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On the Operator $\circledast^k$ Related to Heat Equation

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Abstract

In this paper, we study the equation

$$\frac{\partial}{\partial t}u(x,t) + c^2(-\circledast)^k u(x,t) = 0$$

with the initial condition

$$u(x,0) = f(x)$$

for $x \in \mathbb{R}^n$-the n-dimensional Euclidean space. The operator $(\circledast)^k$ is operator iterated k times , defined by

$$\circledast^k = \left(\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3 + \left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right)^k$$

$p+q=n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is the given generalized function , k is a positive integer and c is a positive constant. Moreover, if we put $q = 0$ and $k = 1$we obtain the solution of equation.

$$\frac{\partial}{\partial t}u(x,t) - c^2\triangle^3 u(x,t) = 0$$

Which is related to the triharmonic heat equation.

Keywords: Fourier transform, Temper distribution, Diamond operator

1. Introduction

It is well known that for the heat equation

$$\frac{\partial}{\partial t}u(x,t) = c^2\triangle u(x,t) \tag{1}$$

with the initial condition

$$u(x,0) = f(x)$$

where $\triangle = \sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}$ is the Laplace operator and $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, we obtain

$$u(x,t) = \frac{1}{(4c^2\pi t)^{n/2}}\int_{\mathbb{R}^n}\exp\left(-\frac{|x-y|^2}{4c^2t}\right)f(y)dy \tag{2}$$

as the solution of (1).

Now, (2) can be written $u(x,t) = E(x,t) * f(x)$ where

$$E(x,t) = \frac{1}{(4c^2\pi t)^{n/2}}\exp\left(-\frac{|x|^2}{4c^2t}\right). \tag{3}$$

$E(x,t)$ is called *the heat kernel*, where $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$ and $t > 0$, see (F. John, p208-209).

In 1996, A. Kananthai (1997) has introduced the Diamond operator $\diamond$ defined by

$$\diamond = \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2, \ p+q=n$$

or $\diamond$ can be written as the product of the operators in the form $\diamond = \triangle\Box = \Box\triangle$ where $\triangle = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian and $\Box = \sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} - \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}$ is the ultra-hyperbolic. The Fourier transform of the Diamond operator also has been studied and the elementary solution of such operator, see (A. Kananthai, 1999).

Next, K. Nonlaopon and A. Kananthai (2003) study the equation

$$\frac{\partial}{\partial t} u(x,t) = c^2 \Box u(x,t)$$

We obtain the ultra-hyperbolic heat kernel

$$E(x,t) = \frac{i^q}{(4c^2\pi t)^{n/2}} \exp\left(-\frac{\sum_{i=1}^{p} x_i^2 - \sum_{j=p+1}^{p+q} x_j^2}{4c^2 t}\right). \tag{4}$$

Next, W. Satsanit has been first introduced the $\circledast^k$ operator where $\circledast^k$ defined by

$$\begin{aligned}
\circledast^k &= \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^3 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^3\right)^k \\
&= \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^k \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)\left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right) + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\
&= (\triangle)^k (\triangle^2 - \frac{3}{4}(\triangle + \Box)(\triangle - \Box))^k \\
&= (\frac{3}{4}\diamond\Box + \frac{1}{4}\triangle^3)^k
\end{aligned}$$

where

$$\begin{aligned}
\triangle &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \ldots + \frac{\partial^2}{\partial x_n^2} \\
\Box &= \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \frac{\partial^2}{\partial x_{p+1}^2} - \frac{\partial^2}{\partial x_{p+2}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2} \\
\diamond &= \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_p^2}\right)^2 - \left(\frac{\partial^2}{\partial x_{p+1}^2} + \frac{\partial^2}{\partial x_{p+2}^2} + \cdots + \frac{\partial^2}{\partial x_{p+q}^2}\right)^2
\end{aligned}$$

