ISSN: 1916-9795
E-ISSN: 1916-9809

Level $n-$Fuzzy Bounded Set and $\alpha-$Completeness in Fuzzy $n-$Normed Linear Space

Iqbal H. Jebril (Corresponding author)
Department of Mathematics, King Faisal University, Saudi Arabia
E-mail: iqbal501@hotmail.com

Abstract

In the present paper, we introduce $\alpha-$completeness, level $n-$fuzzy bounded set, and level $n-$fuzzy closed set in a fuzzy n-normed space.

Keywords: Fuzzy n-normed space, $\alpha-$convergent sequence, $\alpha-$Cauchy sequence, $\alpha-$completeness, Level $n-$fuzzy bounded set, Level $n-$fuzzy closed set

AMS Mathematics Subject Classification: 46S40, 03E72.

1. Introduction

A satisfactory theory of 2-norm on linear space has been introduced and developed by Gähler (Gähler, 1964, p.1-43). A systematic development of n-normed linear spaces is due to S. S. Kim and Y. J. Cho. (Kim, 1996, p.739-744), R. Malceski (Malceski, 1997, p.81-102), A. Misiak (Misiak, 1989, p.299-319), and H. Gunawan and M. Mashadi (Gunawan, 2001, p.631–639). A detailed theory of fuzzy normed linear spaces can be found in (Bag, 2003, p.687-705), (Chang, 1994, p.429-436), (Felbin, 1993, p.428-440), (Felbin, 1999, p.117-131), and (Krishna, 1994, p.207-217). Al Narayanan and S. Vijayabalaji (Narayanan, 2005, p.3963-3977) extended the notion of n-normed linear space to fuzzy n-normed linear space.

Convergence and completeness in fuzzy n-normed linear space were discussed by S. Vijayabalaji and N. Thillaigovindan (Vijayabalaji, 2007, p.119-126) by generalizing it for fuzzy n-normed linear space in terms of $\alpha-$convergence and $\alpha-$completeness.

In the present paper, after an introduction to fuzzy n-normed linear spaces, we shall introduce the notion of $\alpha-$convergent sequence and $\alpha-$Cauchy sequence in fuzzy n-normed linear space. We also introduce the concept of $\alpha-$completeness which would provide a more general framework to study the $\alpha-$completeness of the fuzzy n-normed linear space. Then we defined level $n-$fuzzy bounded set and level $n-$fuzzy closed set in a fuzzy n-normed space.

2. Preliminaries

This section is devoted to a collection of basic definitions and results which will be needed in the sequel.

Definition 2.1. (Gunawan, 2001, p.631–639). Let $n \in \mathbb{N}$ (natural numbers) and X be a real vector space of dimension $d \geq n$. A real valued function $\|\bullet, \bullet, ..., \bullet\|$ on $\underbrace{X \times ... \times X}_{n} = X^n$, satisfying the following properties:

(1) $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent,

(2) $\|x_1, x_2, ..., x_n\|$ is invariant under any permutation of $x_1, x_2, ..., x_n$,

(3) $\|x_1, x_2, ..., \alpha x_n\| = |\alpha| \, \|x_1, x_2, ..., x_n\|$, where $\alpha \in \mathbb{R}$ (set of real numbers),

(4) $\|x_1, x_2, ..., x_{n-1}, y + z\| \leq \|x_1, x_2, ..., x_{n-1}, y\| + \|x_1, x_2, ..., x_{n-1}, z\|$,

$\|\bullet, \bullet, ..., \bullet\|$ is called an n-norm on X and the pair $(X, \|\bullet, \bullet, ..., \bullet\|)$ is called an n-normed linear space.

Definition 2.2. (Gunawan, 2001, p.631–639). A sequence $\{x_n\}$ in an n-normed linear space $(X, \|\bullet, \bullet, ..., \bullet\|)$ is said to be:

i. Converge to an $x \in X$ (in the n-norm), whenever

$$\lim_{n\to\infty} \|x_1, x_2, ..., x_{n-1}, x_n - x\| = 0.$$

ii. Cauchy sequence, if

$$\lim_{n\to\infty} \left\|x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}\right\| = 0, \forall p = 1, 2, 3, ...$$

iii. Complete, if every Cauchy sequence in it is convergent.

