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Fuzzy Topological Dynamical Systems

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Abstract

In this paper by considering a fuzzy continuous action of a fuzzy topological group on a fuzzy topological space we have fuzzifiifed the notion of topological dynamical system. Some properties of this fuzzy structure are investigated. Then we have constructed mixed fuzzy topological dynamical system from two given fuzzy topological dynamical systems.

Keywords: Fuzzy topologies, Fuzzy dynamical system, Fuzzy neighborhoods, Fuzzy orbit, Mixed fuzzy topology

1. Introduction

In general, the theory of dynamical systems deals with the action of groups of continuous transformations of topological spaces. A classical dynamical system(Wieslaw Szlenk, 1984) is a structure (π, G, X,) where G is a topological group, X is a topological space and π is a continuous function from $G \times X \to X$ satisfying $\pi(0, x) = x$ and $\pi(s, (t, x)) = \pi(s + t, x)$, where 0 is the identity of G. In this paper we fuzzify the above concept as a natural transition from the corresponding crisp structure. For this fuzzification we will consider a fuzzy topological group(Wieslaw Szlenk, 1984), a fuzzy topological space and a fuzzy continuous map from $G \times X \to X$ satisfying the above stated conditions. In (N.R. Das, 1995, p77-784) Das and Baishya introduced the notion of fuzzy mixed topology and in (N.R. Das, 2000, p401-408) they constructed fuzzy mixed topological group. In this paper we will construct fuzzy mixed topological dynamical system. Throughout our discussion the fuzzy topology on any set will contain all the constant fuzzy subsets. In other words we will use Lowen(R. Lowen, 1976, p621-633) definition of fuzzy topology.

2. Preliminaries

In this section we recall some preliminary definitions and results to be used in the sequel.

Let X be a non-empty set. A fuzzy set in X is an element of the set I^X of all functions from X into the unit interval I. A fuzzy point of a set X is a fuzzy subset which takes non-zero value at a single point and zero at every other point. The fuzzy point which takes value $\alpha \neq 0$ at $x \in X$, and zero elsewhere is denoted by x_α. Let λ be a fuzzy subset of X. Suppose $\lambda(x) = \alpha$ for $x \in X$. Then λ can be expressed as union of all its fuzzy points, i.e, $\lambda = \vee_{x \in X} x_\alpha$. Here $\vee$ denote union. We will use the same notation $\vee$ to denote supremum of a set of numbers. Similarly $\wedge$ will be used to denote intersection of fuzzy sets as well as infimum of a set of real numbers.

Let S be a set with a binary operation *. Then for any fuzzy subsets λ and μ of S we define $(\lambda * \mu)(x * y) = \sup\{\min(\lambda(x), \mu(y))\}$.

Clearly then $(\lambda * \mu)(x * y) \geq \min(\lambda(x), \mu(y))$.

Let λ and μ be fuzzy subsets of X, then we write $\lambda \subseteq \mu$ whenever $\lambda(x) \leq \mu(x)$. Let λ be a fuzzy subset of a group $(G, +)$. Then we define a fuzzy subset $-\lambda$ as $-\lambda(x) = \lambda(-x)$.

If f is a function from X into Y and $\mu \in I^Y$, then $f^{-1}(\mu)$ is the fuzzy set in X defined by $f^{-1}(\mu)(x) = \mu(f(x))$. Equivalently, $f^{-1}(\mu) = \mu \circ f$. Also, for $\rho \in I^X$, $f(\rho)$, is the member of I^Y which is defined by

$$f(\rho)(y) = \begin{cases} \sup\{\rho(x) : x \in f^{-1}[y]\} & if\ f^{-1}[y]\ is\ not\ empty \\ 0 & otherwise \end{cases}$$

For the definition of a fuzzy topology, we will use the one given by Lowen (R. Lowen, 1976, p621-633) since his definition is more appropriate in our case. So, throughout this paper, by a fuzzy topology on a set X we will mean a sub-collection τ of I^X satisfying the following conditions:

(i) τ contains every constant fuzzy subset in X ;

(ii) If $\mu_1,\ \mu_2 \in \tau$, then $\mu_1 \wedge \mu_2 \in \tau$;

(iii) If $\mu_i \in \tau$ for each $i \in A$, then $\vee_{i \in A}\mu_i \in \tau$.

A fuzzy topological space is a set X on which there is given a fuzzy topology τ. The elements of τ are the open fuzzy sets in X. Complement of an open fuzzy set is called a closed fuzzy set. Interior of a fuzzy set λ is the union of all the open fuzzy sets contained in λ and the closure of λ is the intersection of all fuzzy sets containing λ. The interior and closure of λ will be denoted by λ^o and clλ respectively. A map f from a fuzzy topological space X to a fuzzy topological space Y, is called continuous if $f^{-1}(\mu)$ is open in X for each open fuzzy set μ in Y. Let X be a fuzzy topological space and $x \in X$. A fuzzy set μ in X is called a neighborhood of x if there exists an open fuzzy set ρ with $\rho \subseteq \mu$ and $\rho(x) = \mu(x) > 0$. Given a crisp topological space (X, T), the collection $\varpi(T)$, of all fuzzy sets in X which are lower semicontinuous, as functions from X to the unit interval $I = [0, 1]$ equipped with the usual topology, is a fuzzy topology on X ((R. Lowen, 1976, p621-633)). We will refer to the fuzzy topology $\varpi(T)$ as the fuzzy topology generated by the usual topology T. If $(X, T_j)_{j \in J}$ is a family of crisp topological spaces and T the product topology on $X = \Pi_{j \in J} X_j$, then $\varpi(T)$ is the product of the fuzzy topologies $\varpi(T_j)$, $j \in J$, (R. Lowen, 1997, p11-21).

