}$, then the statement is true by Theorem 11. Otherwise, assume that $0 \leq t_1 \leq t_2$. Since a union of a minimum clique partition of $G_1 - K_2$ and a minimum clique partition of G_2 is a clique partition of $G_1 \underset{K_2}{\oplus} G_2$, $cp(G_1 \underset{K_2}{\oplus} G_2) \leq cp(G_1) + cp(G_2) + t_1$. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_2}{\oplus} G_2$. By Proposition 2, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. Since $t_1, t_2 \geq 0$ and by Corollary 14, the K_2-clone is not in $\mathscr{P}$ and then $\mathscr{P}$ is partitioned into $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$. We have two cases.

Case 1. $\mathscr{P}[G_1]$ is a clique partition of G_1 and $\mathscr{P}[G_2]$ is a clique partition of $G_2 - K_2$. Thus, $cp(G_1 \underset{K_2}{\oplus} G_2) = |\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| \geq cp(G_1) + cp(G_2) + t_2 \geq cp(G_1) + cp(G_2) + t_1$. Hence, $cp(G_1 \underset{K_2}{\oplus} G_2) = cp(G_1) + cp(G_2) + t$ where $t = \min\{t_1, t_2\}$.

Case 2. $\mathscr{P}[G_2]$ is a clique partition of G_2 and $\mathscr{P}[G_1]$ is a clique partition of $G_1 - K_2$. Thus, $cp(G_1 \underset{K_2}{\oplus} G_2) = |\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| \geq cp(G_1) + cp(G_2) + t_1$. Hence, $cp(G_1 \underset{K_2}{\oplus} G_2) = cp(G_1) + cp(G_2) + t$ where $t = \min\{t_1, t_2\}$. □

Now a characterization of $G_1 \underset{K_2}{\oplus} G_2$ satisfying the upper bound in equation (2) which states that $cp(G_1 \underset{K_2}{\oplus} G_2) = cp(G_1) + cp(G_2) + s - 2$ where $s \geq 3$ is obtained in the following theorem.

Theorem 13 *Let $G_1 \underset{K_2}{\oplus} G_2$ be a glued graph of G_1 and G_2 at K_2-clone, and s the order of the smallest clique containing the K_2-clone among all of the minimum clique partitions of G_1 and G_2 where $s \geq 3$. Then $cp(G_1 \underset{K_2}{\oplus} G_2) = cp(G_1) + cp(G_2) + s - 2$ if and only if, $cp(G_i - K_2) \geq cp(G_i) + s - 2$ for each $i \in \{1, 2\}$.*

Proof. Assume that $cp(G_1 \underset{K_2}{\oplus} G_2) = cp(G_1) + cp(G_2) + s - 2$. We have that $cp(G_1 \underset{K_2}{\oplus} G_2) \leq cp(G_j) + cp(G_i - K_2)$ for all $i \neq j \in \{1, 2\}$. It follows that $cp(G_i - K_2) \geq cp(G_i) + s - 2$ for all $i \in \{1, 2\}$.

Conversely, assume that $cp(G_i - K_2) \geq cp(G_i) + s - 2$ for all $i \in \{1, 2\}$. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_2}{\oplus} G_2$. By Proposition 2, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. Since $s \geq 3$ and by Corollary 14, the K_2-clone is not in $\mathscr{P}$. Then $|\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]|$. Without loss of generality, let $\mathscr{P}[G_1]$ be a clique partition of G_1. Then $\mathscr{P}[G_2]$ is a clique partition of $G_2 - K_2$. Thus, $cp(G_1 \underset{K_2}{\oplus} G_2) = |\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| \geq cp(G_1) + cp(G_2 - K_2) \geq cp(G_1) + cp(G_2) + s - 2$. Together with Theorem 10, we have $cp(G_1 \underset{K_2}{\oplus} G_2) = cp(G_1) + cp(G_2) + s - 2$ as desired. □

4. Clique Partitions of Glued Graphs at K_3-Clones

Pimpasalee obtains bounds of clique covering numbers of any glued graphs at K_3-clones as in Theorem 14. We study these analogous statements for clique partition numbers in this section.