Definition 2.3. (Narayanan, 2005, p.3963-3977). Let X be a linear space over a real field F. A fuzzy subset N of $X^n \times \mathbb{R}$ is called a fuzzy n-norm on X if and only if:

(N1) For all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1, x_2, ..., x_n, t) = 0$.:

(N2) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

(N3) $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

(N4) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., cx_n, t/|c|)$ if $c \neq 0$, $c \in F$.

(N5) For all $s, t \in \mathbb{R}$, $N(x_1, x_2, ..., x_n + x'_n, s + t) \geq \min\{N(x_1, x_2, ..., x_n, s), N(x_1, x_2, ..., x'_n, t)\}$.

(N6) $N(x_1, x_2, ..., x_n, t)$ is a non-decreasing function of $t \in \mathbb{R}$ and $\lim_{t\to\infty} N(x_1, x_2, ..., x_n, t) = 1$.

Then (X, N) is called a fuzzy n-normed linear space.

Theorem 2.4. (Narayanan, 2005, p.3963-3977). Let (X, N) be a fuzzy n-normed linear space. Assume further that

(N7) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) > 0$,

implies that $x_1, x_2, ..., x_n$ are linearly dependent.

Define $\|x_1, x_2, ..., x_n\|_\alpha = \inf\{t : N(x_1, x_2, ..., x_n, t) \geq \alpha\}$, $\alpha \in (0, 1)$.

Then $\{\|\bullet, \bullet, ..., \bullet\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of n-norms on X. These n-norms are called $\alpha-$n-norms on X corresponding to fuzzy n-norm on X.

Definition 2.5. (Menger, 1942, p.535-537). A binary operation $* : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ is a continuous t-norm if $*$ satisfies the following conditions:

(1) $*$ is commutative and associative.

(2) $*$ is continuous.

(3) $a * 1 = a$, for all $a \in [0, 1]$.

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0, 1]$.

In (Vijayabalaji, 2007, p.119-126) redefine the notion of fuzzy n-normed linear space using t-norm.

Definition 2.6. (Vijayabalaji, 2007, p.119-126). Let X be a linear space over a real field F. A fuzzy subset N of $X^n \times \mathbb{R}$ is called a fuzzy n-norm on X if and only if:

(N1′) For all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1, x_2, ..., x_n, t) = 0$.:

(N2′) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

(N3′) $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

(N4′) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., cx_n, t) = N(x_1, x_2, ..., cx_n, t/|c|)$ if $c \neq 0$, $c \in F$.

(N5′) For all $s, t \in \mathbb{R}$, $N(x_1, x_2, ..., x_n + x'_n, s + t) \geq N(x_1, x_2, ..., x_n, s) * N(x_1, x_2, ..., x'_n, t)$.

(N6′) $N(x_1, x_2, ..., x_n, t)$ is left continuous and non-decreasing function such that $\lim_{t\to\infty} N(x_1, x_2, ..., x_n, t) = 1$.

To strengthen the above definition, see the following examples.

Example 2.7. (Narayanan, 2005,p.3963-3977) Let $(X, \|\bullet, \bullet, ..., \bullet\|)$ be an n-normed space, where $(x_1, x_2, ..., x_n) \in \underbrace{X \times ... \times X}_{n}$.

Define $a * b = \min\{a, b\}$ and

$$N(x_1, x_2, ..., x_n, t) = \begin{cases} \frac{t}{t+\|x_1,x_2,...,x_n\|}, & \text{when } t > 0,\ t \in \mathbb{R}, \\ 0, & \text{when } t \leq 0. \end{cases}$$

Then (X, N) is an f-n-NLS.

Example 2.8. For $(x_1, x_2, ..., x_n) \in \underbrace{X \times ... \times X}_{n}$, we define $a * b = \min\{a, b\}$ and

$$N(x_1, x_2, ..., x_n, t) = \begin{cases} \frac{t}{t+k\|x_1,x_2,...,x_n\|}, & \text{when } t > 0,\ t \in \mathbb{R},\ k > 0 \\ 0, & \text{when } t \leq 0. \end{cases}$$

Then (X, N) is an f-n-NLS.

Proof:

(N1′) For all $t \in \mathbb{R}$ with $t \leq 0$, $N(x_1, x_2, ..., x_n, t) = 0$.

(N2′) For all $t \in \mathbb{R}$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 1$

$\Longleftrightarrow \frac{t}{t+k\|x_1,x_2,...,x_n\|} = 1 \Longleftrightarrow \|x_1, x_2, ..., x_n\| = 0 \Longleftrightarrow x_1, x_2, ..., x_n$ are linearly dependent.