Now, the purpose of this work is to study the equation

$$\frac{\partial}{\partial t} u(x,t) + c^2(-\circledast)^k u(x,t) = 0 \tag{5}$$

with the initial condition

$$u(x,0) = f(x)$$

for $x \in \mathbb{R}^n$-the n-dimensional Euclidean space. The $\circledast^k$ operator is defined by

$$\begin{aligned}
\circledast^k &= \left(\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^3 + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^3\right)^k \\
&= \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2} + \sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^k \left[\left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)^2 - \left(\sum_{i=1}^{p} \frac{\partial^2}{\partial x_i^2}\right)\left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right) + \left(\sum_{j=p+1}^{p+q} \frac{\partial^2}{\partial x_j^2}\right)^2\right]^k \\
&= (\triangle)^k (\triangle^2 - \frac{3}{4}(\triangle + \Box)(\triangle - \Box))^k \\
&= (\frac{3}{4}\diamond\Box + \frac{1}{4}\triangle^3)^k
\end{aligned}$$

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$p + q = n$ is the dimension of the Euclidean space $\mathbb{R}^n$, $u(x,t)$ is an unknown function for $(x,t) = (x_1, x_2, \ldots, x_n, t) \in \mathbb{R}^n \times (0,\infty)$, $f(x)$ is the given generalized function and c is a positive constant. We obtain $u(x,t) = E(x,t) * f(x)$ as a solution of (5), where

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{R^n} \exp\left[-c^2 \left[\left(\sum_{i=1}^{p} \xi_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^3\right]^k t + i(\xi, x)\right] d\xi. \tag{6}$$

The function $E(x,t)$ is the elementary solution of (5).

All properties of $E(x,t)$ will be studied in details.

Now, if we put $q = 0$ and $k = 1$ in (5), then (5) reduces to the equation

$$\frac{\partial}{\partial t} u(x,t) - c^2 \triangle^3 u(x,t) = 0$$

which is related to the triharmonic heat equation.

Before going that points, the following definitions and some concepts are needed.

2. Preliminaries

Definition 2.1 Let $f(x) \in \mathrm{L}_1(\mathbb{R}^n)$-the space of integrable function in $\mathbb{R}^n$. The Fourier transform of $f(x)$ is defined by

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i(\xi,x)} f(x)\, dx \tag{7}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $(\xi, x) = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n$ and $dx = dx_1\, dx_2 \ldots dx_n$.

Also, the inverse of Fourier transform is defined by

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{i(\xi,x)} \widehat{f}(\xi)\, d\xi. \tag{8}$$

If f is a distribution with compact supports f can be written as (See A. H. Zemanian, 1965, Theorem 7.4-3, p.187 Eq.(2.1))

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \left\langle f(x), e^{-i(\xi,x)} \right\rangle. \tag{9}$$

Lemma 1 *Given the function*

$$f(x) = \exp\left(-\left(\left(\sum_{i=1}^{p} x_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q} x_j^2\right)^3\right)^k\right)$$

where

$$(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n, \quad p + q = n,$$

and k is the positive number . Then

$$\left| \int_{\mathbb{R}^n} f(x) dx \right| \leq \frac{\pi^{n/2}}{9k^2} \cdot \frac{\Gamma(\frac{p}{6k})\Gamma(\frac{q}{6k})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}$$

where $\frac{p+q}{2} = \frac{n}{2}$ *and* Γ *denoted the gamma function. That is* $\int_{\mathbb{R}^n} f(x)dx$ *is bounded.*

Proof.

$$\int_{\mathbb{R}^n} f(x)dx \quad = \quad \int_{\mathbb{R}^n} \exp\left(-\left(\left(\sum_{i=1}^{p} x_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q} x_j^2\right)^3\right)^k\right) dx$$

for k is a positive integer. By changing the coordinate.Now , put

$$x_1 = y_1, \quad x_2 = y_2, \ldots, \quad x_p = y_p$$

$$dx_1 = dy_1, \quad dx_2 = dy_2, \ldots, \quad dx_p = dy_p$$

and

$$x_{p+1} = y_{p+1}, \quad x_{p+2} = y_{p+2}, \ldots, \quad x_{p+q} = y_{p+q}$$

$$dx_{p+1} = dy_{p+1}, \quad dx_{p+2} = dy_{p+2}, \ldots, \quad dx_{p+q} = dy_{p+q}$$

Then we obtain

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \exp\left(-\left(\left(\sum_{j=p+1}^{p+q} y_j^2\right)^3 + \left(\sum_{i=1}^{p} y_i^2\right)^3\right)^k\right)dy \tag{10}$$