Result 2.1 (A. K. Katsaras, 1981, p85-95) Let (X_i, T_i), $i = 1, 2, 3$, be crisp topological spaces, $X = X_1 \times X_2$, T the product of the topologies T_1, T_2 and $f : (X, T) \to (X_3, T_3)$ a continuous map. If δ is the product of the fuzzy topologies $\varpi(T_1)$ and $\varpi(T_2)$, then

$$f : (X, \delta) \to (X_3, \varpi(T_3))$$

is fuzzy continuous.

Result 2.2 Let (π, G, X) be a classical dynamical system. If we equip G and X with the induced fuzzy topologies and $G \times X$, with the corresponding product fuzzy topology, then the mapping $\pi : G \times X \to X$ is fuzzy continuous.

Proof. It follows from the previous result.

Result 2.5 (Liu Yingming, 1997) Let $(X, \delta), (Y, \tau)$ and (Z, k) be fuzzy topological spaces and $f : X \to Y$ and $g : Y \to Z$ be any mappings. Then f, g are fuzzy continuous $\Rightarrow$ gof is fuzzy continuous.

Definition 2.6 If σ is a fuzzy subset of X and η is a fuzzy subset of Y, then the fuzzy subset $\sigma \times \eta$ on $X \times Y$ is defined as $(\sigma \times \eta)(x, y) = \min\{\sigma(x), \eta(y)\}$.

Definition 2.7 (N. Palaniappan, 2005) Let (X, δ) and (Y, τ) be two fuzzy topological spaces. Then $f : X \to Y$ is fuzzy open (closed) if the image of every fuzzy open(closed) subset of X is fuzzy open(closed) in Y.

Definition 2.8 (Rajesh Kumar, 1993). Let (G, +) be a group. Then a fuzzy subset λ is said to be a fuzzy subgroup of G if $\lambda(x + y) \geq \min\{\lambda(x), \lambda(y))$ and $\lambda(-x) = \lambda(x)$

Definition 2.9 (N. Palaniappan. 2005) A fuzzy topological space (X, τ) is said to be product related to another fuzzy topological space (Y, δ) if for any fuzzy set υ of X and ζ of Y whenever $\lambda^c \not\geq \upsilon$ and $\mu^c \not\geq \zeta$ implies $(\lambda^c \times 1) \vee (1 \times \mu^c) \geq \upsilon \times \zeta$, where $\lambda \in \tau$ and $\mu \in \delta$, then there exist $\lambda_1 \in \tau$ and $\mu_1 \in \delta$ such that $\lambda_1^c \geq \upsilon$ or $\mu_1^c \geq \zeta$ and $(\lambda^c \times 1) \vee (1 \times \mu^c) = (\lambda_1^c \times 1) \vee (1 \times \mu_1^c)$.

Result 2.10 (N. Palaniappan. 2005) Let (X, τ) be product related to (Y, δ). Then for any fuzzy subset λ of X and a fuzzy subset μ of Y, $cl(\lambda \times \mu) = cl\lambda \times cl\mu$.

Definition 2.11 (A. K. Katsaras, 1981). The fuzzy usual topology on **K** is the fuzzy topology generated by the usual topology of **K**.

Definition 2.12 (Liu, Yingming, 1997) : A fuzzy topological space (X, τ) is called fuzzy regular if for any $\mu \in \tau$ there exists a sub-collection δ of τ such that $\sup\{\eta : \eta \in \delta\} = \mu$ and $cl\eta \leq \mu$ for all $\eta \in \delta$.

Dewan M. Ali (1990) proved the following equivalence.

Result 2.13 A fuzzy topological space (X, τ) is called fuzzy regular iff for $x \in X$, $t \in (0, 1)$ and $\mu \in \tau$ with $t < \mu(x)$ there exists $\eta \in \tau$ such that $t < \eta(x)$ and $cl\eta \leq \mu$.

3. Fuzzy topological Dynamical systems

In this section we will introduce the concept of fuzzy topological dynamical systems.

Definition 3.1 Let X be a fuzzy topological space, G be a fuzzy topological group.

If $\pi : G \times X \to X$ satisfies

(i) $\pi(0, x) = x$

(ii) $\pi(s, (t, x)) = \pi(s + t, x)$

(iii) π is fuzzy continuous

then (π, G, X) is called a fuzzy topological dynamical system

Throughout this section X will stand for a fuzzy topological dynamical system.

Definition 3.2 Let $t \in G$, then the t-transition of (π, G, X) denoted by π^t is the mapping : $\pi^t : X \to X$ such that $\pi^t(x) = \pi(t, x)$.

