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Discrete First-Order Three-Point Boundary Value Problem

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Abstract

We study difference equations which arise as discrete approximations to three-point boundary value problems for systems of first-order ordinary differential equations. We obtain new results of the existence of solutions to the discrete problem by employing Euler's method. The existence of solutions are proven by the contraction mapping theorem and the Brouwer fixed point theorem in Euclidean space. We apply our results to show that solutions to the discrete problem converge to solutions of the continuous problem in an aggregate sense. We also give some examples to illustrate the existence of a unique solution of the contraction mapping theorem.

Keywords: Discrete three-point boundary value problem, Contraction mapping theorem, Brouwer fixed point theorem

1. Introduction

In this paper, we study the three-point boundary value problem

$$Dx_k = f(t_k, x_k), \quad k = 1, \cdots, n, \tag{1}$$
$$Mx_0 + Nx_{\hat{s}} + Rx_n = \alpha. \tag{2}$$

which arises as a discrete approximation to the continuous problem

$$x'(t) = f(t, x), \; t \in [a, c], \tag{3}$$
$$Mx(a) + Nx(b) + Rx(c) = \alpha, \tag{4}$$

Here f is a continuous, vector-valued and possibly nonlinear function, the step size $h = (c-a)/n$ and grid points $t_k = a+kh$ for $k = 0, \cdots, n$. M, N and R are given $d \times d$ matrices, and $\alpha \in \mathbb{R}^d$. Let $s \in \{0, 1, \cdots, n-1\}$ be such that $s < b < s+1$. Choose a $\theta \in [0, 1]$ so that $x_{\hat{s}} = \theta x_{s+1} + (1-\theta)x_s$. Thus we approximate $x_{\hat{s}}$ by linear interpolation (see McCormick, 1964, p. 50-51).

Numerical solutions to (3), (4) involve discretization. The numerical methods of interest provide solutions that closely approximate the exact solutions for sufficiently small step size (see Keller, 1991). Since the Euler method is the simplest numerical scheme for solving initial value problems, we employ this method for approximating the solution of (3), (4), requiring an extremely small step size.

The discretized boundary value problems and the 'effect' that this discretization may have on possible solutions when compared with solutions to the original continuous boundary value problem, have been researched by (Agarwal, 1985), (Gaines, 1974) and (Lasota, 1968). For example, (Agarwal, 1985) provides some examples showing that even though the continuous boundary value problem may have a solution, its discretization may have no solution. Thus we formulate a convergence theorem which is a generalization of Theorem 2.5, (Gaines, 1974) showing that if solutions to the continuous problem (3), (4) are unique, then solutions to the discrete problem (1), (2) converge to solutions of the continuous problem.

The primary motivation here for the research in this paper is the work by (Ma, 2002) who studied the existence and uniqueness of solutions of three-point boundary value problems using the Leray-Schauder continuation theorem when f is a Carath*é*odory function. In this work, by employing Euler's method, we obtain new results of the existence of solutions to (1), (2) with uniqueness as well as the existence of solutions to (1), (2) without uniqueness. We prove existence and uniqueness results for nonlinear boundary value problems using the contraction mapping theorem and the Brouwer fixed point theorem in Euclidean space. We also give some examples to illustrate the existence of a unique solution of the contraction mapping theorem.

2. Notation and Preliminary Results

Denote $\mathbf{X} = \mathbb{R}^{(n+1)d} = \mathbb{R}^d \times \cdots \times \mathbb{R}^d \simeq \{0, h, \cdots, nh\} \times \mathbb{R}^d$. Let $a, b, c \in \mathbb{R}$ with $a < b < c$ and $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{R}^d$. Let B be a $d \times d$ matrix with elements b_{ij}, with the norm