Let us transform to bipolar coordinates defined by

$$y_1 = rw_1, y_2 = rw_2, \ldots, \quad y_p = rw_p$$

and $$y_{p+1} = sw_{p+1}, y_{p+2} = sw_{p+2}, \ldots, y_{p+q} = sw_{p+q}, \quad p+q = n$$

where $w_1^2 + w_2^2 + \cdots + w_p^2 = 1$ and $w_{p+1}^2 + w_{p+2}^2 + \cdots + w_{p+q}^2 = 1$, Thus

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}^n} \exp\left(-\left(r^6 + s^6\right)^k\right) r^{p-1} s^{q-1} drdsd\Omega_p d\Omega_q \tag{11}$$

where

$$dy = r^{p-1} s^{q-1} drdsd\Omega_p d\Omega_q$$

$d\Omega_p$ and $d\Omega_q$ are the elements of surface area on the unit sphere in R^p and R^q respectively.By computing directly , we obtain

$$\int_{\mathbb{R}^n} f(x)dx = \Omega_p\Omega_q \int_0^\infty \int_0^\infty \exp\left(-\left(r^6 + s^6\right)^k\right) r^{p-1} s^{q-1} drds \tag{12}$$

where $\Omega_p = \dfrac{2\pi^{p/2}}{\Gamma(p/2)}$ and $\Omega_q = \dfrac{2\pi^{q/2}}{\Gamma(q/2)}$.
Since $(r^6 + s^6)^k \geq r^{6k} + s^{6k}$, then

$$\exp(-(r^6 + s^6)^k) \leq \exp(-(r^{6k} + s^{6k})$$

Thus

$$\begin{aligned} |\int_{\mathbb{R}^n} f(x)dx| &\leq \Omega_p\Omega_q \int_0^\infty \int_0^\infty \exp\left(-r^{6k} - s^{6k}\right) r^{p-1} s^{q-1} drds \\ &= \Omega_p\Omega_q \int_0^\infty \exp\left(-r^{6k}\right) r^{p-1} dr \int_0^\infty \exp\left(-s^{6k}\right) s^{q-1} ds \end{aligned}$$

Put $u = r^{6k}, dr = \frac{1}{6k} u^{\frac{1}{6k}-1} du$ and $v = s^{6k}, ds = \frac{1}{6k} v^{\frac{1}{6k}-1} dv$
Then we obtain

$$\begin{aligned} |\int_{\mathbb{R}^n} f(x)dx| &\leq \frac{\Omega_p\Omega_q}{(6k)^2} \int_0^\infty e^{-u} u^{\frac{p}{6k}-1} du \int_0^\infty e^{-v} v^{\frac{q}{6k}-1} dv \\ &= \frac{\Omega_p\Omega_q}{(6k)^2} \Gamma\left(\frac{p}{6k}\right)\Gamma\left(\frac{q}{6k}\right) \\ &= \frac{2\pi^{p/2}2\pi^{q/2}}{(6k)^2} \frac{\Gamma(\frac{p}{6k})\Gamma(\frac{q}{6k})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \\ &= \frac{\pi^{n/2}}{9k^2} \frac{\Gamma(\frac{p}{6k})\Gamma(\frac{q}{6k})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})} \end{aligned}$$

Where $\frac{p+q}{2} = \frac{n}{2}$. That is $\int_{\mathbb{R}^n} f(x)dx$ is bounded.

Lemma 2 *(The Fourier transform of $(-\circledast)^k\delta$)*

$$\mathcal{F}(-\circledast)^k\delta = \frac{(-1)^{4k}}{(2\pi)^{n/2}} \left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^3 + \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^3\right]^k$$

where $\mathcal{F}$ is the Fourier transform defined by Eq.(7) and if the norm of ξ is given by $\|\xi\| = \left(\xi_1^2 + \xi_2^2 + \ldots + \xi_n^2\right)^{1/2}$ then

$$\mathcal{F}(-\circledast)^k\delta \leq \frac{3^k}{(2\pi)^{n/2}} \|\xi\|^{6k}$$

that is $\mathcal{F}(-\circledast)^k$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution. Moreover, by Eq.(8)

$$(-\circledast)^k\delta = \mathcal{F}^{-1} \frac{1}{(2\pi)^{n/2}} \left[\left(\xi_1^2 + \xi_2^2 + \ldots + \xi_p^2\right)^3 + \left(\xi_{p+1}^2 + \xi_{p+2}^2 + \ldots + \xi_{p+q}^2\right)^3\right]^k$$