(N3′) As $\|x_1, x_2, ..., x_n\|$ is invariant under any permutation of $x_1, x_2, ..., x_n$, it follow that $N(x_1, x_2, ..., x_n, t)$ is invariant

ISSN: 1916-9795
E-ISSN: 1916-9809

under any permutation of $x_1, x_2, ..., x_n$. under any permutation of $x_1, x_2, ..., x_n$.

(N4′) For all $t \in \mathbb{R}$ with $t > 0$ and $c \in \mathbb{R}\setminus\{0\}$,

$$N(x_1, x_2, ..., cx_n, t/|c|) = \frac{t/|c|}{t/|c|+k\|x_1,x_2,...,x_n\|} = \frac{t/|c|}{\frac{t+k|c|\|x_1,x_2,...,x_n\|}{|c|}}$$

$$= \frac{t}{t+k|c|\|x_1,x_2,...,x_n\|} = \frac{t}{t+k\|x_1,x_2,...,cx_n\|} = N(x_1, x_2, ..., cx_n, t/|c|).$$

Thus $N(x_1, x_2, ..., cx_n, t/|c|) = N(x_1, x_2, ..., cx_n, t/|c|)$.

(N5′) For all $s, t \in \mathbb{R}$,

If $s + t < 0$, $s = t = 0$, and $s + t > 0$; ($s > 0, t < 0$ or $s < 0, t > 0$), then

$$N(x_1, x_2, ..., x_n + x_n', s + t) \geq N(x_1, x_2, ..., x_n, s) * N(x_1, x_2, ..., x_n', t).$$

If $s > 0, t > 0, s + t >$ then assume that

$$N(x_1, x_2, ..., x_n', t) \leq N(x_1, x_2, ..., x_n, s)$$

$$\Rightarrow \frac{t}{t+k\|x_1,x_2,...,x_n'\|} \leq \frac{s}{s+k\|x_1,x_2,...,x_n\|}$$

$$\Rightarrow t\,(s + k\,\|x_1, x_2, ..., x_n\|) \leq s\left(t + k\left\|x_1, x_2, ..., x_n'\right\|\right)$$

$$\Rightarrow t\,\|x_1, x_2, ..., x_n\| \leq s\left\|x_1, x_2, ..., x_n'\right\|$$

$$\Rightarrow \|x_1, x_2, ..., x_n\| \leq \tfrac{s}{t}\left\|x_1, x_2, ..., x_n'\right\|$$

Therefore,

$$\|x_1, x_2, ..., x_n\| + \left\|x_1, x_2, ..., x_n'\right\| \leq \tfrac{s}{t}\left\|x_1, x_2, ..., x_n'\right\| + \left\|x_1, x_2, ..., x_n'\right\|$$

$$\leq \left(\tfrac{s}{t} + 1\right)\left\|x_1, x_2, ..., x_n'\right\| = \left(\tfrac{s+t}{t}\right)\left\|x_1, x_2, ..., x_n'\right\|.$$

But,

$$\left\|x_1, x_2, ..., x_n + x_n'\right\| \leq \|x_1, x_2, ..., x_n\| + \left\|x_1, x_2, ..., x_n'\right\| \leq \left(\tfrac{s+t}{t}\right)\left\|x_1, x_2, ..., x_n'\right\|$$

Then,

$$1 + \frac{k\left\|x_1,x_2,...,x_n+x_n'\right\|}{s+t} \leq 1 + \frac{k\left\|x_1,x_2,...,x_n'\right\|}{t} \Rightarrow \frac{s+t}{s+t+k\|x_1,x_2,...,x_n+x_n'\|} \leq \frac{t}{t+k\|x_1,x_2,...,x_n'\|}$$

$$\Rightarrow N(x_1, x_2, ..., x_n + x_n', s + t) \geq N(x_1, x_2, ..., x_n, s) * N(x_1, x_2, ..., x_n', t).$$

(N6′) Clearly $N(x_1, x_2, ..., x_n, t)$ is left continuous function. Suppose that $t_2 > t_1 > 0$ with $t_1, t_2 \in [0, 1)$ then,

$$\frac{t_2}{t_2+k\|x_1,x_2,...,x_n\|} - \frac{t_1}{t_1+k\|x_1,x_2,...,x_n\|} = \frac{k\|x_1,x_2,...,x_n\|(t_2-t_1)}{(t_2+k\|x_1,x_2,...,x_n\|)(t_1+k\|x_1,x_2,...,x_n\|)} \geq 0$$

for all $(x_1, x_2, ..., x_n) \in X^n$

$$\Rightarrow \frac{t_2}{t_2+k\|x_1,x_2,...,x_n\|} \geq \frac{t_1}{t_1+k\|x_1,x_2,...,x_n\|} \Rightarrow N(x_1, x_2, ..., x_n, t_2) \geq N(x_1, x_2, ..., x_n, t_1)$$

Thus $N(x_1, x_2, ..., x_n, t)$ is non-decreasing function of $t \in [0, 1)$.

