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# Existence of Solution to Initial-Boundary Value Problems of the Cahn-Hilliard Equation with Nonlocal Terms

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**Abstract**

In this paper, inspired from the study on denoising, segmentation and reconstruction in image processing, and combining with the theories of two phase flows, we introduce one class of initial-boundary value problem of the Cahn-Hilliard equation with nonlocal terms. Then, by using the Schauder fixed point theorem, we obtain the existence of weak solutions to this initial boundary value problem for the nonlocal Cahn-Hilliard equation.

**Keywords:** Nonlocal term, Cahn-Hilliard equation, Initial-boundary value problem

## 1. Introduction

### *1.1 Cahn-Hilliard equation*

In order to describes the complicated phase separation and coarsening phenomena in a melted alloy, Cahn J.W., Hilliard E., 1 958, P. 258-297 proposed the following initial-boundary problem:

$$\begin{cases} u_t = -M\Delta(\epsilon\Delta u - \frac{1}{\epsilon}\varphi'(u)), \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad \frac{\partial(\varphi'(u)-\epsilon^2\Delta u)}{\partial n}|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0 \end{cases} \tag{1}$$

where $\Omega \subseteq R^n$ is a domain with smooth boundary $\partial\Omega$, $\vec{n}$ is the outer normal vector, $t > 0$, $\epsilon$ serves as a measure of the transition region between two metals in an alloy. $M$ is a positive constant, and $\varphi(u)$ is the double potential well function, e.g.,

$$\varphi(u) = (u^2 - 1)^2,$$

where $u = \pm 1$ is the equivalent state.

There are a lot of research work on this topic, for example, Elliott, C.M., Zheng, S. 1986, P.339-357, have established the existence and uniqueness of the solution to this initial boundary value problem. And Pego, R.L., 1989, P.261-278 has analyzed the asymptotic property of the solution in the case of $\epsilon \to 0$, and the author proved that the limit of the solution satisfy the Hele-Shaw initial problems with free edge conditions£Then Alikakos, N.D., Bates, P.W. and Chen, X., 1994, P.165-205, Alikakos, N.D., Fusco, G., 1993, P.637-674, Chen, X., 1993, P.117-151, 1994, P.1371-1395, and Stoth, B., 1996, P.154-183. have formally investigated the mathematical property of the solution. In recent years, there are some progress in the study of dynamic behavior and long time behavior of the solution (refer to Schimperna, G., 2007, P. 2365-2387 and references therein). More recently, Qian, T., Wang, X., Sheng, P., 2003, P.1-15 propose to describe the two-phase fluid property by coupling Cahn-Hilliard equation and Navier-Stokes equation. In Betes, P.W. and Han, J., 2005, P.235-277, the author has investigated a class of long distance disturb term Cahn-Hilliard equation with nonlocal term, they proved the existence, uniqueness and the stability property of the solution by the energy methods.

### *1.2 The partial differential equations in image processing*

In recent years, with the development of partial differential equation and the information sciences, the PDES methods has been applied affectively and widely in the image processing such as denoising and reconstruction. For example, Perona, P., Malik, J., 1990, P.629-639 proposed the well-known anisotropic diffusion model. Even though this model has made great progress in the theory and improved the filtering property of the original methods£there are still some limitations

with this mothed. For example, if there exists white noise in the image, then the term $|\nabla u|$ may be very large such that the diffusion coefficient is very small, thus these noise point remains and the denoise is inefficient. In order to correct this drawback, Catte, F., Lions, P.L., Morel, J.M. 1992, P.182-193 proposed the following selective smoohing model,

$$\begin{cases} u_t = div(F(|\nabla u * G_\sigma|^2)\nabla u), \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{2}$$

where $u$ denotes the grey value of the image, the $\Omega \subseteq R^2$ is the bounded domain with smooth boundary $\partial\Omega$, $\vec{n}$ is the outer normal vector, and $F$ is a smooth non-increasing function with $F(0) = 1,\ F(x) \geq 1$, and $F(x)$ tending to zero at infinity.

