Journal of Mathematics Research; Vol. 6, No. 4; 2014
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

Effect of Some Geometric Transfers on Homology Groups

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Received: September 17, 2014 Accepted: October 21, 2014 Online Published: October 30, 2014

doi:10.5539/jmr.v6n4p90 URL: http://dx.doi.org/10.5539/jmr.v6n4p90

Abstract

In this work, we introduce the results of some geometric transformation of the manifold on the homology group. Some types of folding and unfolding on a wedge sum of manifolds which are determined by their homology group are obtained. Also, the homology group of the limit of folding and unfolding on a wedge sum of 2-manifolds is deduced.

Keywords: manifolds, homology group, folding, unfolding.

Mathematics Subject Classification: 51H20, 57N10, 57T10.

1. Introduction

The notion of folding on manifolds has been introduced by (Robertson, 1977). The conditional folding of manifold and a graph folding have been defined by (El-Kholy, 1981-2005). Also, the unfolding of a manifold has been defined and discussed by (M.El-Ghoul, 1988). Many authors have studied the folding of many types of manifolds. The homology groups of some types of a manifold are discussed by (M.El-Ghoul, 1990; M.Abu-Saleem, 2010). (Abu-Saleem, 2007) introduced the results of some geometric transformation of the manifold on the fundamental group. In this paper, we introduce the folding and unfolding of some types of manifolds which are determined by their homology group and we study and discuss the homology group of the limit of folding and unfolding on a wedge sum of 2-manifolds.

2. Preliminaries

In this section, we introduce some necessary definitions which are needed especially in this paper.

Definition 2.1 The n-dimension manifold is a Hausdorff space such that each point has an open neighborhood homeomorphic to the open n-dimensional disc $U_n = \{x \in R^n : |x| < 1\}$, where n is positive integer (W. S. Massey, 1976).

Definition 2.2 An abstract simplicial complex is a pair $X = (V, S)$ where V is a finite set whose elements are called the vertices of X and S is a set of non-empty subsets of V. Each element $s \in S$ has precisely $n+1$ elements $(n \geq 0)$, s is called an n-simplex. (Thus an abstract simplex is merely the set of its vertices). The simplexes of X satisfy the following conditions;

(1) $v \in V \Rightarrow \{v\} \in S$;

(2) $s \in S, t \subset s, t \neq \phi \Rightarrow t \in S$.

The dimension of S is n and the dimension of X is the largest of the dimensions of its simplexes (P.J. Giblin, 1977).

Definition 2.3 Let $s = [v^0,...,v^n]$ be an oriented n-simplex of S for some $n > 0$. The boundary homomorphism ∂_n of s is $(n-1)$-chain

$\partial_n[v^0,...,v^n] = \sum_{i=0}^{n} (-1)^i [v^0,...,v^{j-1}, v^{j+1},...,v^n]$ i.e. $\partial_n : C_n \to C_{n-1}$ and for $n = 0$, ∂_0 is the null function (P.J. Giblin, 1977).

Definition2.4 The sequence $\cdots \xrightarrow{\partial_{n+2}} C_{n+1}(S) \xrightarrow{\partial_{n+1}} C_n(S) \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_2} C_1(S) \xrightarrow{\partial_1} C_0(S)$ is called the chain complex of S. For any n, $\partial_n \circ \partial_{n+1} = 0$. In the sequence, $\ker \partial_n$ is denoted by $Z_n(S)$, and elements of $Z_n(S)$ are called n-cycles. Also, $\operatorname{Im} \partial_{n+1}$ is denoted by $B_n(S)$, and elements of $B_n(S)$ are called n-boundaries. And since $B_n \subset Z_n$ there is a quotient group $H_n = Z_n / B_n$, called the n-th homology group of S (P.J. Giblin, 1977).

Notation: $H_n(S)$ is measure the number of independent n-dimensions of holes in S, where $0 \le n \le \dim S$.

Definition 2.5 Let M and N be two manifolds of dimensions m and n respectively. A map $f : M \to N$ is said to be an isometric folding of M into N if, for every piecewise geodesic path $\gamma : I \to M, (I = [0,1] \subseteq R)$, the induced path $f \circ \gamma : I \to N$ is piecewise geodesic and of the same length as γ. If f does not preserve length, it is called a topological folding (E.El-Kholy, 1981; S.A.Robertson, 1977).