Also,

$$\lim_{n\to\infty} N(x_1, x_2, ..., x_n, t) = \lim_{n\to\infty} \frac{t}{t+k\|x_1,x_2,...,x_n\|} = \lim_{n\to\infty} \frac{t}{t}\left(1 + \frac{1}{tk\|x_1,x_2,...,x_n\|}\right) = 1.$$

Hence (X, N) is called f-n-LNS.

Definition 2.9. (Vijayabalaji, 2007,p.119-126). Let (X, N) be a f-n-NLS and $\{x_n\}$ be a sequence in X then $\{x_n\}$ is said to be convergent if given $r > 0$, $t > 0$, $0 < r < 1$, there exists an integer $n_0 \in \mathbb{N}$ such that $N(x_1, x_2, ..., x_{n-1}, x_n - x, t) > 1 - r$ for all $n \geq n_0$. In this case x is called the limit of the sequence $\{x_n\}$.

Definition 2.10. (Vijayabalaji, 2007,p.119-126). Let (X, N) be a f-n-NLS and $\{x_n\}$ be a sequence in X. The sequence $\{x_n\}$ is said to be convergent if and only if

$$N(x_1, x_2, ..., x_{n-1}, x_n - x, t) \to 1 \text{ as } n \to \infty.$$

Definition 2.11. (Vijayabalaji, 2007,p.119-126). Let (X, N) be a f-n-NLS and $\{x_n\}$ be a sequence in X then $\{x_n\}$ is said to be a Cauchy sequence if given $\varepsilon > 0$ with $0 < \varepsilon < 1$, $t > 0$, there exists an integer $n_0 \in \mathbb{N}$ such that $N(x_1, x_2, ..., x_{n-1}, x_n - x_k, t) > 1 - \varepsilon$ for all $n, k \geq n_0$.

Definition 2.12. (Vijayabalaji, 2007,p.119-126). A f-n-NLS is said to be complete if every Cauchy sequence in it is convergent.

3. $\alpha-$Completeness in f-n-NLS

In this section we generalize the notions of convergence and completeness in f-n-NLS by introducing the notions of

$\alpha-$convergence, $\alpha-$Cauchyness and $\alpha-$completeness in f-n-NLS and studying the $\alpha-$completeness of fuzzy n-normed linear space.

Definition 3.1. Let (X, N) be a f-n-NLS and $\alpha \in (0,1)$. A sequence $\{x_n\}$ in X is said to be $\alpha-$convergent to x if $\lim\limits_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x, t) > \alpha$, for all $t > 0$.

Theorem 3.2. Let (X, N) be a f-n-NLS satisfying (N7). If $\{x_n\}$ is an $\alpha-$convergent sequence in.(X, N) , then

$$\lim_{n\to\infty} \|x_1, x_2, ..., x_{n-1}, x_n - x\|_\alpha = 0, \forall\alpha \in (0,1).$$

Proof: Let $\{x_n\}$ be an $\alpha-$convergent sequence in.(X, N) and suppose that it converges to x. Thus

$\lim\limits_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x, t) > \alpha, \forall\ t > 0, .\alpha \in (0,1).$

$\Longrightarrow \forall t > 0, \exists n_0\,(t)$ such that $N(x_1, x_2, ..., x_{n-1}, x_n - x, t) > \alpha, \forall t > 0.$

$\Longrightarrow \forall t > 0, \exists n_0\,(t)$ such that $\|x_1, x_2, ..., x_{n-1}, x_n - x\|_\alpha \leq t, \forall n \geq n_0\,(t)$.