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Proof. By Eq. (9)

$$\begin{aligned}
\mathcal{F}(-\circledast)^k\delta &= \frac{1}{(2\pi)^{n/2}}\left\langle(-\circledast)^k\delta, e^{-i(\xi.x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^k e^{-i(\xi.x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}(-\circledast)e^{-i(\xi.x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}\left(-\frac{3}{4}\diamond\Box - \frac{1}{4}\triangle^3\right)e^{-i(\xi.x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}\left(-\frac{3}{4}\diamond\Box\right)e^{-i(\xi.x)}\right\rangle + \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}\left(-\frac{1}{4}\Box^3\right)e^{-i(\xi.x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}\frac{3}{4}(-1)^4\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right]\cdot\right. \\
&\quad \left.\left(\sum_{i=1}^{p}\xi_i^2 - \sum_{j=p+1}^{p+q}\xi_j^2\right)e^{-i(\xi.x)}\right\rangle + \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}\frac{1}{4}(-1)^4\left(\sum_{i=1}^{n}\xi_i^2\right)^3 e^{-i(\xi.x)}\right\rangle \\
&= \frac{1}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}\frac{3}{4}(-1)^4\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^2 - \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^2\right]\left(\sum_{i=1}^{p}\xi_i^2 - \sum_{j=p+1}^{p+q}\xi_j^2\right)\right. \\
&\quad \left. +(\frac{1}{4}(-1)^4\left(\sum_{i=1}^{n}\xi_i^2\right)^3 e^{-i(\xi.x)}\right\rangle \\
&= \frac{(-1)^4}{(2\pi)^{n/2}}\left\langle\delta, (-\circledast)^{k-1}\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)e^{-i(\xi.x)}\right\rangle
\end{aligned}$$

By keeping on operator $(-\circledast)$ with $k-1$ times , we obtain

$$\mathcal{F}(-\circledast)^k\delta = \frac{(-1)^{4k}}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3 + \left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right]^k$$

Now,

$$\begin{aligned}
|\mathcal{F}(-\circledast)^k\delta| &= \frac{1}{(2\pi)^{n/2}}\left|\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3 + \left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right|^k \\
&\leq \frac{1}{(2\pi)^{n/2}}\left|\xi_1^2+\ldots+\xi_n^2\right|^k\left|\left(\xi_1^2+\ldots+\xi_n^2\right)^2 + \left(\xi_1^2+\ldots+\xi_n^2\right)^2 + \left(\xi_1^2+\ldots+\xi_n^2\right)^2\right|^k \\
&\leq \frac{3^k}{(2\pi)^{n/2}}\|\xi\|^{6k}
\end{aligned}$$

where $\|\xi\| = \left(\xi_1^2+\xi_2^2+\ldots+\xi_n^2\right)^{1/2}$, $\xi_i(i=1,2,\ldots,n)\in\mathbb{R}$. Hence we obtain $\mathcal{F}(-\circledast)^k\delta$ is bounded and continuous on the space $\mathcal{S}'$ of the tempered distribution.

Since $\mathcal{F}$ is $1-1$ transformation from the space $\mathcal{S}'$ of the tempered distribution to the real space $\mathbb{R}$, then by (8)

$$\circledast\delta = \mathcal{F}^{-1}\frac{1}{(2\pi)^{n/2}}\left[\left(\xi_1^2+\xi_2^2+\ldots+\xi_p^2\right)^3 + \left(\xi_{p+1}^2+\xi_{p+2}^2+\ldots+\xi_{p+q}^2\right)^3\right].$$

That completes the proof.