Definition 2.6 Let M and N be two manifolds of the same dimensions. A map $g : M \to N$ is said to be unfolding of M into N if, for every piecewise geodesic path $\gamma : I \to M$, the induced path $g \circ \gamma : I \to N$ is piecewise geodesic with length greater than γ (M.El-Ghoul, 1988).

Definition 2.7 Let M and N be two manifolds of the same dimensions and $unf : M \to M'$ be any unfolding of M into M'. Then, a map $\overline{unf} : H_n(M) \to H_n(M')$ is said to be an induced unfolding of $H_n(M)$ into $H_n(M')$ if

$\overline{unf}(H_n(M)) = H_n(unf(M))$ (M.Abu-Saleem, 2010).

Definition 2.8 Let X and Y be spaces, and choose points $x_0 \in X$, $y_0 \in Y$ in each space. The wedge sum $X \vee Y$ is the quotient of the disjoint union $X \cup Y$ obtained by identifying x_0 and y_0 to a single point $X \vee Y = X \bigcup_{x_0 \sim y_0} Y \approx (X \coprod Y) / x_0 \sim y_0$ (A.Hatcher, 2001, http://www.math.coronell.edu/hatcher).

3. The Main Results

In this section, we will introduce the following:

Lemma 3.1 The homology group $H_n(S^2)$ of any folding of S^2 is either isomorphic to Z or identity group.

Proof. First, for folding with singularity of S^2 as in Figure 1(a) then

$H_0(F(S^2)) \approx Z$ and for $n > 0$, $H_n(F(S^2)) = 0$. Also, folding without singularity of S^2 is a manifold homeomorphic to S^2 as in Figure 1(b) and so $H_0(F(S^2)) \approx Z$, $H_1(F(S^2)) = 0$, $H_2(F(S^2)) \approx Z$ and for $n > 2$, $H_n(F(S^2)) = 0$.

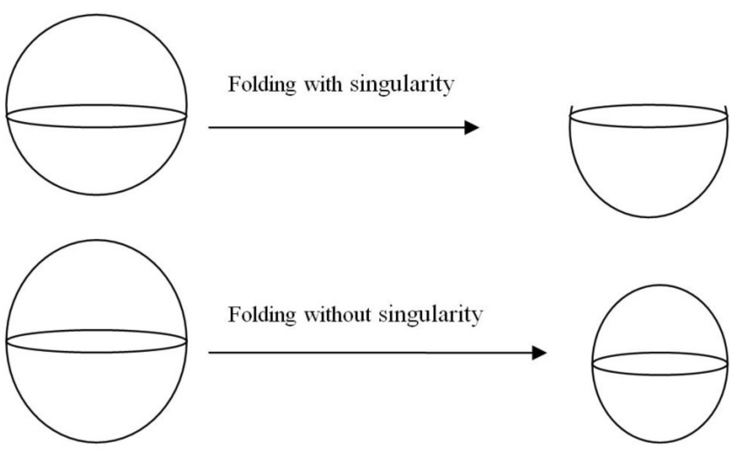

Figure 1

Corollary 3.2 The homology group of the limit of folding and unfolding of a manifold which is homeomorphic to $S^2, n > 2$ is the identity group.

Theorem 3.3 The homology group $H_n(T)$ of any folding of T is either isomorphic to Z , $Z \times Z$ or identity group.

Proof. First, for folding with singularity of T as in figure 2(a) then $H_0(F(T)) \approx Z$, $H_1(F(T)) \approx Z$ and for $n \geq 2$, $H_n(F(T)) = 0$. Also, folding without singularity of T is a manifold homeomorphic to T as in figure 2(b) and so $H_0(F(T)) \approx Z$, $H_1(F(T)) = Z \times Z$, $H_2(F(T)) \approx Z$, and $H_n(F(T)) = 0$, for $n > 2$.

Therefore any folding of T is either isomorphic to Z , $Z \times Z$ or identity group.