$$\| B \|_\infty = \max_{1 \le i \le d} \sum_{j=1}^{d} | b_{ij} | . \tag{5}$$

Let $\mathbf{e} = (e_1, \cdots, e_n)$, where $e_k \in \mathbb{R}^d$, $k = 1, \cdots, n$ with norms $\| e_j \| = \max_{1 \le i \le d} | e_{ji} |$, where $| \, . \, |$ denotes the modulus of $e_{ji} \in \mathbb{R}$. By abuse of notation we let $\| e \|_\infty = \max_{1 \le j \le n} \| e_j \|$. Let $p_k \in \mathbb{R}$, $k = 1, \cdots, n$, where $p : [a, b] \to \mathbb{R}$ is continuous with norm $\| p \| = \max_{1 \le k \le n} | p_k |$, and set $\| x_k \| = \max_{1 \le i \le d} | x_{ki} |$, where $| \, . \, |$ denotes the modulus of $\mathbb{R}$. Set $\| \mathbf{x} \|_\infty = \max_{0 \le k \le n} \| x_k \|$ for each $\mathbf{x} = (x_0, \cdots, x_n) \in \mathbf{X}$ which defines a norm on $\mathbf{X}$. If $\mathbf{x} = (x_0, \cdots, x_n) \in \mathbb{R}^{(n+1)d}$, set $\triangle x_k = (x_k - x_{k-1})$ for $k = 1, \cdots, n$ and the difference quotient $Dx_k = \triangle x_k / h$. If $c \in \mathbb{R}^d$ is a constant then $\mathbf{c}$ satisfies $\mathbf{c}_k = c$ for all $k = 0, 1, \cdots, n$. Thus define $\| c \| = | c |$, where $| \, . \, |$ denotes the modulus of $\mathbb{R}$. By a solution to (3), we mean a vector function $x \in C^1([a, c]; \mathbb{R}^d)$ satisfying (3) for all $t \in [a, c]$ and (4). By a solution to (1), we mean a vector $\mathbf{x} = (x_0, x_1, \cdots, x_n) \in \mathbb{R}^{(n+1)d}$ satisfying (1) for all $k = 1, \cdots, n$ and (2). The value of the k^{th} component x_k of a solution $\mathbf{x}$ of (1) is expected to approximate $x(t_k)$, for some solution x of (3). We assume the following:

Assumption (A1). M, N and R are constant square matrices of order d such that

$$\det(M + N + R) \neq 0. \tag{6}$$

Lemma 1 Let (**A1**) hold. Let $\alpha \in \mathbb{R}^d$, $\mathbf{e} = (e_1, \cdots, e_n)$, where $e_k \in \mathbb{R}^d$, $k = 1, \cdots, n$, and $x_k \in \mathbb{R}^d$, $k = 0, 1, \cdots, n$ be such that

$$\begin{aligned} Dx_k &= e_k, \ \ k = 1, \cdots, n, \\ Mx_0 + Nx_{\hat{s}} + Rx_n &= \alpha. \end{aligned}$$

Then

$$x_k = \sum_{j=1}^{k} he_j + x_0, \ \ k = 0, 1, \cdots, n,$$

where,

$$x_0 = (M + N + R)^{-1} \Big(\alpha - N\theta he_{s+1} - N \sum_{j=1}^{s} he_j - R \sum_{j=1}^{n} he_j \Big), \tag{7}$$

and $\sum_{j=1}^{0} he_j = 0$ by definition.

Proof We have $x_s = x_0 + \sum_{j=1}^{s} he_j$, $x_{s+1} = x_0 + \sum_{j=1}^{s} he_j + he_{s+1}$, $x_{\hat{s}} = x_0 + \theta he_{s+1} + \sum_{j=1}^{s} he_j$.
Combining this with (2), we conclude that

$$\alpha = Mx_0 + N\Big(x_0 + \theta he_{s+1} + \sum_{j=1}^{s} he_j\Big) + R\Big(x_0 + \sum_{j=1}^{n} he_j\Big),$$

where

$$x_0 = \Big(M + N + R\Big)^{-1}\Big(\alpha - N\theta h e_{s+1} - N\sum_{j=1}^{s} he_j - R\sum_{j=1}^{n} he_j\Big).$$

Therefore

$$x_k = \sum_{j=1}^{k} he_j + x_0, \quad k = 0, 1, \cdots, n,$$

where $\sum_{j=1}^{0} he_j = 0$ by definition and

$$x_0 = (M + N + R)^{-1}\Big(\alpha - N\theta h e_{s+1} - N\sum_{j=1}^{s} he_j - R\sum_{j=1}^{n} he_j\Big).$$

This completes the proof.

Lemma 2 Let (**A1**) hold. Let $\alpha \in \mathbb{R}^d$, $\mathbf{e} = (e_1, \cdots, e_n)$, where $e_k \in \mathbb{R}^d$, $k = 1, \cdots, n$, and $x_k \in \mathbb{R}^d$, $k = 0, 1, \cdots, n$ be such that

$$\begin{aligned} Dx_k &= e_k, \quad k = 1, \cdots, n, \\ Mx_0 + Nx_{\hat{s}} + Rx_n &= 0. \end{aligned}$$