Result 3.3 (i) π^0 is the identity mapping of X.

(ii) $\pi^s\pi^t = \pi^{s+t}$ for s, $t \in G$

(iii) π^t is one-to-one mapping of X onto X and $-(\pi^t) = \pi^{-t}$

(iv) For $t \in G$, π^t is a fuzzy homeomorphism of X onto X.

Proof. Straightforward.

Definition 3.4 The transition group of (π, G, X) is the set $G^t = \{\pi^t : t \in G\}$. The transition projection of (π, G, X) is the mapping $\theta : G \to G^t$ defined as $\theta(t) = \pi^t$.

Definition 3.5 (π, G, X) is said to be effective if $t \in G$ with $t \neq 0 \Rightarrow \pi^t(x) \neq x$ for some x.

Result 3.6 (i) G^t is a group of fuzzy homeomorphisms of X onto X

(ii) θ is a group homomorphism of G onto G^t.

(iii) θ is one-one iff (π, G, X) is effective.

Proof. Straightforward.

Definition 3.7 Let $x \in X$, then the x-motion of (π , G, X) is the mapping $\pi_x : G \to X$ such that $\pi_x(t) = \pi(t, x)$.

Result 3.8 π_x is a fuzzy continuous mapping of G into X.

Proof. Straightforward.

Notation: We will denote $\pi(\lambda \times \mu)$ by $\lambda\mu$

Result 3.9 (Tazid Ali and Sampa Das, 2009) (i) For $t \in G$ and a fuzzy subset μ of X, $cl\pi(t \times \mu) = \pi(t \times cl\mu)$

(ii) Let G and X be product related, then for a fuzzy subset λ of G and a fuzzy subset μ of X, $\pi(cl\lambda \times cl\mu) \subseteq cl\pi(\lambda \times \mu)$ and $cl\pi(cl\lambda \times \mu) = cl\pi(\lambda \times cl\mu) = cl\pi(\lambda \times \mu)$.

(iii) $\pi^t\mu = \mu\pi^{-t}$ for any $t \in G$.

(iv) $\pi^t\mu^c = 1 - \pi^t\mu$.

(v) If $\mu \in I^X$ is a fuzzy open (closed), then $t\mu$ is fuzzy open(closed).

Result 3.10 Let α be a constant fuzzy subset of G and $\mu \in I^X$ be fuzzy open. Then $\pi(\alpha \times \mu)$ is fuzzy open.

Proof. We have for any $u \in X$, $\pi(\alpha \times \mu)(u) = \sup\{(\alpha \times \mu)(t, x) : \pi(t, x) = u\}$

$= \sup\{(\alpha(t) \wedge \mu(x) : \pi(t, x) = u\} = \sup\{(\alpha \wedge \mu(x) : \pi(t, x) = u\}$

$= \alpha \wedge \sup\{\mu(x) : \pi^t(x) = u\} = \alpha \wedge \sup\{\mu(\pi^{-t}(u)) : \pi^{-t}(u) = x\}$

$= \alpha \wedge \sup\{\pi^t\mu(u) : \pi^{-t}(u) = x\}$, since $\mu\pi^{-t} = \pi^t\mu$.

$= \alpha \wedge \{\vee\{\pi^t\mu(u)\}$ where $\pi^{-t}(u) = x$

$= \{\alpha \wedge \{\vee(\pi^t\mu)\}\}(u)$, where $\pi^{-t}(u) = x$

Thus $\pi(\alpha \times \mu) = \alpha \wedge \{\vee(\pi^t\mu)\}$. Now each π^t is open and μ is open so $\pi^t\mu$ is open. Also by definition of fuzzy topology α is open. Consequently $\alpha \wedge \{\vee(\pi^t\mu)\}$ is open. Hence $\pi(\alpha \times \mu)$ is open.

Corollary 3.11 Let μ be a fuzzy open subset of X. then for any fuzzy point t_α of G, $\pi(t_\alpha \times \mu)$ is fuzzy open.

Corollary 3.12 Let λ be any fuzzy subset of G and $\mu \in I^X$ be fuzzy open, then $\pi(\lambda \times \mu)$ is fuzzy open.

Corollary 3.13 Let $t \in G$ and $\mu \in I^X$ be a fuzzy nbd. of x, for some $x \in X$. Then $\pi(t_\alpha \times \mu)$ is a fuzzy nbd. of $\pi(t_\alpha \times x)$

Result 3.14 Let μ be a fuzzy closed subset of X. then for any fuzzy point t_α of G, $\pi(t_\alpha \times \mu)$ is fuzzy closed.

Proof. We have for any $u \in X$, $\pi(t_\alpha \times \mu)(u) = \sup\{(t_\alpha \times \mu)(s, x) : \pi(s, x) = u\}$

$= \sup\{t_\alpha(s) \wedge \mu(x) : \pi(s, x) = u\}$

$= \alpha \wedge \mu(x) : \pi(t, x) = u$, since $t_\alpha(s) \neq 0$ only when $s = t$.

$= \alpha \wedge \mu(x) : \pi^t(x) = u$

$= \alpha \wedge \mu(\pi^{-t}(u))$

$= \alpha \wedge \pi^t\mu(u)$, since $\mu\pi^{-t} = \pi^t\mu$.