Since $t > 0$ is arbitrary, then

$$\lim_{n\to\infty} \|x_1, x_2, ..., x_{n-1}, x_n - x\|_\alpha = 0, \forall\alpha \in (0,1).$$

Theorem 3.3. Let (X, N) be a f-n-NLS satisfying (N7) and $\{x_n\}$ be a sequence in (X, N) . Then $\{x_n\}$ is convergent to x (see Definition 2.9) if and only if

$$\lim_{n\to\infty} \|x_1, x_2, ..., x_{n-1}, x_n - x\|_\alpha = 0, \forall\alpha \in (0,1).$$

Proof: Let $\{x_n\}$ be a convergent sequence in (X, N) to x. Choose $\alpha \in (0,1)$. There exists $n_0 \in \mathbb{N}$ such that $N(x_1, x_2, ..., x_{n-1}, x_n - x, t) > 1 - \alpha$ for all $n \geq n_0$.

It follows that

$$\|x_1, x_2, ..., x_{n-1}, x_n - x\|_{1-\alpha} \leq t, \forall n \geq n_0.$$

Thus

$$\lim_{n\to\infty} \|x_1, x_2, ..., x_{n-1}, x_n - x\|_{1-\alpha} = 0, \forall\alpha \in (0,1)\,.$$

Conversely, let

$\lim\limits_{n\to\infty} \|x_1, x_2, ..., x_{n-1}, x_n - x\|_\alpha = 0$, for every $\alpha \in (0,1)$ .Fix $\alpha \in (0,1)$ and $t > 0$. There exists $n_0 \in \mathbb{N}$ such that

$$\inf\{r : N(x_1, x_2, ..., x_{n-1}, x_n - x, r) \geq 1 - \alpha\} < t,$$

i.e. the sequence $\{x_n\}$ is convergent to x.

Definition 3.4. Let (X, N) be a f-n-NLS and $\alpha \in (0,1)$. A sequence $\{x_n\}$ in X is said to be $\alpha-$Cauchy if

$\lim\limits_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}, t) \geq \alpha$, for all $t > 0$, $p = 1, 2, ...$

Theorem 3.5. Let (X, N) be a f-n-NLS satisfiying (N7). Then every Cauchy sequence in $(X, \|\bullet, \bullet, ..., \bullet\|_\alpha)$ is an $\alpha-$Cauchy sequence in (X, N) , where $\|\bullet, \bullet, ..., \bullet\|_\alpha$ denotes the $\alpha-$n-norm of N, $\forall\alpha \in (0,1)$.

Proof: Let $\alpha_0 \in (0,1)$ and $\{x_n\}$ be a Cauchy sequence in $\left(X, \|\bullet, \bullet, ..., \bullet\|_{\alpha_0}\right)$.

Then,

$$\lim_{n\to\infty} \left\|x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}\right\|_{\alpha_0} = 0, p = 1, 2, 3, ...$$

Thus for a given $\varepsilon > 0$, there exist a positive integer $N\,(\varepsilon)$ such that

$\left\|x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}\right\|_{\alpha_0} < \varepsilon, \forall n \geq N\,(\varepsilon)\,, p = 1, 2, 3, ...$

$\Longrightarrow \inf\left\{t > 0 : N(x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}, t) \geq \alpha_0\right\} < \varepsilon, \forall n \geq N\,(\varepsilon)\,, p = 1, 2, 3, ...$

$\Longrightarrow \forall n \geq N\,(\varepsilon)\,, p = 1, 2, 3, ..., \exists t\,(n, p, \varepsilon) < \varepsilon$ such that $N(x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}, t\,(n, p, \varepsilon)) \geq \alpha_0$

$\Longrightarrow N(x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}, \varepsilon) \geq \alpha_0, \forall n \geq N\,(\varepsilon)\,, p = 1, 2, 3, ...$

Since $\varepsilon > 0$ is arbitrary, then

$\lim\limits_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}, \varepsilon) \geq \alpha_0, \forall t > 0.$

$\Longrightarrow \{x_n\}$ is an α_0-Cauchy sequence in (X, N) .

Since $\alpha_0 \in (0,1)$ is arbitrary, then every Cauchy sequence in $(X, \|\bullet, \bullet, ..., \bullet\|_\alpha)$ is an $\alpha-$Cauchy sequence in (X, N) for each $\alpha \in (0,1)$.

Definition 3.6. In f-n-NLS (X, N), every $\alpha-$convergent sequence is an $\alpha-$Cauchy sequence.