3. Main Results

Theorem 1 *Given the equation*

$$\frac{\partial}{\partial t}u(x,t) + c^2(-\circledast)^k u(x,t) = 0 \tag{13}$$

with the initial condition

$$u(x,0) = f(x) \tag{14}$$

The operator

$$\begin{aligned}(\circledast)^k &= \left(\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^3+\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^3\right)^k\\ &= \left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}+\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^k\left[\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)^2-\left(\sum_{i=1}^{p}\frac{\partial^2}{\partial x_i^2}\right)\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)+\left(\sum_{j=p+1}^{p+q}\frac{\partial^2}{\partial x_j^2}\right)^2\right]^k\\ &= (\frac{3}{4}\diamond\Box+\frac{1}{4}\triangle^3)^k\end{aligned}$$

$p+q=n$ is the dimension of Euclidean space $\mathbb{R}^n$, k is a positive integer, $u(x,t)$ is an unknown function for $(x,t)=(x_1,x_2,\ldots,x_n,t)\in\mathbb{R}^n\times(0,\infty)$, $f(x)$ is the given generalized function, and c is a positive constant. Then we obtain

$$E(x,t)=\frac{1}{(2\pi)^n}\int_{R^n}\exp\left[-c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3+\left(\sum_{j=1}^{p+q}\xi_j^2\right)^3\right]^k t+i(\xi,x)\right]d\xi. \tag{15}$$

as a elementary solution of (13).

Proof. *Taking the Fourier transform defined by (7) to both sides of (13), we obtain*

$$\frac{\partial}{\partial t}\widehat{u}(\xi,t)+c^2\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]^k\widehat{u}(\xi,t)=0,$$

(see Lemma 8). Thus

$$\widehat{u}(\xi,t)=K(\xi)\exp\left[-c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right] \tag{16}$$

where $K(\xi)$ is constant and $\widehat{u}(\xi,0)=K(\xi)$.
Now , for k is positive number.Thus $\widehat{u}(\xi,t)$ in (16) is bound and can be written by

$$\widehat{u}(\xi,t)=K(\xi)\exp\left[-c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right] \tag{17}$$

Now , by (14) we have

$$K(\xi)=\widehat{u}(\xi,0)=\widehat{f}(\xi)=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{-i(\xi,x)}f(x)\,dx \tag{18}$$

and by the inversion (8)in (17) and (18) we obtain

$$\begin{aligned}u(x,t)&=\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}e^{i(\xi,x)}\widehat{u}(\xi,t)\,d\xi\\ &=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i(\xi,x)}e^{-i(\xi,y)}f(y)\exp\left[-c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right]dy\,d\xi.\end{aligned}$$

Thus

$$u(x,t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}e^{i(\xi,x-y)}\exp\left[-c^2t\left(\left(\sum_{i=1}^{p}\xi_i^2\right)^3+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right)^k\right]f(y)\,dy\,d\xi$$

or

$$u(x,t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\exp\left[-c^2t\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3+\left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]^k+i(\xi,x-y)\right]f(y)\,dy\,d\xi. \tag{19}$$

Set

$$E(x,t)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\exp\left[-c^2t\left[\left(\sum_{i=1}^{p}\xi_i^2\right)^3+\left(\sum_{j=p+1}^{p+q}\xi_i^2\right)^3\right]^k+i(\xi,x)\right]d\xi. \tag{20}$$

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Thus (19) can be written in the convolution form

$$u(x,t) = E(x,t) * f(x).$$

Now , the $E(x,t)$ given by (15) exists.

We obtain $u(x,t) = E(x,t) * f(x)$ is a solution of (13), where $E(x,t)$ is defined by (15). and k is positive integer.

Theorem 2 *The kernel $E(x,t)$ defined by (15) has the following properties :*

(1) $E(x,t) \in C^\infty$*-the space of continuous function for $x \in \mathbb{R}^n$, $t > 0$ with infinitely differentiable.*

(2) $\left(\frac{\partial}{\partial t} + c^2(-\circledast)^k\right)E(x,t) = 0$ *for* $t > 0$*.and k is positive integer.*

(3) $E(x,t) > 0$ *for* $t > 0$.

(4) $|E(x,t)| \leq \frac{1}{9.2^n \pi^{n/2} k^2 (c^2 t)^{\frac{n}{6k}}} \frac{\Gamma(\frac{p}{6k})\Gamma(\frac{q}{6k})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}, \quad$ *for* $\quad t > 0,$
Thus $E(x,t)$ is bounded for any fixed $t > 0$.

(5) $\lim_{t\to 0} E(x,t) = \delta$.