For example, we can take

$$F(s) = \frac{1}{1 + \frac{s}{k^2}}$$

with $k$ is the parameter and

$$G_\sigma = \frac{1}{4\pi\sigma} e^{-\frac{|x|^2}{4\sigma^2}}$$

is the Gaussian function and $u_0$ is the original image with noise.

Recently, researchers use multi-phase movement model in fluid mechanics for reference to study the edge detection, segmentation and restoration in image processing. Bertozzi, A.L., Esedoglu, S., & Gillette,2007, P. 285-291 propose the following initial boundary value problems based on Cahn-Hilliard equation,

$$\begin{cases} u_t = -\Delta(\epsilon\Delta u - \frac{1}{\epsilon}\varphi'(u)) + \lambda\chi_{\Omega\setminus D}(f - u), \\ \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \quad \frac{\partial\nabla u}{\partial n}|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{3}$$

and use this model to do image restoration and edge detection. The use of the model for inpainting based on the Cahn-Hilliard equation ,which allowed for fast,efficient inpaining. The existence and uniqueness of a solution to the initial boundary value problem(3) is investigated, however, just like the Gauss denoise used by Witkin, A.P., 1983, P.1019-1021. the edge is smoothed in the time of denoise such that the image loose the fidelity.

*1.3 The initial-boundary value problem of Cahn-Hilliard equation*

Inspired from the Catte, F., Lions, P.L., Morel, J.M., 1992, P.182-193, and Bertozzi, A.L., Esedoglu, S., & Gillette, 2007,P. 285-291, we propose the following initial boundary equation:

$$\begin{cases} u_t = -\Delta\left(\epsilon div(F(|\nabla u * G_\sigma|^2)\nabla u) - \frac{1}{\epsilon}\varphi'(u)\right) + \lambda \cdot \chi_{\Omega|D}(f - u), \\ u|_{\partial\Omega} = 0, \quad \frac{\partial u}{\partial n}|_{\partial\Omega} = 0, \\ u|_{t=0} = u_0 \end{cases} \tag{4}$$

$\Omega \subseteq R^2$ is the domain with smooth boundary $\partial\Omega$. $D \subseteq \Omega$ is the inpainting domain, and $\vec{n}$ is outer normal vector. Our motivation of choosing the right hand side is as follows: The first term: $F$ and $G_\sigma$ are the same with the corresponding function used in (2). The introduction of $G_\sigma$ is to denoise the image. The chosen of $F$ is to produce the anistropic denoise effect. Just as Catte, F., Lions, P.L., Morel, J.M., 1992, P.182-193 pointed out, this kind of model can well preserve the edge of the image.

The second term: $\varphi(u)$ is a double potential well function, which is the same with its role in (1). For example, we can choose $\varphi(u) = (u^2 - 1)^2$. For the convenience of the following discussion, we suppose $\varphi(u)$ satisfies

$$|\varphi'(u)| \leq C_1|u| + C_2|u|^3, \tag{5}$$

with $C_1$ and $C_2$ are two positive constant.

The third term: We try to preserve the original image in the domain $\Omega\setminus D$, $\lambda > 0$ is the parameter.

The main difference between our equation and the classical Cahn-Hilliard equation is that there is a nonlocal term in main term of the equation,

$$div(F(|\nabla u * G_\sigma|^2)\nabla u).$$

**2. Existence of weak solutions of the Cahn-Hilliard equation with nonlocal terms**

In the following, We shall discuss the existence of a weak solution of (4) by a classical fixed point theorem of Schauder.