Proof: Suppose that $\{x_n\}$ is $\alpha-$convergent to x and $\alpha \in (0, 1)$, then we have

$$\lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x, t) > \alpha, \text{ for all } t > 0.$$

Now, for all $p = 1, 2, 3, ...$

$N(x_1, x_2, ..., x_{n-1}, x_n-x_{n+p}, t) = N(x_1, x_2, ..., x_{n-1}, x_n-x+x-x_{n+p}, t/2+t/2) \geq N(x_1, x_2, ..., x_{n-1}, x_n-x, t/2)*N(x_1, x_2, ..., x_{n-1}, x-x_{n+p}, t/2)$.

Therefore,

$\lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}, t) \geq \lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x, t/2)*$
$\lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x - x_{n+p}, t/2) > \alpha.$

Hence $\{x_n\}$ is an $\alpha-$Cauchy sequence in (X, N).

The converse of the above theorem is not necessarily true. This is justified by the following example.

Example 3.7. Let $(X, \|\bullet, \bullet, ., ., \bullet\|)$ be an n-normed space and define $a * b = \min\{a, b\}$, for all $a, b \in [0, 1]$. Define

$$N(x_1, x_2, ..., x_n, t) = \begin{cases} \frac{t}{t+k\|x_1, x_2, ..., x_n\|}, & \text{when } t > 0, \ t \in \mathbb{R}, \\ 0, & \text{when } t \leq 0, \end{cases}$$

where $k > 0$. Then (X, N) is an f-n-NLS (see Example 2.8). We now show that

a)$\{x_n\}$ is a Cauchy sequence in $(X, \|\bullet, \bullet, ..., \bullet\|)$ if and only if $\{x_n\}$ is an $\alpha-$Cauchy sequence in (X, N).

b)$\{x_n\}$ is a convergent sequence in $(X, \|\bullet, \bullet, ..., \bullet\|)$ if and only if $\{x_n\}$ is an $\alpha-$convergent sequence in (X, N).

Proof: a) Let $\{x_n\}$ be a Cauchy sequence in $(X, \|\bullet, \bullet, ..., \bullet\|)$

$\Leftrightarrow \lim_{n\to\infty} \left\|x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}\right\| = 0$, for all $p = 1, 2, 3, ...$

$\Leftrightarrow \lim_{n\to\infty} N\left(x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}\right) =$

$\lim_{n\to\infty} \frac{t}{t+k\left\|x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}\right\|} = 1 > \alpha$, for all $t, k > 0$.

$\Leftrightarrow \lim_{n\to\infty} N\left(x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}\right) > \alpha$

$\Leftrightarrow \{x_n\}$ is an $\alpha-$Cauchy sequence in (X, N).

b) $\{x_n\}$ is a convergent sequence in $(X, \|\bullet, \bullet, ..., \bullet\|)$

$\Leftrightarrow \lim_{n\to\infty} \|x_1, x_2, ..., x_{n-1}, x_n - x\| = 0$

$\Leftrightarrow \lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x) =$

$\lim_{n\to\infty} \frac{t}{t+k\|x_1, x_2, ..., x_{n-1}, x_n - x\|} = 1 > \alpha$,for all $t, k > 0$.

$\Leftrightarrow \lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x) > \alpha$,

$\Leftrightarrow \{x_n\}$ is an $\alpha-$convergent sequence in (X, N).

Remark 3.8. If there exist a Cauchy sequence in n-normed linear space which is not convergent then there may exist a Cauchy sequence in R-n-LNS which is not convergent. Thus if there exists an n-normed linear space $(X, \|\bullet, \bullet, ..., \bullet\|)$ which is not complete then the fuzzy n-norm induced by such a crisp n-norm $\|\bullet, \bullet, ..., \bullet\|$ on an incomplete n-normed linear space X is an $\alpha-$incomplete f-n-NLS

Theorem 3.9. In f-n-LNS (X, N) in which every $\alpha-$Cauchy sequence has an $\alpha-$convergent subsequence is $\alpha-$complete, where $a * b = \min\{a, b\}$ .and $\alpha \in (0, 1)$.