Then

$$\| \mathbf{x} \|_\infty \leq \Gamma_0 h(n + \theta) \| e \|_\infty, \tag{8}$$

where

$$\begin{aligned} \Gamma_0 = \max\{ & \| (M + N + R)^{-1} R \|_\infty, \| (M + N + R)^{-1} M \|_\infty, \| (M + N + R)^{-1} N \|_\infty, \\ & \| (M + N + R)^{-1}(N + R) \|_\infty, \| (M + N + R)^{-1}(M + N) \|_\infty, \\ & \| (M + N + R)^{-1} M \|_\infty + \| (M + N + R)^{-1} R \|_\infty, \\ & \| (M + N + R)^{-1}(N + R) \|_\infty + \| (M + N + R)^{-1} R \|_\infty \\ & \| (M + N + R)^{-1} M \|_\infty + \| (M + N + R)^{-1} R \|_\infty, \\ & \| (M + N + R)^{-1} M \|_\infty + \| (M + N + R)^{-1}(M + N) \|_\infty\}, \end{aligned} \tag{9}$$

and $\| \,.\, \|_\infty$ is given in (5).

Proof Setting $\alpha = 0$ in (7) we have

$$\begin{aligned} x_k \;=\; & \sum_{j=1}^{k} he_j + (M + N + R)^{-1}\Big(- N\theta h e_{s+1} \\ & -N\sum_{j=1}^{s} he_j - R\sum_{j=1}^{n} he_j\Big). \end{aligned} \tag{10}$$

For $t_0 \leq t_k \leq t_s \leq t_{s+1} \leq t_n$, we have

$$\begin{aligned} x_k \;=\; & \sum_{j=1}^{k} he_j + (M + N + R)^{-1}\Bigg(- N\theta h e_{s+1} \\ & -N\Big(\sum_{j=1}^{k} he_j + \sum_{j=k+1}^{s} he_j\Big) - R\Big(\sum_{j=1}^{k} he_j \\ & + \sum_{j=k+1}^{s} he_j + \sum_{j=s+1}^{n} he_j\Big)\Bigg) \\ =\; & (M + N + R)^{-1} M \sum_{j=1}^{k} he_j - (M + N + R)^{-1}(N + R) \sum_{j=k+1}^{s} he_j \\ & -(M + N + R)^{-1} R \sum_{j=s+1}^{n} he_j - (M + N + R)^{-1} N\theta h e_{s+1}, \\ & k = 0, 1, \cdots, n, \end{aligned}$$

where $\sum_{j=1}^{0} he_j = 0$ by definition. Thus

$$\begin{aligned} \| \mathbf{x} \|_\infty &\leq \Gamma_{1,0} h(\sum_{j=1}^{n} \| e_j \| + \theta \| e_{s+1} \|) \\ &\leq \Gamma_{1,0} h(n+\theta) \| e \|_\infty, \end{aligned} \tag{11}$$

where

$$\begin{aligned} \Gamma_{1,0} = \max\{&\| (M+N+R)^{-1}M \|_\infty, \| (M+N+R)^{-1}(N+R) \|_\infty, \\ &\| (M+N+R)^{-1}R \|_\infty, \| (M+N+R)^{-1}N \|_\infty, \\ &\| (M+N+R)^{-1}M \|_\infty + \| (M+N+R)^{-1}R \|_\infty, \\ &\| (M+N+R)^{-1}(N+R) \|_\infty + \| (M+N+R)^{-1}R \|_\infty\}, \end{aligned} \tag{12}$$

and $\| \,.\, \|_\infty$ is given in (5). For $t_0 \leq t_s \leq t_{s+1} \leq t_k \leq t_n$, we have

$$\begin{aligned} x_k &= \sum_{j=1}^{s} he_j + \sum_{j=s+1}^{k} he_j + (M+N+R)^{-1}\Big(-N\theta he_{s+1} - N\sum_{j=1}^{s} he_j \\ &\quad -R\Big(\sum_{j=1}^{s} he_j + \sum_{j=s+1}^{k} he_j + \sum_{j=k+1}^{n} he_j\Big)\Big) \\ &= (M+N+R)^{-1}M\sum_{j=1}^{s} he_j + (M+N+R)^{-1}(M+N)\sum_{j=s+1}^{k} he_j \\ &\quad -(M+N+R)^{-1}R\sum_{j=k+1}^{n} he_j - (M+N+R)^{-1}N\theta he_{s+1}, \\ &\quad k = 0, 1, \cdots, n, \end{aligned}$$

where $\sum_{j=s+1}^{0} he_j = 0$ by definition. Thus

$$\begin{aligned} \| \mathbf{x} \|_\infty &\leq \Gamma_{2,0} h(\sum_{j=1}^{n} \| e_j \| + \theta \| e_{s+1} \|) \\ &\leq \Gamma_{2,0} h(n+\theta) \| e \|_\infty, \end{aligned} \tag{13}$$