**Theorem 1.** For any $\epsilon > 0$, $u_0 \in L^2(\Omega)$, $f \in L^2(0, T_0; L^2(\Omega))$, then there exists $0 \le T \le T_0$ such that the initial-boundary value problem (4) has a solution in the bounded function space, i.e.,

$$u \in L^2(0, T; H_0^2(\Omega)) \cap L^\infty(0, T; L^2(\Omega)) \cap H^1(0, T; H_0^2(\Omega)').$$

**Proof.** Firstly, let us introduce the space

$$W(0, T) = \{w \in L^2(0, T; H_0^2(\Omega)) \cap L^\infty(0, T) \times \Omega), \partial t_w \in L^2(0, T; H_0^2(\Omega)')\}$$

for any fixed $\forall w \in W(0, T)$, and let us define the weak form: i.e., search $u \in W(0, T)$ satisfy

$$(E_w) : \begin{cases} \left(\frac{\partial u}{\partial t}, \nu\right) = -\epsilon(div(F(|\nabla w * G_\sigma|^2)\nabla u),\ \triangle\nu) + \frac{1}{\epsilon}(\varphi'(w), \triangle\nu) + (\lambda \cdot \chi_{\Omega|D}(f - u),\ \nu) \\ u|_{t=0} = u_0, \end{cases}$$

for any $v \in H_0^2(\Omega)$, which is now linear in $w$. Here $\langle ., . \rangle$ specifies the $L^2$ inner product.

In the above problem, taking $v = u$, then we have

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2_{L^2(\Omega)} = -\int_\Omega \epsilon(div(F(|\nabla u * G_\sigma|^2)\nabla u)\Delta u dxdy + \frac{1}{\epsilon}\varphi'(w)\Delta u dxdy + \lambda\chi_{\Omega/D}\int_\Omega (f - u)u dxdy. \tag{6}$$

From the first term in (7), we have

$$-\int_\Omega (\epsilon div(F(|\nabla w * G_\sigma|^2)\nabla u)\Delta u) = -\epsilon\int_\Omega F(|\nabla u * G_\sigma|^2)|\Delta u|^2 - \epsilon\int_\Omega \nabla F(|\nabla u * G_\sigma|^2)\nabla u\Delta u,$$

Putting the last two equalities together, we have

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2_{L^2(\Omega)} = -\epsilon\int_\Omega F(|\nabla u * G_\sigma|^2)|\Delta u|^2 - \epsilon\int_\Omega \nabla F(|\nabla u * G_\sigma|^2)\nabla u\Delta u + \frac{1}{\epsilon}\varphi'(w)\Delta u dxdy + \lambda\chi_{\Omega/D}\int_\Omega (f - u)u dxdy, \tag{7}$$

First, writing the second term above as

$$\begin{array}{rcl} -\epsilon\int_\Omega \nabla F(|\nabla u * G_\sigma|^2)\nabla u\Delta u & \le & C\epsilon\int_\Omega \nabla(|\nabla u * G_\sigma|^2)\nabla u\Delta u \\ & \le & C\sigma_1\epsilon\int_\Omega |\Delta u|^2 + \frac{\epsilon}{C}||H||^2_{L^\infty(\Omega)}\int_\Omega |\nabla u|^2 \\ & \le & C\sigma_1\epsilon\int_\Omega |\Delta u|^2 + \frac{\epsilon}{C_3\sigma_1}\int_\Omega |\nabla u|^2 \end{array}$$

where $H = |\nabla w * G_\sigma|^2$. Then based on the interpolation inequality, we have

$$-\epsilon\int_\Omega \nabla F(|\nabla u * G_\sigma|^2)\nabla u\Delta u \le C\sigma_1\epsilon\int_\Omega |\Delta u|^2 + \frac{\epsilon\eta}{B\sigma_1}\int_\Omega |\Delta u|^2 + \frac{\epsilon}{B\sigma_1\eta}\int_\Omega |u|^2.$$

since

$$|\nabla w * G_\sigma|_{L^\infty(\Omega)} \le ||G_\sigma||_{L^\infty(\Omega)}|w|_{L^1(\Omega)}.$$