Proof: Let $\{x_n\}$ be a $\alpha-$Cauchy sequence in (X, N) and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ that $\alpha-$converges to x. We prove that $\{x_n\}$ $\alpha-$converges to x. Since $\{x_n\}$ is an $\alpha-$Cauchy sequence, there exists an integer $n_0 \in \mathbb{N}$ such that

$\lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x_{n+p}, t) \geq \alpha$, for all $t > 0$, $p = 1, 2, ...$

Since $\{x_{n_k}\}$ $\alpha-$converges to x, then

$$\lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_{n_{i_k}} - x, t/2) > \alpha, \text{ for all } t > 0.$$

Now,

$N(x_1, x_2, ..., x_{n-1}, x_n - x, t) = N(x_1, x_2, ..., x_{n-1}, x_n - x_{n_{i_k}} + n_{i_k} - x, t/2 + t/2)$

$\geq N(x_1, x_2, ..., x_{n-1}, x_n - x_{n_{i_k}}, t/2) * N(x_1, x_2, ..., x_{n-1}, n_{i_k} - x, t/2) \Longrightarrow$

$\lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x, t) \geq \lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x_{n_{i_k}}, t/2)*$

$\lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, n_{i_k} - x, t/2) > \alpha * \alpha = \alpha.$

Therefore $\{x_n\}$ $\alpha-$converges to x in (X, N) and hence it is $\alpha-$complete.

4. Level Fuzzy bounded sets in f-n-NLS

In this section, we define level $n-$fuzzy bounded set and level $n-$fuzzy closed set in a fuzzy n-normed space.

Definition 4.1. Let (X, N) be a f-n-NLS. X is said to be level $n-$fuzzy bounded ($l - n-$fuzzy) if for any $\alpha \in (0, 1)$, there exist $t(\alpha)$ such that $N(x_1, x_2, ..., x_{n-1}, x_n, t(\alpha)) > \alpha$, for all $(x_1, x_2, ..., x_n) \in X^n$.

Theorem 4.2. Let (X, N) be a f-n-NLS satisfying (N7). Then X is $l - n-$fuzzy bounded iff X is bounded with respect to $\|\bullet, \bullet, ..., \bullet\|_\alpha$ for all $\alpha \in (0, 1)$, where $\|\bullet, \bullet, ..., \bullet\|_\alpha$ denotes the $\alpha - n - norm$ of N.

Proof: If X is an $l - n-$fuzzy bounded then for any $\alpha \in (0, 1)$, there exist $t(\alpha)$ such that $N(x_1, x_2, ..., x_{n-1}, x_n, t(\alpha)) > \alpha$, for all $(x_1, x_2, ..., x_n) \in X^n$.

Therefore $\|x_1, x_2, ..., x_n\|_\alpha \leq t(\alpha)$,for all $(x_1, x_2, ..., x_n) \in X^n$ and $\alpha \in (0, 1)$. This implies that X is bounded with respect to $\|\bullet, \bullet, ..., \bullet\|_\alpha$ for all $\alpha \in (0, 1)$.

Conversely, let X be bounded with respect to $\|\bullet, \bullet, ..., \bullet\|_\alpha$ for all $\alpha \in (0, 1)$

$\Longrightarrow \|x_1, x_2, ..., x_n\|_\alpha \leq t(\alpha)$ for all $(x_1, x_2, ..., x_n) \in X^n$,

$\Longrightarrow \|x_1, x_2, ..., x_n\|_\alpha \leq t(\alpha) \leq t(\alpha) + 1$, for all $\alpha \in (0, 1)$,

$\Longrightarrow N(x_1, x_2, ..., x_{n-1}, x_n, t(\alpha)) > \alpha$, for all $(x_1, x_2, ..., x_n) \in X^n$,

$\Longrightarrow X$ is $l - n-$fuzzy bounded.

Definition 4.1. Let (X, N) be a f-n-NLS. A subset A of X is said to be $l - n-$fuzzy closed if for any $\alpha \in (0, 1)$ and $\{x_n\}$ in X , for all $(x_1, x_2, ..., x_n) \in X^n$, $\lim_{n\to\infty} N(x_1, x_2, ..., x_n - x, t) \geq \alpha, \forall t > 0 \Longrightarrow x \in A$.

Theorem 4.2. Let (X, N) be a f-n-NLS satisfying (N7) and $A \subset X$. Then A is $l-n-$fuzzy closed iff A is closed with respect to $\|\bullet, \bullet, ..., \bullet\|_\alpha$ for all $\alpha \in (0, 1)$.

Proof: Let $\alpha_0 \in (0, 1)$ and $\{x_n\}$ be a sequence in $\left(X, \|\bullet, \bullet, ..., \bullet\|_{\alpha_0}\right)$.