where

$$\begin{aligned} \Gamma_{2,0} = \max\{&\| (M+N+R)^{-1}M \|_\infty, \| (M+N+R)^{-1}(M+N) \|_\infty, \\ &\| (M+N+R)^{-1}R \|_\infty, \| (M+N+R)^{-1}N \|_\infty, \\ &\| (M+N+R)^{-1}M \|_\infty + \| (M+N+R)^{-1}R \|_\infty, \\ &\| (M+N+R)^{-1}M \|_\infty + \| (M+N+R)^{-1}(M+N) \|_\infty\}, \end{aligned} \tag{14}$$

and $\| \,.\, \|_\infty$ is given in (5).

Combining (11) with (13), we obtain

$$\| \mathbf{x} \|_\infty \leq \Gamma_0 h(n+\theta) \| e \|_\infty, \tag{15}$$

where $\Gamma_0 = \max\{\Gamma_{1,0}, \Gamma_{2,0}\}$.

Lemma 3 Let (**A1**) hold. Let $x_k \in \mathbb{R}^d$, $k = 0, 1, \cdots, n$, $\alpha \in \mathbb{R}^d$, and $w \in \mathbb{R}^d$. Then the problem

$$\begin{aligned} Dx_k = 0,\ k = 1, \cdots, n, \\ Mx_0 + Nx_{\hat{s}} + Rx_n = \alpha \end{aligned}$$

has a unique solution $\mathbf{x}_k = w$ where

$$w = (M+N+R)^{-1}\alpha. \tag{16}$$

Lemma 4 Let (**A1**) hold. Let $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous, $x_k \in \mathbb{R}^d$, $k = 0, 1, \cdots, n$, $\alpha \in \mathbb{R}^d$, and w be defined by (16). Then the problem

$$Dx_k = f(t_k, x_k),\ k = 1, \cdots, n, \tag{17}$$

$$Mx_0 + Nx_{\hat{s}} + Rx_n = \alpha \tag{18}$$

has a unique solution $\mathbf{x} = \mathbf{u} + w$ if and only if $\mathbf{u}$ is the only solution of

$$Dx_k = f(t_k, u_k + w),\ k = 1, \cdots, n, \tag{19}$$
$$Mx_0 + Nx_{\hat{s}} + Rx_n = 0. \tag{20}$$

Proof Suppose $\mathbf{u}$ is the only solution of (19), (20). Define $\mathbf{x} = \mathbf{u} + w$. It is clear that $Dx_k = f(t_k, x_k)$, $k = 1, \cdots, n$, and

$$\begin{aligned} Mx_0 + Nx_{\hat{s}} + Rx_n &= M[u_0 + w] + N[u_{\hat{s}} + w] + R[u_n + w] \\ &= Mu_0 + Nu_{\hat{s}} + Ru_n + (M + N + R)w \\ &= 0 + \alpha. \end{aligned}$$

Conversely, suppose $\mathbf{x} = \mathbf{u} + w$ is the only solution of (17), (18). Then it is clear that

$$Du_k = f(t_k, u_k),\ k = 1, \cdots, n,$$
$$Mu_0 + Nu_{\hat{s}} + Ru_n = 0.$$

Thus the proof is complete.

3. Existence Results

In this section, first to obtain an existence theorem without uniqueness of the solution, we will apply the Brouwer Fixed Point Theorem which is given in (Keller, 1991, p. 382). Then we shall use the contraction mapping theorem which is given in (Keller, 1991, p. 372) to establish the existence of a unique solution to the boundary value problem (1), (2).

Theorem 1 Let (**A1**) hold. Let $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function and $p_k \in \mathbb{R}$, $k = 1, \cdots, n$ such that

$$\| f(t_k, u) - f(t_k, v) \| \le p_k \| u - v \|,\ \ k = 1, \cdots, n, \tag{21}$$

for all $u, v \in \mathbb{R}^d$, where $\| f(t_k, u) - f(t_k, v) \| = \max_{1\le i\le d} | f_i(t_k, u) - f_i(t_k, v) |$. If

$$\Gamma_0 h(n + \theta) \| p \| < 1, \tag{22}$$

then the three-point boundary value problem (1), (2) has at least one solution.