Theorem 14 (Pimpasalee, 2008) *For any graphs G_1 and G_2 containing K_3 as a subgraph,*

$$cc(G_1) + cc(G_2) - 2 \leq cc(G_1 \underset{K_3}{\oplus} G_2) \leq cc(G_1) + cc(G_2).$$

Definition 15 *Let $\mathscr{P}$ be a minimum clique partition of a glued graph $G_1 \underset{K_n}{\oplus} G_2$. We define*

$$E_1[\mathscr{P}] = \{e \in E(K_n) \mid e \text{ is not covered by any clique in } \mathscr{P}[G_1]\} \text{ and}$$
$$E_2[\mathscr{P}] = \{e \in E(K_n) \mid e \text{ is not covered by any clique in } \mathscr{P}[G_2]\}.$$

Note that $\mathscr{P}[G_1] \cup E_1[\mathscr{P}]$ and $\mathscr{P}[G_2] \cup E_2[\mathscr{P}]$ are clique partitions of G_1 and G_2, respectively.

Theorem 16 *Let G_1 and G_2 be graphs containing K_3 as a subgraph. Then*

$$cp(G_1) + cp(G_2) - 3 \leq cp(G_1 \underset{K_3}{\oplus} G_2). \tag{4}$$

Proof. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_3}{\oplus} G_2$. Since the clone is a complete graph, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. So, $\mathscr{P} = [(\mathscr{P}[G_1] \cup E_1[\mathscr{P}]) \cup (\mathscr{P}[G_2] \cup E_2[\mathscr{P}])] \setminus (E_1[\mathscr{P}] \cup E_2[\mathscr{P}])$. Thus,

$$cp(G_1 \underset{K_3}{\oplus} G_2) = |\mathscr{P}| = |[(\mathscr{P}[G_1] \cup E_1[\mathscr{P}]) \cup (\mathscr{P}[G_2] \cup E_2[\mathscr{P}])] \setminus (E_1[\mathscr{P}] \cup E_2[\mathscr{P}])|$$

$$= |\mathscr{P}[G_1] \cup E_1[\mathscr{P}])| + |\mathscr{P}[G_2] \cup E_2[\mathscr{P}])| - |(\mathscr{P}[G_1] \cap \mathscr{P}[G_2]) \cup (E_1[\mathscr{P}] \cup E_2[\mathscr{P}])|.$$

Since $|\mathscr{P}[G_1] \cup E_1[\mathscr{P}]| \geq cp(G_1)$ $|\mathscr{P}[G_2] \cup E_2[\mathscr{P}]| \geq cp(G_2)$ and $|(\mathscr{P}[G_1] \cap \mathscr{P}[G_2]) \cup (E_1[\mathscr{P}] \cup E_2[\mathscr{P}])| \leq 3$, we have

$$cp(G_1 \underset{K_3}{\oplus} G_2) \geq cp(G_1) + cp(G_2) - |(\mathscr{P}[G_1] \cap \mathscr{P}[G_2]) \cup (E_1[\mathscr{P}] \cup E_2[\mathscr{P}])|$$
$$\geq cp(G_1) + cp(G_2) - 3.$$

□

Definition 17 *Let G be a graph containing a triangle T and $\mathscr{P}$ a minimum clique partition of G. Then we say that*

1. *$\mathscr{P}$ is **type 1 with respect to the triangle** T, if $\mathscr{P}$ contains the triangle T;*

2. *$\mathscr{P}$ is **type 2 with respect to the triangle** T, if $\mathscr{P}$ contains a clique of order at least 4 covering the triangle T;*

3. *Otherwise, $\mathscr{P}$ is **type 3 with respect to the triangle** T, that is, each edge of the triangle T is covered by different cliques in $\mathscr{P}$.*

Remark 18 *Let $G_1 \underset{K_3}{\oplus} G_2$ be a glued graph at K_3-clone and $\mathscr{P}$ a minimum clique partition of $G_1 \underset{K_3}{\oplus} G_2$.*

(i) If $\mathscr{P}$ is type 1 with respect to the K_3-clone, then $\mathscr{P}[G_1] \cap \mathscr{P}[G_2] = \{K_3\}$, and hence, $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$ are clique partitions of G_1 and G_2, respectively.

(ii) If $\mathscr{P}$ is type 2 with respect to the K_3-clone, then $\mathscr{P}[G_1] \cap \mathscr{P}[G_2] = \emptyset$, furthermore, $\mathscr{P}[G_i]$ and $\mathscr{P}[G_j]$ are clique partitions of G_i and $G_j - K_3$, respectively, for some $i \neq j \in \{1, 2\}$.