Proof.
(1) From (15), since

$$\frac{\partial^n}{\partial x^n} E(x,t) = \frac{1}{(2\pi)^n} \int_{R^n} \frac{\partial^n}{\partial x^n} \exp\left[-c^2\left[\left(\sum_{i=1}^{p} \xi_i^2\right)^3 + \left(\sum_{j=1}^{p+q} \xi_j^2\right)^3\right]^k t + i(\xi, x)\right] d\xi.$$

Thus $E(x,t) \in C^\infty$ for $x \in \mathbb{R}^n$, $t > 0$.
(2) By computing directly, we obtain

$$\left(\frac{\partial}{\partial t} + c^2(-\circledast)^k\right)E(x,t) = 0.$$

for $t > 0$ where $E(x,t)$ is defined by (15).
(3) $E(x,t) > 0$ for $t > 0$ is obvious by (15).
(4) We have

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{R^n} \exp\left[-c^2 t\left[\left(\sum_{i=1}^{p} \xi_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q} \xi_j^2\right)^3\right]^k + i(\xi, x)\right] d\xi.$$

put

$$\xi_1 = y_1, \ \ \xi_2 = y_2, \ldots, \ \ \xi_p = y_p$$
$$d\xi_1 = dy_1, \ \ d\xi_2 = dy_2, \ldots, \ \ d\xi_p = dy_p$$

and

$$\xi_{p+1} = y_{p+1}, \ \ \xi_{p+2} = y_{p+2}, \ldots, \ \ \xi_{p+q} = y_{p+q}$$
$$d\xi_{p+1} = dy_{p+1}, \ \ d\xi_{p+2} = dy_{p+2}, \ldots, \ \ d\xi_{p+q} = dy_{p+q}$$

Thus ,we obtain

$$E(x,t) = \frac{1}{(2\pi)^n} \int_{R^n} \exp\left[-c^2 t\left[\left(\sum_{i=1}^{p} \xi_i^2\right)^3 + \left(\sum_{i=1}^{p} \xi_i^2\right)^3\right]^k + i(\xi, x)\right] dy.$$

$$|E(x,t)| \leq \frac{1}{(2\pi)^n} \int_{R^n} \exp\left[-c^2 t\left(\left(\sum_{i=1}^{p} y_i^2\right)^3 + \left(\sum_{j=p+1}^{p+q} y_j^2\right)\right)^k\right] dy. \tag{21}$$

The same process as Lemma 2.1 then (21) becomes

$$|E(x,t)| \leq \frac{1}{9.2^n \pi^{n/2} k^2 (c^2 t)^{\frac{n}{6k}}} \frac{\Gamma(\frac{p}{6k})\Gamma(\frac{q}{6k})}{\Gamma(\frac{p}{2})\Gamma(\frac{q}{2})}$$

(5)We have

$$E(x,t) = \frac{1}{(2\pi)^n}\int_{R^n} \exp\left[-c^2 t\left[\left(\sum_{i=1}^{p}\xi_j^2\right)^3 + \left(\sum_{j=p+1}^{p+q}\xi_j^2\right)^3\right]^k + i(\xi,x)\right] d\xi.$$

Since $E(x,t)$ exists , then

$$\begin{aligned}\lim_{t\to 0} E(x,t) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{i(\xi,x)}\, d\xi \\ &= \delta(x), \qquad \text{for} \quad x \in \mathbb{R}^n. \end{aligned} \tag{22}$$

See (R. Haberman,1983 , p396, Eq.(10.2.19b)). then

$$\lim_{t\to 0} u(x,t) = u(x,0) = \delta * f(x) = f(x)$$

which satisfies (14)

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References

A. H. Zemanian. (1965). *Distribution Theory and Transform Analysis*, McGraw-Hill, New York.

A. Kananthai. (1997). On the Solution of the n-Dimensional Diamond Operator. *Applied Mathematics and Computational*, 88:27-37.

A. Kananthai. (1999). On the Fourier Transform of the Diamond Kernel of Marcel Riesz. *Applied Mathematics and Computation*, 101:151-158.

F. John. (1982). Partial Differential Equations, 4th Edition, Springer-Verlag, New York.

K. Nonlaopon, A. Kananthai. (2003). On the Ultra-hyperbolic heat kernel. *Applied Mathematics*, Vol. 13, No. 2, 215-225.

R. Haberman. (1983). Elementary Applied Partial Differential Equations, 2nd Edition, Prentice-Hall International, Inc.