Proof In view of Lemma 4, to prove that (1), (2) has a solution $\mathbf{x} = \mathbf{u} + w$, it suffices to prove the following problem

$$Du_k = f(t_k, u_k + w),\ k = 1, \cdots, n, \tag{23}$$
$$Mu_0 + Nu_{\hat{s}} + Ru_n = 0 \tag{24}$$

has a solution in $\mathbf{u}$, where w is defined by (16). The general solution of (23), (24) is

$$\begin{aligned} u_k = & \sum_{j=1}^{k} hf(t_j, u_j + w) + (M + N + R)^{-1}\Big(- N\theta hf(t_{s+1}, u_{s+1} + w) \\ & -N \sum_{j=1}^{s} hf(t_j, u_j + w) - R \sum_{j=1}^{n} hf(t_j, u_j + w)\Big),\ k = 0, 1, \cdots, n. \end{aligned} \tag{25}$$

Let $l > 0$, set $\Omega = \{\mathbf{u} \in R^{(n+1)d} : \| \mathbf{u} \|_\infty \le l\}$ so that Ω is a closed subset of $\mathbf{X} = \mathbb{R}^{(n+1)d}$, where

$$l \ge \frac{\Gamma_0 h(n + \theta)\{\| p \| \| w \| + \max_{1\le k\le n} \| f(t_k, 0) \|\}}{1 - \Gamma_0 h(n + \theta) \| p \|}.$$

Let $\| f \|_{\infty,\Omega} = \max_{\mathbf{u}\in\Omega} \max_{1\le k\le n} \| f(t_k, u_k + w) \|$. Define T on Ω by

$$\begin{aligned} Tu_k = & \sum_{j=1}^{k} hf(t_j, u_j + w)\ +\ (M + N + R)^{-1}\Big(- N\theta hf(t_{s+1}, u_{s+1} + w) \\ & -N \sum_{j=1}^{s} hf(t_j, u_j + w)\ -\ R \sum_{j=1}^{n} hf(t_j, u_j + w)\Big),\ k = 0, 1, \cdots, n, \end{aligned} \tag{26}$$

where $\sum_{j=1}^{0} hf(t_j, u_j + w) = 0$ by definition. By the continuity of f, T is continuous. We now show that $T : \Omega \longrightarrow \Omega$. Then by (21) we have

$$\begin{aligned} \| f \|_{\infty,\Omega} &\leq \max_{\mathbf{u}\in\Omega}\max_{1\leq k\leq n}\{\| f(t_k, u_k + w) - f(t_k, 0) \| + \| f(t_k, 0) \|\} \\ &\leq \max_{\mathbf{u}\in\Omega}\max_{1\leq k\leq n}\{p_k(\| u_k \| + \| w \|) + \| f(t_k, 0) \|\},\ k = 1, \cdots, n, \\ &\leq \| p \| (l + \| w \|) + \max_{1\leq k\leq n} \| f(t_k, 0) \| . \end{aligned} \tag{27}$$

For $t_0 \leq t_k \leq t_s \leq t_{s+1} \leq t_n$ we have

$$\begin{aligned} Tu_k &= (M + N + R)^{-1} M \sum_{j=1}^{k} hf(t_j, u_j + w) \ - \ (M + N + R)^{-1}(N + R) \\ &\quad \times \sum_{j=k+1}^{s} hf(t_j, u_j + w) - (M + N + R)^{-1} R \sum_{j=s+1}^{n} hf(t_j, u_j + w) \\ &\quad -(M + N + R)^{-1} N\theta hf(t_{s+1}, u_{s+1} + w), \end{aligned}$$

where $\sum_{j=1}^{0} hf(t_j, u_j + w) = 0$ by definition. Thus

$$\begin{aligned} \| T\mathbf{u} \|_\infty &\leq \Gamma_{1,0} h \max_{\mathbf{u}\in\Omega}\{\sum_{j=1}^{n} \| f(t_j, u_j + w) \| + \theta \| f(t_{s+1}, u_{s+1} + w) \|\} \\ &\leq \Gamma_{1,0} h(n + \theta) \| f \|_{\infty,\Omega} \\ &\leq \Gamma_{1,0}(n + \theta)h\{\| p \| (l + \| w \|) + \max_{1\leq k\leq n} \| f(t_k, 0) \|\}, \end{aligned} \tag{28}$$

where $\Gamma_{1,0}$ is given in (12). Also, for $t_0 \leq t_s \leq t_{s+1} \leq t_k \leq t_n$ we have

$$\begin{aligned} Tu_k &= (M + N + R)^{-1} M \sum_{j=1}^{s} hf(t_j, u_j + w) + (M + N + R)^{-1}(M + N) \\ &\quad \times \sum_{j=s+1}^{k} hf(t_j, u_j + w) - (M + N + R)^{-1} R \sum_{j=k+1}^{n} hf(t_j, u_j + w) \\ &\quad -(M + N + R)^{-1} N\theta hf(t_{s+1}, u_{s+1} + w), \\ &\quad \text{for } k = 0, 1, \cdots, n, \end{aligned}$$