Then,

$$\lim_{n\to\infty} \|x_1, x_2, ..., x_{n-1}, x_n - x\|_{\alpha_0} = 0,$$

Thus for a given $\varepsilon > 0$, there exist a positive integer $N(\varepsilon)$ such that

$\|x_1, x_2, ..., x_{n-1}, x_n - x\|_{\alpha_0} < \varepsilon, \forall n \geq N(\varepsilon)$.

$\Longrightarrow N(x_1, x_2, ..., x_{n-1}, x_n - x, \varepsilon) \geq \alpha_0$.

$\Longrightarrow \lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x, t) \geq \alpha_0, \forall t > 0$ (since ε is arbitrary).

$\Longrightarrow x \in A$

$\Longrightarrow A$ is closed with respect to $\|\bullet, \bullet, ..., \bullet\|_\alpha$ for all $\alpha \in (0, 1)$.

Since $\alpha_0 \in (0, 1)$ is arbitrary, it follows that A is closed with respect to $\|\bullet, \bullet, ..., \bullet\|_\alpha$, $\alpha \in (0, 1)$.

Conversely, suppose that A is closed with respect to $\|\bullet, \bullet, ..., \bullet\|_\alpha$, for each $\alpha \in (0, 1)$.

Choose an arbitrary $\beta_0 \in (0, 1)$. Let $\{x_n\}$ be a sequence in A such that

$\lim_{n\to\infty} N(x_1, x_2, ..., x_{n-1}, x_n - x, t) \geq \beta_0, \forall t > 0.$

Then for a given $\varepsilon > 0$ with $\beta_0 - \varepsilon > 0$ and for a given $t > 0$. There exist a positive integer $N(\varepsilon, t)$ such that,

$N(x_1, x_2, ..., x_{n-1}, x_n - x, t) \geq \beta_0 - \varepsilon, \forall n \geq N(\varepsilon, t)$.

$\Longrightarrow \|x_1, x_2, ..., x_{n-1}, x_n - x\|_{\beta_0-\varepsilon} \leq t, \forall n \geq N(\varepsilon, t)$.

$\Longrightarrow \lim_{n\to\infty} \|x_1, x_2, ..., x_{n-1}, x_n - x\|_{\beta_0-\varepsilon} = 0.$

$\Longrightarrow x \in A.$

Since $\beta_0 \in (0, 1)$ is arbitrary, it follows that A is $l - n-$fuzzy closed

ISSN: 1916-9795
E-ISSN: 1916-9809

Acknowledgment

It is a pleasure to thank Prof. Dr. S. Vijayabalaji and Prof. Dr. N. Thillaigovindan for their valuable comments and suggestions in rewriting the paper in the present form.

References

Bag, T. & Samanta, S. K. (2003). Finite dimensional fuzzy normed linear spaces. *The Journal of Fuzzy Mathematics*, 11(3), 687-705.

Chang, S. C. & Mordesen, J. N. (1994). Fuzzy linear operators and fuzzy normed linear spaces. *Bull. Cal. Math. Soc.*, 86, 429-436.

Felbin, C. (1993). The completion of fuzzy normed linear space. *Journal of Mathematical Analysis and Applications*, 174(2), 428-440.

Felbin, C. (1999). Finite dimensional fuzzy normed linear spaces II, *Journal of Analysis*, 7, 117-131.

Gähler, S. (1964). Lineare 2-normierte Raume. *Math. Nachr.* 28, 1-43.

Gunawan, H. & Mashadi, M. (2001). On n-normed space. *Int. J. Math. Math. Sci.,* 27(10), 631–639.

Kim, S. S. & Cho, Y. J. (1996). Strict convexity in linear n-normed spaces. *Demonstratio Math.*, 29(4), 739-744.

Krishna, S. V. & Sarma, K. K. M. (1994). Separation of fuzzy normed linear spaces. *Fuzzy Sets and Systems*, 63, 207-217.

Malceski, R.(1997). Strong n-convex n-normed spaces. *Mat.Bilten*, 21, 81-102.

Menger, K. (1942). Statistical metrics space. *Proc. Nat. Acad. Sci. USA,* 28, 535–537.

Misiak, A. (1989). n-inner product spaces. *Math. Nachr.*, 140, 299-319.

Narayanan, A. & Vijayabalaji, S. (2005). Fuzzy n-normed linear space. *Int. J. Math. Math. Sci.*, 24, 3963-3977.

Vijayabalaji, S. & Thillaigovindan, N. (2007). Complete fuzzy n-normed linear space. *Journal of Fundamental Sciences*, 3, 119-126.