$= (\alpha \wedge \pi^t\mu)(u)$, considering α as a constant fuzzy subset on X

Thus $\pi(\alpha \times \mu) = \alpha \wedge \pi^t\mu$. Now π^t is closed and μ is closed so $\pi^t\mu$ is closed. Also by definition of fuzzy topology α is closed. Consequently $\alpha \wedge \pi^t\mu$ is fuzzy closed. Hence $\pi(t_\alpha \times \mu)$ is closed.

Corollary 3.15 Let λ be any fuzzy subset of G and $\mu \in I^X$ be fuzzy closed. If suppλ is finite, then $\pi(\lambda\times\mu)$ is fuzzy closed.

Proof. We have $\lambda = \vee t_\alpha$, where $\alpha = \lambda(x)$. So $\pi(\lambda \times \mu) = \pi(\vee t_\alpha \times \mu) = \vee\pi(t_\alpha \times \mu)$. As already proved each $\pi(t_\alpha \times \mu)$ is closed. Also since suppλ is finite, the union is over finite number of closed fuzzy subsets. Hence $\pi(\lambda \times \mu)$ is closed.

Result 3.16 Let μ be a neighborhood of $z = \pi(t,\ x)$ in X. Then for each real number θ with $0 < \theta < \mu(z)$ there exist open neighborhoods $\mu_1,\ \mu_2$ of the points t, x respectively, such that $\pi(\mu_1 \times \mu_2) \subseteq \mu$ and $\min\{\mu_1(t),\ \mu_2(x)\} > \theta$.

Proof. Without loss of generality, we may assume that μ is open. Since the map $\pi : G \times X \to X$ is continuous, the fuzzy set $\pi^{-1}(\mu)$ is open in $G \times X$. Since $\pi^{-1}(\mu)(t,\ x) = \mu(z) > \theta$ there exist open fuzzy sets $\mu_1,\ \mu_2$ in G and X respectively with $\mu_1 \times \mu_2 \le \pi^{-1}(\mu)$ and $(\mu_1 \times \mu_2)(t,\ x) > \theta$. Clearly $\mu_1,\ \mu_2$ are open neighborhoods of $t,\ x$ respectively and $\pi(\mu_1 \times \mu_2) \subseteq \mu$.

Remark I. Let (π, G, X) be a fuzzy topological dynamical system. Then for any $\mu \in I^X$, $\pi(s,\ \pi(t,\ \mu) = \pi(s + t,\ \mu)$.

We have for s, t $\in$ G,

$\pi(s,\ \pi(t,\ \mu))(x) = \sup\{(s,\ \pi(t,\ \mu))(r,\ u) : \pi(r,\ u) = x\}$

$= \sup\{\pi(t,\ \mu)(u) : \pi(s,\ u) = x\} = \sup[\sup\{(t,\ \mu)(m,\ y) : \pi(m,\ y) = u\} : \pi(s,\ u) = x]$

$\sup[\sup\{\mu(y) : \pi(t,\ y) = u\} : \pi(s,\ u) = x]$

$= \sup[\sup\{\mu(y) : \pi(s,\ \pi(t,\ y)) = x\}]$

$= \sup[\sup\{\mu(y) : \pi(s + t,\ y) = x\}] = \pi(s + t,\ \mu)(x)$.

In the following results we will consider the topological group $(K,\ +)$(R or C) equipped with usual fuzzy topology.

Result 3.17. Let X be a fuzzy topological dynamical system, $a \in X$ and μ a neighborhood of a. Then, for each $0 < \theta < \mu(a)$ there exists an open neighborhood of zero in K such that $\alpha(0) > \theta$ and $\alpha(t) \le \mu\pi(t,\ a)$ for all $t \in K$.

Proof. We have $\pi : K \times X \to X$ is continuous. Since $\pi(0,\ a) = a$, $\pi^{-1}(\mu)$ is a neighborhood of (0, a) in $K \times X$. Hence there exist an open neighborhood α_1 of zero in K and an open neighborhood μ_1 of a in X such that $\alpha_1 \times \mu_1 \le \pi^{-1}(\mu)$ and $\min\{\alpha_1(0),\ \mu_1(a)\} > \theta$.

Choose a real number θ_1 such that $\theta < \theta_1 < \min\{\alpha_1(0),\ \mu_1(a)\}$. The set

$$V = \{t \in K : \alpha_1(t) > \theta_1\}$$

is an open subset of K for the usual topology of K. Since $0 \in V$, there exists $\varepsilon > 0$ such that

$$\{t : |t| < \varepsilon\} \subset V.$$

Let $\alpha : K \to I$ be a continuous function, $0 \le \alpha \le \theta_1,\ \alpha(0) = \theta_1,\ \alpha(t) = 0$ if $|t| \ge \varepsilon$. We will show that $\alpha(t) \le \mu(ta)$ for all $t \in K$. In fact, if $|t| \ge \varepsilon$, then $\alpha(t) = 0$.

For $|t| < \varepsilon$ we have $\alpha_1(t) > \theta_1$ and hence

$\mu(ta) = \mu(\pi(t,\ a)) = \pi^{-1}(\mu)(t,\ a) \ge \min\{\alpha_1(t),\ \mu_1(a)\} > \theta_1 \ge \alpha(t)$.