where $\sum_{j=s+1}^{0} hf(t_j, u_j + w) = 0$ by definition. Thus

$$\begin{aligned} \| T\mathbf{u} \|_\infty &\leq \Gamma_{2,0} h \max_{\mathbf{u}\in\Omega}\{\sum_{j=1}^{n} \| f(t_j, u_j + w) \| + \theta \| f(t_{s+1}, u_{s+1} + w) \|\} \\ &\leq \Gamma_{2,0} h(n + \theta) \| f \|_{\infty,\Omega} \\ &\leq \Gamma_{2,0} h(n + \theta)\{\| p \| (l + \| w \|) + \max_{1\leq k\leq n} \| f(t_k, 0) \|\}, \end{aligned} \tag{29}$$

where $\Gamma_{2,0}$ is given in (14). Combining (28) with (29), we obtain

$$\begin{aligned} \| T\mathbf{u} \|_\infty &\leq \Gamma_0 h(n + \theta)\{\| p \| (l + \| w \|) + \max_{1\leq k\leq n} \| f(t_k, 0) \|\} \\ &\leq l \end{aligned} \tag{30}$$

where Γ_0 is given in (9) and

$$l \geq \frac{\Gamma_0 h(n + \theta)\{\| p \| \| w \| + \max_{1\leq k\leq n} \| f(t_k, 0) \|\}}{1 - \Gamma_0 h(n + \theta) \| p \|}$$

provided

$$\Gamma_0 h(n + \theta) \| p \| < 1.$$

Hence $T\mathbf{u} \in \Omega$, and the conclusion follows from the Brouwer Fixed Point Theorem that there exists at least one solution to the boundary value problem (1), (2) in $\Omega \subseteq \mathbf{X} = \mathbb{R}^{(n+1)d}$.

Theorem 2 Let (**A1**) hold. Let $f : [a, b] \times \mathbb{R}^d \to \mathbb{R}^d$ be a continuous function and $p_k \in \mathbb{R}$, $k = 1, \cdots, n$ such that

$$\| f(t_k, u) - f(t_k, v) \| \leq \ p_k \| u - v \|, \quad k = 1, \cdots, n, \tag{31}$$

for all $u, v \in \mathbb{R}^d$, where $\| f(t_k, u) - f(t_k, v) \| = \max_{1\le i\le d} | f_i(t_k, u) - f_i(t_k, v) |$. If

$$\Gamma_0 h(n+\theta) \| p \| < 1, \tag{32}$$

then the three point boundary value problem (1), (2) has a unique solution.

Proof The proof is similar to the proof of Theorem 1. Let $l > 0$, and set $\Omega = \{\mathbf{u} \in R^{(n+1)d} : \| \mathbf{u} \|_\infty \le l\}$ so that Ω is a closed subset of $\mathbf{X} = \mathbb{R}^{(n+1)d}$, where

$$l \ge \frac{\Gamma_0 h(n+\theta)\{\| p \| \| w \| + \max_{1\le k\le n} \| f(t_k, 0) \|\}}{1 - \Gamma_0 h(n+\theta) \| p \|}.$$

Let $\| f \|_{\infty,\Omega} = \max_{\mathbf{u}\in\Omega} \max_{1\le k\le n} \| f(t_k, u_k + w) \|$. Define T on Ω by

$$\begin{aligned} Tu_k &= \sum_{j=1}^{k} hf(t_j, u_j + w) + (M + N + R)^{-1}\Big(- N\theta hf(t_{s+1}, u_{s+1} + w) \\ &\quad - N \sum_{j=1}^{s} hf(t_j, u_j + w) - R \sum_{j=1}^{n} hf(t_j, u_j + w)\Big), \ k = 0, 1, \cdots, n. \end{aligned}$$

By the continuity of f, T is continuous. We have already shown that $T : \Omega \longrightarrow \Omega$ in the proof of Theorem 1. Then by (31) we have

$$\begin{aligned} \| f \|_{\infty,\Omega} &\le \max_{\mathbf{u}\in\Omega} \max_{1\le k\le n}\{\| f(t_k, u_k + w) - f(t_k, 0) \| + \| f(t_k, 0) \|\} \\ &\le \max_{\mathbf{u}\in\Omega} \max_{1\le k\le n}\{p_k(\| u_k \| + \| w \|) + \| f(t_k, 0) \|\}, \ k = 1, \cdots, n, \\ &\le \| p \| (l + \| w \|) + \max_{1\le k\le n} \| f(t_k, 0) \| . \end{aligned} \tag{33}$$