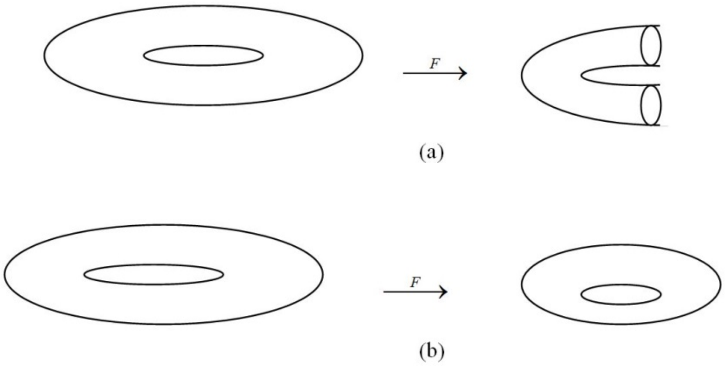

Figure 2

Corollary 3.4 The homology group of the limit of folding and unfolding of a manifold which is homeomorphic to $T, n > 2$ is the identity group.

Theorem 3.5 If $F_i : S_1^2 \vee S_2^2 \to S_1^2 \vee S_2^2, i = 1,2$ are two types of folding such that

$F_i(S_j^2) = S_i^2, j = 1,2$. Then, $H_n(\lim\limits_{n\to\infty} F_{i_n}(S_1^2 \vee S_2^2))$ is isomorphic to Z or identity group.

Proof. If $F_i : S_1^2 \vee S_2^2 \to S_1^2 \vee S_2^2, i = 1, 2$ are two types of foldings such that $F_i(S_j^2) = S_i^2, j = 1,2$, then $\lim\limits_{n\to\infty} F_{i_n}(S_1^2 \vee S_2^2) = S_i^2$ as in Figure (3). Thus, $H_n(\lim\limits_{n\to\infty} F_{i_n}(S_1^2 \vee S_2^2)) = H_n(S_i^2)$. Therefore, $H_n(\lim\limits_{n\to\infty} F_{i_n}(S_1^2 \vee S_2^2))$ is isomorphic to Z or identity group.

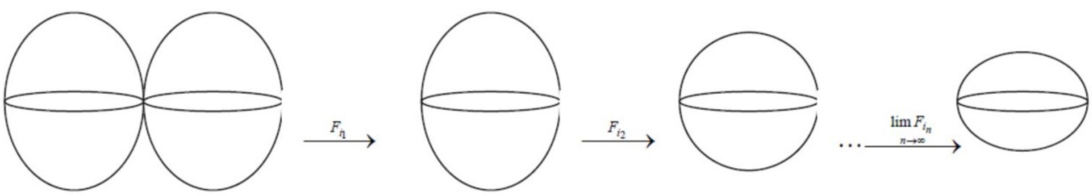

Figure 3

Theorem 3.6 If $F_i : T_1 \vee T_2 \to T_1 \vee T_2$, $i = 1,2$ are two types of foldings such that $F_i(T_j) = T_i, j = 1,2$. Then $H_n(\lim\limits_{n\to\infty} F_{i_n}(T_1 \vee T_2))$ is isomorphic to Z, $Z \times Z$ or identity group, for all n.

Proof. If $F_i : T_1 \vee T_2 \to T_1 \vee T_2, i = 1,2$ are two types of foldings such that $F_i(T_j) = T_i, j = 1,2$, then $\lim_{n\to\infty} F_{i_n}(T_1 \vee T_2)) = T_i$ as in Figure 4, thus $H_n(\lim_{n\to\infty} F_{i_n}(T_1 \vee T_2)) = H_n(T_i)$, since $H_0(T_i) \approx Z, H_1(T_i) \approx Z \times Z, H_2(T_i) \approx Z,$ and if $k > 2, H_n(T_i) \approx 0$ therefore, $H_n(\lim_{n\to\infty} F_{i_n}(T_1 \vee T_2))$ is isomorphic to Z, $Z \times Z$ or identity group.