Corollary 3.18. Given a neighborhood μ of a in a fuzzy topological dynamical system X, and $0 < \theta < \mu(a)$, there exists $\varepsilon > 0$ such that $\mu(ta) > \theta$ if $|t| \le \varepsilon$.

Proof. By the preceding result, there exists an open neighborhood α of zero in $\boldsymbol{K}$ such that $\alpha(0) > \theta$ and $\alpha(t) \le \mu(tx)$ for all $t \in K$. Since the set

$$V = \{t \in \boldsymbol{K} : \alpha(t) > \theta\}$$

is open and contains 0, there exists $\varepsilon > 0$ such that $t \in V$ whenever $|t| \le \varepsilon$.

Clearly

$\mu(ta) > \theta$ if $|t| \le \varepsilon$.

Result 3.19 Let X be a fuzzy topological dynamical system and μ a neighborhood of a. Then, for each real number θ with $0 < \theta < \mu(a)$ there exist an open neighborhood ρ of $a \in X$, with $\rho \le \mu$ and $\rho(a) > \theta$, and a positive real number such that

$\pi(t,\ \rho) \le \mu$ for each $t \in \boldsymbol{K}$ with $|t| \le \varepsilon$.

Proof. Without loss of generality, we may assume that μ is open. The function $\pi : \boldsymbol{K} \times X \to X$ is continuous. Since $\pi^{-1}(\mu)(0,\ a) = \mu(a) > \theta$, there exist an open neighborhood α of zero in $\boldsymbol{K}$ and an open neighborhood μ_1 of a in X such that $\min\{\alpha(0),\ \mu_1(a)\} > \theta$ and $\alpha \times \mu_1 \le \pi^{-1}(\mu)$, Let $\theta < \theta_1 < \alpha(0)$ and set $\rho = \theta_1 \wedge \mu_1 \wedge \mu$. Then ρ is open, $\rho \subseteq \mu$ and $\rho(a) > \theta$. Since α is a lower semicontinuous function on $\boldsymbol{K}$ when $\boldsymbol{K}$ has its usual topology, there exists a positive number ε such that

$$\{t \in K : |t| \le \varepsilon\} \subset \{t : \alpha(t) > \theta_1\}$$

Now, let $|t| \le \varepsilon$. For each $x \in X$ we have

$\mu(tx)) = \mu(\pi(t,\ x)) = \pi^{-1}(\mu)(t,\ x)) \ge (\alpha \times \mu_1)(t,\ x) \ge (\alpha \times \rho)(t,\ x) = \min\{\alpha(t),\ \rho(x)\} = \rho(x)$, since $\rho(x) \le \theta_1 < \alpha(t)$. Since $\mu(\pi(t,\ x)) \ge \rho(x)$ for each $x \in X$, it follows that $\pi(t \times \rho) \le \mu$

We propose the following definition

Definition 3.20 Let $(\pi, \boldsymbol{K}, X,)$ be a fuzzy topological dynamical system. A fuzzy set μ in X is called balanced if $\pi(t,\ \mu) \le \mu$ for each $t \in \boldsymbol{K}$ with $|t| \le 1$.

Result 3.21 μ is balanced iff $\mu(tx) \ge \mu(x)$.

Proof. For any $u \in X$,

we have $(t\mu)(u) = \pi(t \times \mu)(u) = \sup\{(t \times \mu)(s,\ x) : \pi(s,\ x) = u\}$

$= \sup\{\min(t(s),\ \mu(x)) : \pi(t,\ x) = u\} = \sup\{\mu(x) : \pi(t,\ x) = u\}$—(i)

Suppose $\mu(tx) \ge \mu(x)$ for each $t \in K$ with $|t| \le 1$.

i.e., $\mu(\pi(t,\ x)) \ge \mu(x)$ for each $t \in K$ with $|t| \le 1$.

Then for any $u \in X$, $(t\mu)(u) = \sup\{\mu(x) : \pi(t,\ x) = u\}$ from (i)

$\le \sup\{\mu(\pi(t,\ x)) : \pi(t,\ x) = u\}$ (given)

$= \mu(u)$.

Hence $t\mu \le \mu$.

Conversely let μ be balanced. That is $\pi(t,\ \mu) \le \mu$, i.e, $t\mu \le \mu$ for each $t \in K$ with $|t| \le 1$.

We have $t\mu \le \mu \Rightarrow \pi(t,\ \mu) \le \mu \Rightarrow \pi(t,\ \mu)(u) \le \mu(u) \forall u \in X$

$\Rightarrow \sup\{\mu(x) : \pi(t,\ x) = u\} \le \mu(u)$ from (i)

$\Rightarrow \mu(x) \le \mu(u) \forall x : \pi(t,\ x) = u$

$\Rightarrow \mu(x) \le \mu(\pi(t,\ x))$ i.e., $\mu(x) \le \mu(tx)$.

We propose the following definition

Definition 3.22 If in definition 3.1, the fuzzy topological group is replaced by a fuzzy topological semi-group, then the system will be called a fuzzy topological semi-dynamical system.