(iii) If $\mathscr{P}$ is type 3 with respect to the K_3-clone, then an element in $\mathscr{P}[G_1] \cap \mathscr{P}[G_2]$ is a proper subset of $E(K_3)$, consequently, $|\mathscr{P}[G_1] \cap \mathscr{P}[G_2]| = 0, 1$ or 2.

Theorem 19 *Let $G_1 \underset{K_3}{\oplus} G_2$ be a glued graph at K_3-clone. If $G_1 \underset{K_3}{\oplus} G_2$ has a minimum clique partition which is type 1 or type 2 with respect to the K_3-clone, then $cp(G_1 \underset{K_3}{\oplus} G_2) \geq cp(G_1) + cp(G_2) - 1$.*

Proof. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_3}{\oplus} G_2$. By Proposition 2, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$.

Case 1. $\mathscr{P}$ is type 1 with respect to the K_3-clone. By Theorem 7, $cp(G_1 \underset{K_3}{\oplus} G_2) = cp(G_1) + cp(G_2) - 1$.

Case 2. $\mathscr{P}$ is type 2 with respect to the K_3-clone. Without loss of generality, the clique of order at least 4 containing the K_3-clone is in $\mathscr{P}[G_1]$. Then $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2] \cup \{K_3\}$ are clique partitions of G_1 and G_2, respectively. Thus,

$$cp(G_1 \underset{K_3}{\oplus} G_2) = |\mathscr{P}| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2]| = |\mathscr{P}[G_1]| + |\mathscr{P}[G_2] \cup \{K_3\}| - 1 \geq cp(G_1) + cp(G_2) - 1.$$

□

Theorem 20 *Let $G_1 \underset{K_3}{\oplus} G_2$ be a glued graph of G_1 and G_2 at K_3-clone. Then $cp(G_1 \underset{K_3}{\oplus} G_2) = cp(G_1) + cp(G_2) - 3$ if and only if there exist minimum clique partitions $\mathscr{P}_1$ and $\mathscr{P}_2$ of G_1 and G_2, respectively, such that for each edge $e \in E(K_3)$, e must be covered by a 2-clique in $\mathscr{P}_1$ or $\mathscr{P}_2$.*

Proof. First, assume that $cp(G_1 \underset{K_3}{\oplus} G_2) = cp(G_1) + cp(G_2) - 3$. Let $\mathscr{P}$ be a minimum clique partition of $G_1 \underset{K_3}{\oplus} G_2$. By Proposition 2, $\mathscr{P} = \mathscr{P}[G_1] \cup \mathscr{P}[G_2]$. Note that

$$|\mathscr{P}| = |\mathscr{P}[G_1] \cup E_1[\mathscr{P}]| + |\mathscr{P}[G_2] \cup E_2[\mathscr{P}]| - |\mathscr{P}[G_1] \cap \mathscr{P}[G_2]| - |E_1[\mathscr{P}] \cup E_2[\mathscr{P}]|.$$

Moreover, we have that $|\mathscr{P}| = cp(G_1) + cp(G_2) - 3$, $\mathscr{P}[G_i] \cup E_i[\mathscr{P}]$ is a clique partition of G_i for all $i \in \{1, 2\}$ and $|\mathscr{P}[G_1] \cap \mathscr{P}[G_2]| + |E_1[\mathscr{P}] \cup E_2[\mathscr{P}]| \leq 3$. Hence, $|\mathscr{P}[G_i] \cup E_i[\mathscr{P}]| = cp(G_i)$ for all $i \in \{1, 2\}$ and $|\mathscr{P}[G_1] \cap \mathscr{P}[G_2]| + |E_1[\mathscr{P}] \cup E_2[\mathscr{P}]| = 3$. Therefore $\mathscr{P}[G_i] \cup E_i[\mathscr{P}]$ is a minimum clique partition of G_i for all $i \in \{1, 2\}$. Let e be an edge in the K_3-clone of $G_1 \underset{K_3}{\oplus} G_2$. If $e \in \mathscr{P}[G_1] \cap \mathscr{P}[G_2]$, then e is covered by a 2-clique in $\mathscr{P}[G_1]$ and $\mathscr{P}[G_2]$. Thus $e \in \mathscr{P}[G_i] \cup E_i[\mathscr{P}]$ for all $i \in \{1, 2\}$. Suppose that $e \notin \mathscr{P}[G_1] \cap \mathscr{P}[G_2]$. Then there exists a clique C of order more than two in $\mathscr{P}$ covering e. Without loss of generality, assume that $C \in \mathscr{P}[G_1]$. Then $e \in E_2[\mathscr{P}]$, so $e \in \mathscr{P}[G_2] \cup E_2[\mathscr{P}]$.