We obtained

$$\begin{aligned} \| T\mathbf{u} \|_\infty &\le \Gamma_0 h(n+\theta)\{\| p \| (l + \| w \|) + \max_{1\le k\le n} \| f(t_k, 0) \|\} \\ &\le l \end{aligned} \tag{34}$$

where Γ_0 is given in (9) and

$$l \ge \frac{\Gamma_0 (n+\theta) h\{\| p \| \| w \| + \max_{1\le k\le n} \| f(t_k, 0) \|\}}{1 - \Gamma_0 h(n+\theta) \| p \|}$$

provided

$$\Gamma_0 h(n+\theta) \| p \| < 1.$$

Hence $T\mathbf{u} \in \Omega$.

We shall prove that $T : \Omega \longrightarrow \Omega$ is a contraction mapping. Let $\mathbf{u}, \mathbf{v} \in \Omega \subseteq \mathbb{R}^{(n+1)d}$. For $t_0 \le t_k \le t_s \le t_{s+1} \le t_n$ we have

$$\begin{aligned} \| T\mathbf{u} - T\mathbf{v} \|_\infty &\le \Gamma_{1,0} h\{\sum_{j=1}^{n} \| f(t_j, u_j + w) - f(t_j, v_j + w) \| \\ &\quad + \theta \| f(t_{s+1}, u_{s+1} + w) - f(t_{s+1}, v_{s+1} + w) \|\} \\ &\le \Gamma_{1,0} h(n+\theta) \| f(t_j, u_j + w) - f(t_j, v_j + w) \| \\ &\le \Gamma_{1,0} h(n+\theta) p_j \| u_j - v_j \| \\ &\le \Gamma_{1,0} (n+\theta) h \| p \| \| \mathbf{u} - \mathbf{v} \|_\infty . \end{aligned} \tag{35}$$

Also, for $t_0 \le t_s \le t_{s+1} \le t_k \le t_n$ we have

$$\begin{aligned} \| T\mathbf{u} - T\mathbf{v} \|_\infty &\le \Gamma_{2,0} h\{\sum_{j=1}^{n} \| f(t_j, u_j + w) - f(t_j, v_j + w) \| \\ &\quad + \theta \| f(t_{s+1}, u_{s+1} + w) - f(t_{s+1}, v_{s+1} + w) \|\} \\ &\le \Gamma_{2,0} h(n+\theta) \| f(t_j, u_j + w) - f(t_j, v_j + w) \| \\ &\le \Gamma_{2,0} h(n+\theta) p_j \| u_j - v_j \| \\ &\le \Gamma_{2,0} h(n+\theta) \| p \| \| \mathbf{u} - \mathbf{v} \|_\infty . \end{aligned} \tag{36}$$

Combining (35) with (36), we obtain

$$\| T\mathbf{u} - T\mathbf{v} \|_\infty \le \Gamma_0 h(n+\theta) \| p \| \| \mathbf{u} - \mathbf{v} \|_\infty .$$

where Γ_0 is given in (9). It follows that for $\Gamma_0 h(n+\theta) \parallel p \parallel < 1$, $T\mathbf{u} = \mathbf{u}$ has a unique solution $\mathbf{u}$ in $\Omega \subset \mathbf{X} = \mathbb{R}^{(n+1)d}$. This fixed point is the unique solution of the boundary value problem (1), (2). This completes the proof of the theorem.

The following two examples give the existence of a unique solution of Theorem 2.

Example 1

Consider the following discrete boundary value problem

$$Dx_k = (Dx_{k1}, Dx_{k2}) = (t_k x_k, t_k x_k \cos\ x_k), \tag{37}$$
$$= (f_1(t_k, x_k), f_2(t_k, x_k)) = f(t_k, x_k),\ k = 1, \cdots, 20,$$
$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} x_0 + \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} x_{\hat{s}} + \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} x_n = (1,1), \tag{38}$$

where $x_{\hat{s}} = \theta x_{3/2} + (1-\theta)x_{1/2}$. The boundary value problem (37), (38) has a unique solution.