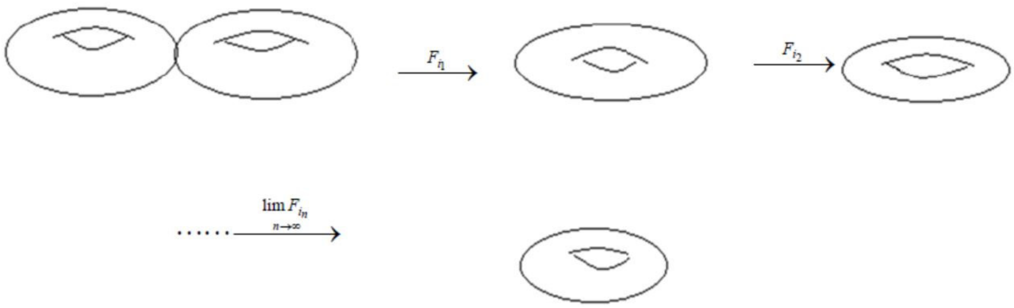

Figure 4

Theorem 3.7 If $F : S_1^2 \vee S_2^2 \to S_1^2 \vee S_2^2$ is a folding by cut such that $F(S_i^2) \neq S_i^2$, for $i = 1, 2$. Then $H_n(\lim_{n\to\infty} F_n(S_1^2 \vee S_2^2))$ is isomorphic to identity group, for all $n > 0$.

Proof. Consider $F(S_i^2) \neq S_i^2, for\, i = 1,2$, then we have the following:

If $\lim_{n\to\infty} F_n(S_1^2 \vee S_2^2)$ as in Figure 5, then $H_n(\lim_{n\to\infty} F_n(S_1^2 \vee S_2^2)) = 0, forall\, n > 0$, therefore $H_n(\lim_{n\to\infty} F_n(S_1^2 \vee S_2^2))$ is isomorphic to identity group, for all $n > 0$.

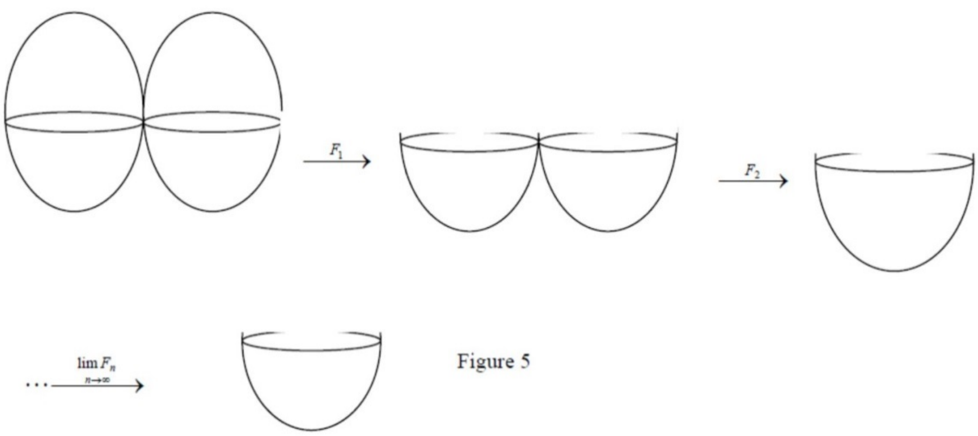

Figure 5

Theorem 3.8 If $F_i : S_1^2 \vee S_2^2 \to S_1^2 \vee S_2^2, i = 1,2$ are two types of foldings such that $F_i(S_i^2) = S_i^2, F_j(S_i^2) \neq S_i^2, j = 1,2, i \neq j$. Then $H_n(\lim_{n\to\infty} F_{i_n}(S_1^2 \vee S_2^2))$ is isomorphic to Z or identity group, for all n.

Proof. Since $F_i(S_i^2) = S_i^2, F_j(S_i^2) \neq S_i^2, j = 1,2, i \neq j$, we have the following:

If $\lim_{n\to\infty} F_{i_n}(S_1^2 \vee S_2^2) = S_i^2$ as in Figure 6. Thus, $H_n(\lim_{n\to\infty} F_{i_n}(S_1^2 \vee S_2^2)) = H_n(S_i^2)$, therefore $H_n(\lim_{n\to\infty} F_{i_n}(S_1^2 \vee S_2^2))$ is isomorphic to Z or identity group, for all n.