For sufficiency, assume that G_1 and G_2 have minimum clique partitions $\mathscr{P}_1$ and $\mathscr{P}_2$, respectively, such that satisfy the condition in the right hand side of the statement. Let $A = \{e \in E(K_3) \mid e \in \mathscr{P}_1\}$ and $B = \{e \in E(K_3) \mid e \in \mathscr{P}_2\}$. Note

that $|A| + |B| - |A \cap B| = e(K_3) = 3$ and $(\mathscr{P}_1 \smallsetminus A) \cup (\mathscr{P}_2 \smallsetminus B) \cup (A \cap B)$ is a clique partition of $G_1 \underset{K_3}{\oplus} G_2$. Thus, $|(\mathscr{P}_1 \smallsetminus A) \cup (\mathscr{P}_2 \smallsetminus B) \cup (A \cap B)| \geq cp(G_1 \underset{K_3}{\oplus} G_2)$. Hence,

$$
\begin{aligned}
cp(G_1) + cp(G_2) - 3 &= |\mathscr{P}_1| + |\mathscr{P}_2| - |A| - |B| + |A \cap B| \\
&= |(\mathscr{P}_1 \smallsetminus A) \cup (\mathscr{P}_2 \smallsetminus B) \cup (A \cap B)| \\
&\geq cp(G_1 \underset{K_3}{\oplus} G_2).
\end{aligned}
$$

By Theorem 16, $cp(G_1 \underset{K_3}{\oplus} G_2) = cp(G_1) + cp(G_2) - 3$. □

The following theorem gives the possible values of $cp(G_1 \underset{K_3}{\oplus} G_2)$ when G_1 or G_2 has a minimum clique partition which is type 1 with respect to the K_3-clone.

Theorem 21 *Let $G_1 \underset{K_3}{\oplus} G_2$ be a glued graph at K_3-clone. If G_1 or G_2 has a minimum clique partition which is type 1 with respect to the K_3-clone, then*

$$
cp(G_1) + cp(G_2) - 3 \leq cp(G_1 \underset{K_3}{\oplus} G_2) \leq cp(G_1) + cp(G_2) - 1.
$$

Proof. It follows directly from Theorems 6 and 16. □

Theorem 22 *Let $G_1 \underset{K_3}{\oplus} G_2$ be any graph at K_3-clone and $\mathscr{P}_i$ a minimum clique partition of G_i for each $i \in \{1, 2\}$. Then*

$$
cp(G_1 \underset{K_3}{\oplus} G_2) \leq cp(G_1) + cp(G_2) - 6 + \min\{\sigma_1, \sigma_2\}
$$

where for each $i \in \{1, 2\}$, s_i is the sum of orders of all cliques in $\mathscr{P}_i$ containing edges of the K_3-clone, and

$$
\sigma_i = \begin{cases} 2s_i & \text{if } \mathscr{P}_i \text{ is type 2 with respect to the } K_3 - \text{clone}, \\ s_i & \text{if } \mathscr{P}_i \text{ is type 3 with respect to the } K_3 - \text{clone}. \end{cases}
$$

Proof. Assume that $\mathscr{P}_1$ and $\mathscr{P}_2$ are minimum clique partitions of G_1 and G_2, respectively.

Here we suppose that $\mathscr{P}_1$ and $\mathscr{P}_2$ are type 2 and type 3 with respect to the K_3-clone, respectively. For other cases, the proof follows similarly. Since $\mathscr{P}_1$ is type 2 with respect to the K_3-clone, there exists a clique $C \in \mathscr{P}_1$ with order $r \geq 4$ containing K_3. Then $G_1 - K_3$ can be partitioned by the union of $\mathscr{P}_1 \smallsetminus \{C\}$ and a minimum clique partition of $C - K_3$. By Theorem 4, $cp(C - K_3) \leq 2r - 5$. Thus $cp(G_1 - K_3) \leq |\mathscr{P}_1| - 1 + 2r - 5 = cp(G_1) + 2r - 6$. Since $\mathscr{P}_2$ is type 3 with respect to the K_3-clone, there exists three cliques in $\mathscr{P}_2$ such that each one covers different edge in the K_3-clone, say Q_1, Q_2 and Q_3 of order q_1, q_2 and q_3, respectively. Then $G_2 - K_3$ can be partitioned by the union of $\mathscr{P}_2 \smallsetminus \{Q_1, Q_2, Q_3\}$ and a minimum clique partition of Q_i deleted an edge in the K_3-clone for all $i \in \{1, 2, 3\}$. We have that $cp(Q_i - e_i) = q_i - 1$ where Q_i covers an edge e_i in the K_3-clone for all $i \in \{1, 2, 3\}$. Thus,

$$
cp(G_2 - K_3) \leq |\mathscr{P}_2| - 3 + (q_1 - 1) + (q_2 - 1) + (q_3 - 1) = cp(G_2) + q_1 + q_2 + q_3 - 6.
$$