Proof We have $M = N = R = I$, $\alpha = (1,1)$, $n = 20$, $h = 0.01$ and $\theta = 0.13$. Let $\Omega = \{\mathbf{x} \in \mathbb{R}^{2(n+1)} : \parallel \mathbf{x} \parallel_\infty < l\}$. By condition (34), we obtain $\parallel T\mathbf{u} \parallel_\infty \leq l$ provided that $l \geq 0.01398$. We have

$$\begin{aligned}\parallel f \parallel_{\infty,\Omega} &\leq \max_{\mathbf{x}\in\Omega}\max_{1\leq k\leq 20}\{\mid t_k \mid\parallel (x_k + w) \parallel, \mid t_k \mid\parallel (x_k + w)\cos(x_k + w) \parallel\}\\ &\leq \max_{\mathbf{x}\in\Omega}\max_{1\leq k\leq 20} \mid t_k \mid \Big(\parallel x_k \parallel + \parallel w \parallel \Big)\\ &\leq \parallel p \parallel (l + \parallel w \parallel),\end{aligned}$$

with $p_k = t_k$, $k = 1, \cdots, 20$, $\parallel w \parallel = 1/3$, and $\parallel p \parallel = \max_{1\leq k\leq 20}\{\mid 0.01k \mid\} = 0.2$. Thus (33) holds. We have $\Gamma_0 = 1$. Condition (32) becomes $\Gamma_0 h(n+\theta) \parallel p \parallel = 0.04026 < 1$. Thus all of the conditions of Theorem 2 hold and we conclude that the boundary value problem (37), (38) has a unique solution.

Example 2 Consider the following discrete boundary value problem

$$Dx_k = (Dx_{k1}, Dx_{k2}) = (t_k^2 x_k \cos\ x_k, t_k^2 x_k \sin\ x_k), \tag{39}$$
$$= (f_1(t_k, x_k), f_2(t_k, x_k)) = f(t_k, x_k),\ k = 1, \cdots, 20,$$
$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} x_0 + \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} x_{\hat{s}} + \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} x_n = (1,1), \tag{40}$$

where $x_{\hat{s}} = \theta x_{3/2} + (1-\theta)x_{1/2}$. The boundary value problem (39), (40) has a unique solution.

Proof We have $M = N = R = I$, $\alpha = (1,1)$, $n = 20$, $h = 0.01$ and $\theta = 0.1$. Let $\Omega = \{\mathbf{x} \in \mathbb{R}^{2(n+1)} : \parallel \mathbf{x} \parallel_\infty < l\}$. By condition (34), we obtain $\parallel T\mathbf{u} \parallel_\infty \leq l$ provided that $l \geq 2.702 \times 10^{-3}$. We have

$$\begin{aligned}\parallel f \parallel_{\infty,\Omega} &= \max_{\mathbf{x}\in\Omega}\max_{1\leq k\leq 20}\{\mid t_k^2 \mid\parallel (x_k + w)\cos(x_k + w) \parallel, \mid t_k^2 \mid\parallel (x_k + w)\sin(x_k + w) \parallel\}\\ &\leq \max_{\mathbf{x}\in\Omega}\max_{1\leq k\leq 20} \mid t_k^2 \mid \Big(\parallel x_k \parallel + \parallel w \parallel \Big)\\ &\leq \parallel p \parallel (l + \parallel w \parallel),\end{aligned}$$

with $p_k = t_k^2$, $k = 1, \cdots, 20$, $\parallel w \parallel = 1/3$, and $\parallel p \parallel = \max_{1\leq k\leq 20}\{\mid (0.01k)^2 \mid\} = 0.04$. Thus (33) holds. As in the previous example, we have $\Gamma_0 = 1$. Condition (32) becomes $\Gamma_0 h(n+\theta) \parallel p \parallel = 0.00804 < 1$. Thus all of the conditions of Theorem 2 hold and we conclude that the boundary value problem (39), (40) has a unique solution.

4. Convergence of Solutions

In this section, the previous results are applied to formulate a convergence theorem. The following is a generalization of Theorem 2.5, (Gaines, 1974).

Theorem 3 Let the assumptions of Theorem 2 hold. Given $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that if $0 < h < \delta$ and $\mathbf{x}$ is the solution of (1), (2), then there is a solution x of (3), (4) such that

$$\max\{\parallel x(t, \mathbf{x}) - x \parallel_\infty : 0 \leq t \leq 1\} \leq \epsilon \tag{41}$$

where $x(t, \mathbf{x}) = x_k + (t - t_k)Dx_{k+1}$ for $t_k \leq t \leq t_{k+1}$.

Proof The proof is similar to that of (Gaines, 1974) and so is omitted.

Remark 1 It follows from Theorem 3 that if the solutions to the continuous problem (3), (4) are unique, then solutions to (1), (2) converge to solutions of the continuous problem in the sense of Theorem 3.

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