(R, .) with usual fuzzy topology is a fuzzy topological semi group.

Result 3.23 Let (π, R, X) be a fuzzy topological semi-dynamical system and μ a neighborhood of a. Then, for each real number θ with $0 < \theta < \mu(a)$ there exist an open neighborhood ρ of $a \in X$, with $\rho \le \mu$ and $\rho(a) > \theta$, and a positive real number ε such that $\pi(t,\ \rho) \le \mu$ for each $t \in K$ with $|t| \le \varepsilon$.

Proof. Without loss of generality, we may assume that μ is open. The function $\pi : \boldsymbol{K} \times X \to X$ is continuous. Since $\pi^{-1}(\mu)(1,\ a) = \mu(a) > \theta$, there exist an open neighborhood α of 1 in $\boldsymbol{K}$ and an open neighborhood μ_1 of a in X such that $\min\{\alpha(1),\ \mu_1(a)\} > \theta$ and $\alpha \times \mu_1 \le \pi^{-1}(\mu)$. Let $\theta < \theta_1 < \alpha(1)$ and set $\rho = \theta_1 \wedge \mu_1 \wedge \mu$. Then ρ is open, $\rho \subseteq \mu$ and $\rho(a) > \theta$. Since α is a lower semicontinuous function on K where K has its usual topology, there exists a positive number ε such that

$$\{t \in K : |t - 1| \le \varepsilon\} \subset \{t : \alpha(t) > \theta_1\}$$

Now, let $|t - 1| \le \varepsilon$. For each $x \in X$ we have

$\mu(tx)) = \mu(\pi(t,\ x)) = \pi^{-1}(\mu)(t,\ x)) \ge (\alpha \times \mu_1)(t,\ x) \ge (\alpha \times \rho)(t,\ x) = \min\{\alpha(t),\ \rho(x)\} = \rho(x)$, since $\rho(x) \le \theta_1 < \alpha(t)$. Since $\mu(\pi(t,\ x)) \ge \rho(x)$ for each $x \in X$, it follows that $\pi(t \times \rho) \le \mu$

Corollary 3.24 Let (π, R, X) be a fuzzy topological semi-dynamical system. Let μ be a neighborhood of a in X. Then, there exist a balanced open neighborhood μ_1 of a such that $\mu_1(a) = \mu(a),\ \mu_1 \subseteq \mu$.

Proof. In view of the Result 3.23, for each $0 < \theta < \mu(a)$ there exist a positive real number ε_θ and an open neighborhood $\rho^{/}_\theta$ of a in X such that $\rho^{/}_\theta(a) > \theta$ and $t\rho^{/}_\theta \leq \mu$ if

$$|t| \leq \varepsilon_\theta$$

Set $\rho_\theta = \sup\{t\rho'_\theta : |t| \leq \varepsilon_\theta\}$

Clearly ρ_θ is open, $\rho_\theta \leq \mu$ and $\rho_\theta(a) > \theta$.

Also ρ_θ is balanced. We have $\rho_\theta = \sup\{t\rho'_\theta : |t| \leq \varepsilon_\theta\}$i.e., $\rho_\theta = \vee\pi(t,\ \rho'_\theta)$ where $|t| \leq \varepsilon_\theta$.

Then for $|s| \leq 1,\ s\rho_\theta = \pi(s,\ \rho_\theta) = \pi(s,\ \ \vee \pi(t,\ \rho'_\theta))$

$= \vee\pi(s,\ \pi(t,\ \rho'_\theta) = \vee\pi(st,\ \rho'_\theta)$ where $|t| \leq \varepsilon_\theta$. (using Remark I)

Since $|s| \leq 1$ and $|t| \leq \varepsilon_\theta$ so $|st| \leq \varepsilon_\theta$. Thus $\vee\{\pi(st,\ \rho'_\theta) : |t| \leq \varepsilon_\theta\} \leq \rho_\theta$ and hence $s\rho_\theta \leq \rho_\theta$.

The result now follows if we take

$$\mu_1 = \sup\{\rho_\theta : 0 < \theta < \mu(a)\}.$$

Definition 3.25 Let x be a point in a fuzzy topological space X. A family $\mathfrak{I}$ of neighborhoods of x is called a base for the system of all neighborhoods of x if for each neighborhood μ of x and each $0 < \theta < \mu(x)$ there exists $\mu_1 \in \mathfrak{I}$ with $\mu_1 \leq \mu$ and $\mu_1(x) > \theta$.

Corollary 3.26 In the fuzzy topological semi-dynamical system (π, R, X), the family of all balanced neighborhoods of x $\in$ X is a base for the system of all neighborhoods of x.

4. Mixed fuzzy topological dynamical system

In this section we will construct mixed fuzzy topological dynamical system from two given systems. First we prove a result, which characterize fuzzy topology in terms of neighbourhood systems.

Result 4.1 Let X be a non-empty set and for any $x \in X$ let N_x be the collection of fuzzy subsets of X such that the following conditions are satisfied.

(i) Each non-zero constant fuzzy set belongs to N_x and if $\mu \in N_x$, then $\mu(x) > 0$.