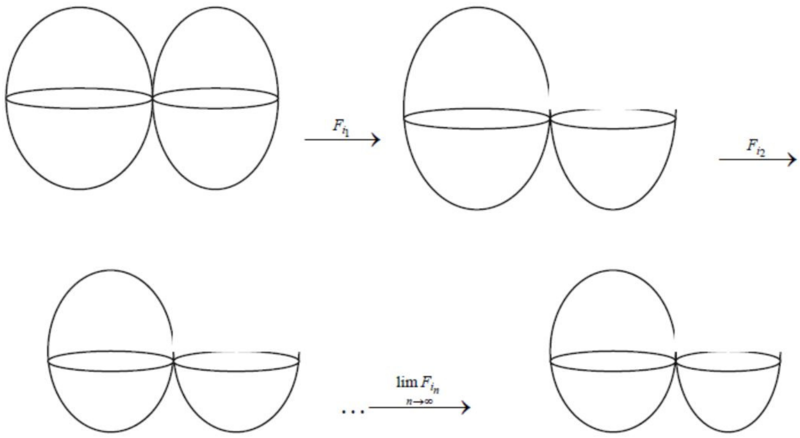

Figure 6

Theorem 3.9 If $F: T_1 \vee T_2 \to T_1 \vee T_2$ is a folding by cut such that $F(T_i) \neq T_i, for\, i = 1,2$. Then $H_n(\lim_{n\to\infty} F_n(T_1 \vee T_2))$ is isomorphic to Z, $Z \times Z$ or identity group, for all $n > 0$.

Proof. Consider $F(T_i) \neq T_i, for\, i = 1,2$, then we have the following:

If $\lim_{n\to\infty} F_n(T_1 \vee T_2) = S_1^1 \vee S_2^1$ as in Figure 7(a), then $H_n(\lim_{n\to\infty} F_n(T_1 \vee T_2)) \approx H_n(S_1^2) \times H_n(S_2^2)$, so $H_n(\lim_{n\to\infty} F_n(T_1 \vee T_2)) \approx Z \times Z$, or 0 for all $n > 0$.

Also, if $\lim_{n\to\infty} F_n(T_1 \vee T_2)$ as in Figure 7(b), then $H_n(\lim_{n\to\infty} F_n(T_1 \vee T_2)) \approx 0$, for all $n > 0$. Moreover, if $\lim_{n\to\infty} F_n(T_1 \vee T_2)$ as in Figure 7(c), then $H_n(\lim_{n\to\infty} F_n(T_1 \vee T_2)) \approx Z$, or 0 for all $n > 0$. Therefore, $H_n(\lim_{n\to\infty} F_n(T_1 \vee T_2))$ is isomorphic to Z, $Z \times Z$ or identity group, for all $n > 0$.

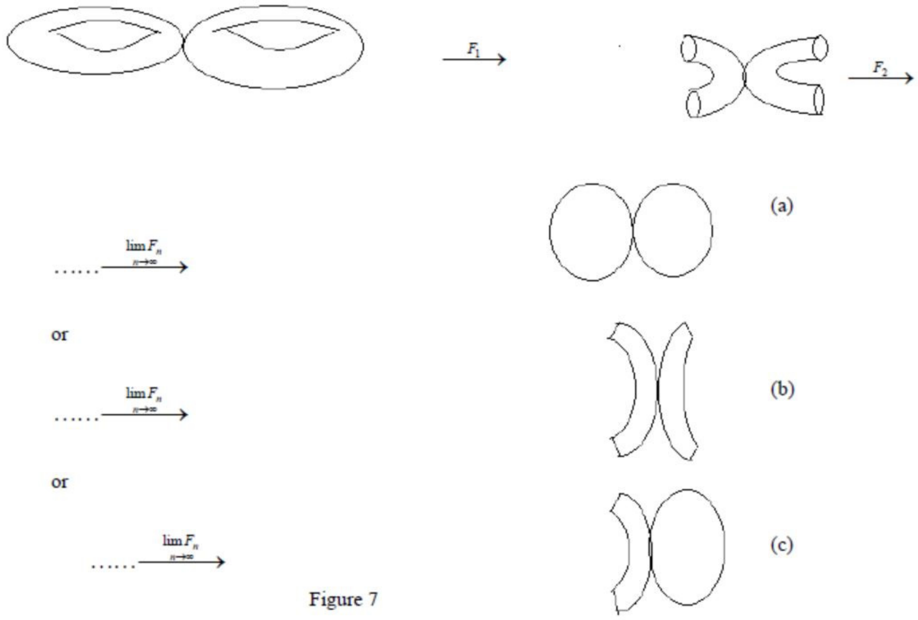

Figure 7

Theorem 3.10 If $F_i : T_1 \vee T_2 \to T_1 \vee T_2, i = 1,2$ are two types of foldings such that $F_i(T_i) = T_i, F_j(T_i) \neq T_i, j = 1,2, i \neq j$. Then $H_n(\lim_{n\to\infty} F_{i_n}(T_1 \vee T_2))$ is isomorphic to $(Z \times Z) \times Z$, $Z \times Z$ or identity group, for all $n > 0$.