Hence, $cp(G_1 \underset{K_3}{\oplus} G_2) \leq cp(G_1) + cp(G_2) - 6 + \min\{\sigma_1, \sigma_2\}$ where $\sigma_1 = 2r$ and $\sigma_2 = q_1 + q_2 + q_3$. □

5. An Application

In general, the problem of computing clique partition numbers is considered to be difficult. Only for some classes of graphs can be obtained their clique partition numbers or even given reasonable bounds. The problem of determining the clique partition numbers of graphs is NP-complete, see Ma et al. (Ma et al., 1988). In an appropriate way, the glue operator could sometimes be used to reduce a large graph into smaller graphs which are easier to find their clique partition numbers than directly find the clique partition number of the large graph.

Example C Consider a graph G shown in Figure 2. Here we illustrate the usefulness of the glue operator to find $cp(G)$. We refer a bold edge in G to be the K_2-clone and reduce the graph G into G_1, G_2 and G_3 as shown as in Figure 2. Then $G \cong (G_1 \underset{K_2}{\oplus} G_2) \underset{K_2}{\oplus} G_3$. Note that $cp(G_1) = 5$, $cp(G_2) = 1$ and $cp(G_3) = 3$. Since G_1 has a minimum clique partition containing the K_2-clone, by Theorem 10, $cp(G_1 \underset{K_2}{\oplus} G_2) = cp(G_1) + cp(G_2) - 1 = 5 + 1 - 1 = 5$. Similarly, G_3 has a minimum clique partition containing the K_2-clone, and again by Theorem 10, $cp((G_1 \underset{K_2}{\oplus} G_2) \underset{K_2}{\oplus} G_3) = cp(G_1 \underset{K_2}{\oplus} G_2) + cp(G_3) - 1 = 5 + 3 - 1 = 7$. Hence, $cp(G) = 7$.

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Uiyyasathian C. and Hemakul W. (2006). Planar Glued Graphs. *Scientific Research (Section T)*, 3, 109-118.
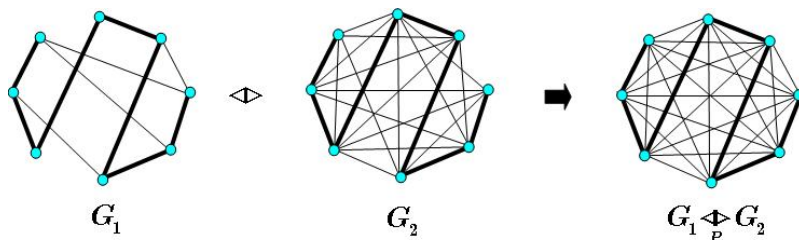
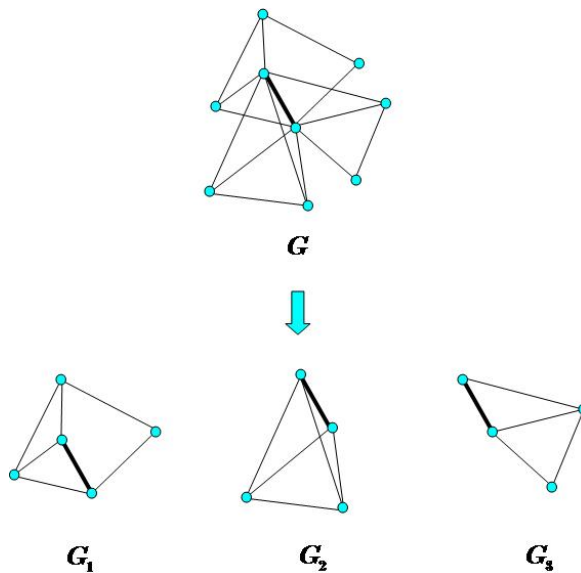Figure 1. A glued graph with clique partition number less than those of both original graph

Figure 2. Picture illustrating Example C