(ii) $\mu,\ \nu \in N_x \Rightarrow \mu \cap \nu \in N_x$

(iii) $\mu \in N_x$ and $\mu \subseteq \nu \Rightarrow \nu \in N_x$

(iv) If $\mu \in N_x$, then $\exists \nu \in N_x : \nu \subseteq \mu$ and $\nu \in N_y$ for any y with $\nu(y) > 0$.

Then the collection $\tau = \{\mu \in I^X : \mu \in N_x \quad \forall x \text{ with } \mu(x) > 0\}$ is a fuzzy topology on X where the neighbourhood system of each x coincides with N_x.

Proof. Since for no $x \in X,\ 0(x) > 0,\ 0 \in \tau$ trivially.

By property (i) all constant fuzzy set belongs to τ.

By Property (ii) τ is closed under finite intersection and by property (iii) τ is closed under arbitrary union. Thus τ is fuzzy topology on X.

Next we show that each member of N_x is a neighbourhood of x. Let $\mu \in N_x$. Then by (iv) we have $\exists \nu \in N_x : \nu \subseteq \mu$ and $\nu \in N_y$ for any y with $\nu(y) > 0$. But then by definition of τ, ν is open. Also as $\nu \in N_x$ by (i) $\nu(x) > 0$. Thus μ contains a member ν of τ with $\nu(x) > 0$. Hence μ is a nbd. of x. Conversely let μ be nbd. of x w.r.t. τ. Then μ contains a member v of τ with $\nu(x) > 0$. But then by definition of τ, $\nu \in N_x$. So by property (iii) $\mu \in N_x$.

Thus the nbds. of x are precisely the members of N_x.

The next result gives the construction of a mixed fuzzy topology.

Result 4.2 Let τ_1 and τ_2 be two fuzzy topologies on a set X such that $\tau_2 \subseteq \tau_1$ and τ_2 is fuzzy regular. For each $x \in X$, let $\mathfrak{I}_{1x}$ and $\mathfrak{I}_{2x}$ denote the system of nbd of x with respect to topologies τ_1 and τ_2, and $\mathfrak{I}_{1x}(\mathfrak{I}_{2x}) = \{\mu \in I^X : \exists\lambda \in \mathfrak{I}_{2x}$ and $cl\lambda \subseteq \mu$, closure w.r.t. $\tau_1\}$.

Then there exists a fuzzy topology $\tau_1(\tau_2)$ w.r.t. which $\mathfrak{I}_{1x}(\mathfrak{I}_{2x})$ is the nbd. system of x.

Proof. It is sufficient to show that $\mathfrak{I}_{1x}(\mathfrak{I}_{2x})$ satisfies all the conditions of Result 4.1.

Since any non-zero constant fuzzy set belongs to $\mathfrak{I}_{2x}$ and is closed w.r.t. any topology, $\mathfrak{I}_{1x}(\mathfrak{I}_{2x})$ contains all non-zero constant fuzzy sets. Also $\mu \in \mathfrak{I}_{1x}(\mathfrak{I}_{2x}) \Rightarrow \mu \in \mathfrak{I}_{2x}$ and hence $\mu(x) > 0$. Thus condition (i) is satisfied.

Let $\mu_1,\ \mu_2 \in \mathfrak{I}_{1x}(\mathfrak{I}_{2x})$. Then $\exists \nu_1,\ \nu_2 \in \mathfrak{I}_{2x}$ and $cl\nu_1 \subseteq \mu_2,\ cl\nu_2 \subseteq \mu_2$, closure being w.r.t. τ_1. Then $\nu_1 \cap \nu_2 \in \mathfrak{I}_{2x}$ and $cl(\nu_1 \cap \nu_2) \subseteq cl\nu_1 \cap cl\nu_2 \subseteq \mu_1 \cap \mu_2$. So condition (ii) is satisfied.

Let $\mu \in \mathfrak{I}_{1x}(\mathfrak{I}_{2x})$ and $\mu \subseteq \nu$. Then $\exists\lambda \in \mathfrak{I}_{2x}$ and $cl\lambda \subseteq \mu$, closure being w.r.t. τ_1. Then $cl\lambda \subseteq \mu \subseteq \nu$. So condition (iii) is satisfied.

Let $\mu \in \Im_{1x}(\Im_{2x})$. Then $\exists\lambda \in \Im_{2x}$ and $cl\lambda \subseteq \mu$, closure being w.r.t. τ_1. Now $\lambda \in \Im_{2x}$ implies $\exists\nu \in \Im_{2x} : \nu \subseteq \lambda$ and $\nu \in \Im_{2y}$ for any y with $\nu(y) > 0$.......(i)

Since τ_2 is fuzzy regular $\exists\rho \in \tau_2$ such that $cl\rho \subseteq \nu$ (closure w.r.t. τ_2). As $\tau_2 \subseteq \tau_1$, τ_1 closure of ρ is contained in τ_2 closure of ρ. Thus we have $\nu \in \Im_{2x}$ and $cl\rho \subseteq \nu$ (closure w.r.t. τ_1). Hence $\nu \in \Im_{1x}(\Im_{2x})$. Also $\nu \subseteq \lambda$ and $cl\lambda \subseteq \mu$ (closure being w.r.t. τ_1) and so $\nu \subseteq \mu$.