Proof. If $F_i : T_1 \vee T_2 \to T_1 \vee T_2, i = 1,2$ are two types of foldings such that $F_i(T_i) = T_i, F_j(T_i) \neq T_i, j = 1,2, i \neq j$, we have the following:

If $\lim_{n\to\infty} F_{i_n}(T_1 \vee T_2) = T_i \vee S_i^1$ as in Figure 8(a). Then, $H_n(\lim_{n\to\infty} F_{i_n}(T_1 \vee T_2)) = H_n(T_i \vee S_i^1) \approx (Z \times Z) \times Z$ or 0, for all $n > 0$.

Also, if $H_n(\lim_{n\to\infty} F_{i_n}(T_1 \vee T_2)) = H_n(T_i) \approx (Z \times Z)$ or 0, for all $n > 0$ as in Figure 8(b). Therefore, $H_n(\lim_{n\to\infty} F_{i_n}(T_1 \vee T_2))$ is isomorphic to $(Z \times Z) \times Z$, $Z \times Z$ or identity group, for all $n > 0$.

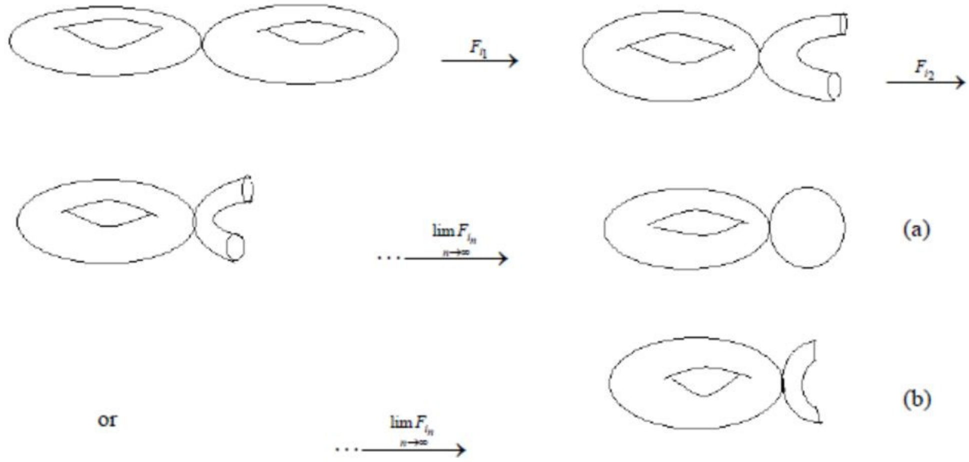

Figure 8

Lemma 3.11 Let M_1, M_2 be two disjoint spheres in R^3. Then there is unfolding $unf: M_1 \cup M_2 \to M_1' \cup M_2'$ which induces unfolding $\overline{unf}: H_n(M_1 \cup M_2) \to H_n(M_1' \cup M_2')$ such that

(1) $\overline{unf}(H_0(M_1 \cup M_2)) \approx \mathrm{Z}$,

(2) $\overline{unf}(H_1(M_1 \cup M_2)) \approx 0$,

(3) $\overline{unf}(H_2(M_1 \cup M_2)) \approx \mathrm{Z} \oplus \mathrm{Z}$,

(4) $\overline{unf}(H_n(M_1 \cup M_2)) \approx 0$, for $n > 2$.