Let $\nu(y) > 0$ for some $y \in X$. Then $\nu \in \Im_{2y}$ by (i). But then $\nu \in \Im_1(\Im_2)y$. So condition (iv) is satisfied. Thus all the conditions of Result 4.1 are satisfied and consequently, we get a topology, say $\tau_1(\tau_2)$ w.r.t. which $\Im_{1x}(\Im_{2x})$ is the neighbourhood system of x.

Result 4.3. Let $(X,\ \tau_1)$ and $(X,\ \tau_2)$ be two fuzzy topological spaces such that $\tau_2 \subseteq \tau_1$ and τ_2 is fuzzy regular. Let $(G,\ \tau)$ be a fuzzy topological group and $\pi : G \times X \to X$ be a mapping.

If $(\pi,\ G,\ X)$ is a fuzzy topological dynamical system with respect to both τ_1 and τ_2, then $(\pi,\ G,\ X)$ is also a fuzzy topological dynamical system with respect to the mixed topology $\tau_1(\tau_2)$.

Proof. By definition we have $\tau_2 \subseteq \tau_1$ and τ_2 is fuzzy regular. We know from Result 6.2 that $(X,\ \tau_1(\tau_2))$ is a fuzzy topological space.

Let $(\alpha,\ x) \in G \times X$ be an arbitrary point and μ be a fuzzy open nbd of $\pi(\alpha,\ x)$, w.r.t $\tau_1(\tau_2)$

Then $\exists\lambda \in \Im_{2x}$ and $cl\lambda \subseteq \mu$, closure being w.r.t. to τ_1.

As $(\pi,\ G,\ X)$ is a fuzzy topological transformation group with respect to τ_2, there exists fuzzy open nbd η of α in G and fuzzy open nbd. ρ_2 of x in X with respect to τ_2 such that $\pi(\eta \times \rho_2) \subseteq \lambda$

Now $\rho_2 \in \Im_{2x}$ and τ_2 is fuzzy regular, therefore, $\exists\rho \in \tau_2$ with $\rho(x) > 0$ such that $cl\rho \subseteq \rho_2[closer\ w.r.t\tau_2]$. Since $\tau_2 \subseteq \tau_1,\ \tau_1 - closer$, of ρ is a subset of τ_2- closer of ρ.

Thus $cl\rho \subseteq \rho_2[closer\ with\ respect\ to\ \tau_1]$ and hence $\rho_2 \in \Im_1(\Im_2)x$.

Then ρ_2 is a $\tau_1(\tau_2)$ nbd of x and η is a nbd. of α such that $\pi(\eta \times \rho_2) \subseteq \lambda \subseteq \mu$.

Hence $(\pi,\ G,\ X)$ is a fuzzy topological transformation system with respect to $\tau_1(\tau_2)$.

Result 4.4. Let $(G,\ \tau_1)$ and $(G,\ \tau_2)$ be two fuzzy topological groups such that $\tau_2 \subseteq \tau_1$ and τ_2 is fuzzy regular. Let $(X,\ \tau)$ be a fuzzy topological space and $\pi : G \times X \to X$ be a mapping.

If $(\pi,\ G,\ X)$ is a fuzzy topological dynamical system with respect to both τ_1 and τ_2, then $(\pi,\ G,\ X)$ is also a fuzzy topological dynamical system with respect to the mixed topology $\tau_1(\tau_2)$.

Proof: By definition we have $\tau_2 \subseteq \tau_1$ and τ_2 is fuzzy regular. We know from $(N.R.Das,\ 2000)(G,\ \tau_1(\tau_2))$ is a fuzzy topological group.

We need to show that $\pi : G \times X \to X$ is continuous with respect to the mixed topology $\tau_1(\tau_2)$. Let $(\alpha,\ x) \in G \times X$ be an arbitrary point and μ be a fuzzy open nbd of $\pi(\alpha,\ x)$.

As $(\pi,\ G,\ X)$ is a fuzzy topological dynamical system with respect to both τ_1 and τ_2, there exists fuzzy open nbd $\lambda_1,\ \lambda_2$ of α in G with respect to $\tau_1\&\tau_2$ and fuzzy open nbd ρ of x such that $\pi(\lambda_1 \times \rho) \subseteq \mu$ and $\pi(\lambda_2 \times \rho) \subseteq \mu$.

Now $\lambda_2 \in \Im_{2x}$ and τ_2 is fuzzy regular, therefore, $\exists\lambda \in \tau_2$ such that $cl\lambda \subseteq \lambda_2[closer\ w.r.t\ \tau_2]$.

Since $\tau_2 \subseteq \tau_1,\ \tau_1 - closer$ of λ is a subset of τ_2- closer of λ.

So $cl\lambda \subseteq \lambda_2[closer\ w.r.t.\ \tau_1]$. Hence λ_2 is a $\tau_1(\tau_2)$ nbd. of α and ρ is a nbd. of x such that $\pi(\lambda_2 \times \rho) \subseteq \mu$.

Hence $(\pi,\ G,\ X)$ is a fuzzy topological dynamical system with respect to $\tau_1(\tau_2)$.

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