Proof. Let $unf: M_1 \cup M_2 \to M_1' \cup M_2'$ be unfolding such that $unf(M_1 \cup M_2) = unf(M_1) \vee unf(M_2)$ as in figure 9, thus we get the induced unfolding $\overline{unf}: H_n(M_1 \cup M_2) \to H_n(M_1' \cup M_2')$ such that $H_n(unf(M_1 \cup M_2)) = H_n(unf(M_1) \vee unf(M_2))$. Now, for $n = 0$, $\overline{unf}(H_0(M_1 \cup M_2)) = H_0(unf(M_1 \cup M_2)) \approx \mathrm{Z}$. And if $n = 1$, $\overline{unf}(H_1(M_1 \cup M_2)) = H_1(unf(M_1 \cup M_2)) \approx 0$. Also, if $n = 2$, $\overline{unf}(H_2(M_1 \cup M_2)) = H_2(unf(M_1 \cup M_2)) \approx H_2(unf(M_1)) \oplus H_2(unf(M_2))$. Since $H_2(unf(M_i) \approx \mathrm{Z}, i = 1,2$ it follows that $\overline{unf}(H_2(M_1 \cup M_2)) \approx \mathrm{Z} \oplus \mathrm{Z}$.

Moreover, if $n \geq 3$, it follows that $H_n(unf(M_i) \approx 0, i = 1,2$. Thus $\overline{unf}(H_n(M_1 \cup M_2)) \approx 0$, for $n > 2$.

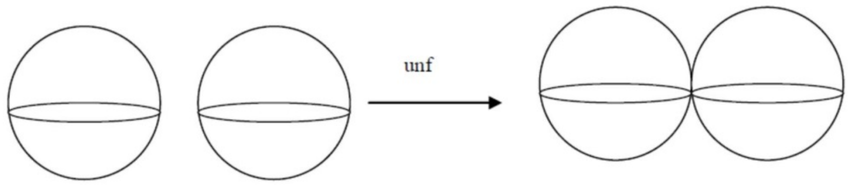

Figure 9

Theorem 3.12 Let D_k be the disjoint union of k-discs on the sphere S^2. Then there is unfolding $unf:(S^2-D_k)\to S^2$ such that $H_2(\lim_{m\to\infty}(unf_m(S^2-D_k)))\approx Z$.

Proof: Let D_k be the disjoint union of k-discs on the sphere S^2. Then, we can define a sequence of unfolding

$$\begin{aligned}
&unf_1: S^2-D_k\to M_1, S^2-D_k\subseteq M_1\subseteq S^2\\
&unf_2: M_1\to M_2, M_1\subseteq M_2\subseteq S^2\\
&\vdots\\
&unf_m: M_{m-1}\to M_m, M_1\subseteq M_2\subseteq\ldots\subseteq M_{m-1}\subseteq M_m\subseteq S^2
\end{aligned}$$

Such that $\lim_{m\to\infty} unf_m(S^2-D_k)=S^2$ as in figure 10 for $k=2$. Hence $H_2(\lim_{m\to\infty}(unf_m(S^2-D_k)))=H_2(S^2)=Z$.

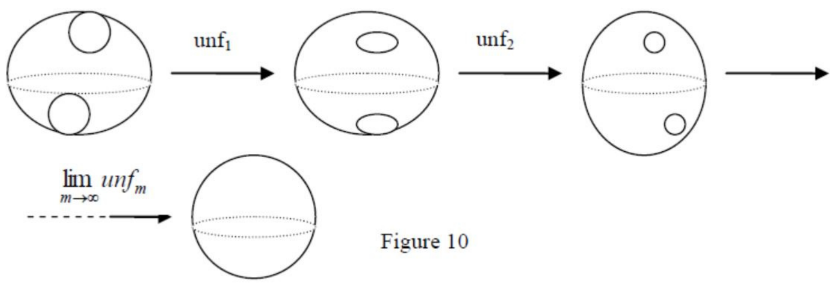

Figure 10

4. Conclusion.

Folding and unfolding problems have been implicit for long time, but have not been studied extensively in the mathematical literature until recently .Over the past few years; there has been a surge of interest in these problems in discrete and computational geometry. This paper gives the folding and unfolding of some types of manifolds, which are determined by their homology group and we discussed the homology group of the limit of folding and unfolding on a wedge sum of 2-manifolds.

The main results can be similarly extended to some other some geometric shapes such as polyhedra .The problems lies: Can all convex polyhedra be edge-unfolded, and can all polyhedra be generally unfolded?

Acknowledgement

The authors are thankful to the referees for a careful reading of the paper and for valuable comments and suggestions.

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