and, there exists a constant $\gamma > 0$ such that

$$F(|\nabla w * G_\sigma|^2) \ge \gamma > 0.$$

Submitting the above term into (8), we have

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2_{L^2(\Omega)} + \epsilon\nu||\Delta u||^2_{L^2(\Omega)} \le C\sigma_1\epsilon||\Delta u||^2_{L^2(\Omega)} + \frac{\epsilon\eta}{B\sigma_1}||\Delta u||^2_{L^2(\Omega)} + \frac{\epsilon}{B\sigma_1\eta}||\Delta u||^2_{L^2(\Omega)} + \frac{1}{\epsilon}\varphi'(w)\Delta u dxdy + \lambda\chi_{\Omega/D}\int_\Omega (f - u)u dxdy. \tag{8}$$

By the assumption made on $\varphi'(w) \le C_1|w| + C_3|w|^3$, submitting it into (9), then we have

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2_{L^2(\Omega)} + \epsilon\nu||\Delta u||^2_{L^2(\Omega)} \le (C\sigma_1\epsilon + \frac{\epsilon\eta}{B\sigma_1} + \frac{C_2\sigma_2}{2\epsilon})||\Delta u||^2_{L^2(\Omega)} + \frac{\epsilon||u||^2_{L^2(\Omega)}}{B\sigma_1\eta} + \frac{C_1||w||^2_{L^2(\Omega)}}{2\epsilon} + \frac{C_2||w||^6_{L^2(\Omega)}}{2\epsilon\sigma_2} + \frac{\lambda\chi_{\Omega/D}}{2}\int_\Omega (f - u)u dxdy. \tag{9}$$

Also, we have

$$\lambda\chi_{\Omega/D}\int_\Omega (f - u)u dx \le \frac{\lambda\chi_{\Omega/D}}{2}\int_\Omega f^2 - \frac{\lambda\chi_{\Omega/D}}{2}\int_\Omega u^2,$$

Hence

$$\frac{1}{2}\frac{d}{dt}||u(t)||^2_{L^2(\Omega)} + \epsilon\nu||\Delta u||^2_{L^2(\Omega)} \le (C\sigma_1\epsilon + \frac{\epsilon\eta}{B\sigma_1} + \frac{C_2\sigma_2}{2\epsilon})||\Delta u||^2_{L^2(\Omega)} + \frac{\epsilon||u||^2_{L^2(\Omega)}}{B\sigma_1\eta} + \frac{C_1||w||^2_{L^2(\Omega)}}{2\epsilon} + \frac{C_2||w||^6_{L^2(\Omega)}}{2\epsilon\sigma_2} + \frac{\lambda\chi_{\Omega/D}}{2}(||f||^2_{L^2(\Omega)} - ||u||^2_{L^2(\Omega)}). \tag{10}$$

We now try to satisfy the following conditions with our choice of the constants,

(1). $C\epsilon\sigma_1 < \frac{\epsilon v}{4}$, *i.e.* $\sigma_1 < \frac{v}{4C}$

(2). $\frac{\epsilon\eta}{B\sigma_1} < \frac{\epsilon v}{4}$, *i.e.* $\eta < \frac{Bv\sigma_1}{4}$

(3). $\frac{C_2\sigma_2}{2\epsilon} < \frac{\epsilon v}{4}$, *i.e.* $\sigma_2 < \frac{v\epsilon^2}{2C_2}$,

To satisfy the first condition, take $\sigma_1 = \frac{v}{8C}$. Then, to satisfy the second, we can choose $\eta = \frac{v\sigma_1 B}{4}$. To satisfy the third, choose $\sigma_2 = \frac{v\epsilon^2}{4C_2}$. With these choices, we end up with the following inequality:

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{L^2(\Omega)} + \|\Delta u\|^2_{L^2(\Omega)} \le C_3\|f\|^2_{L^2(\Omega)} + C_4\|u\|^2_{L^2(\Omega)} + C_5(\|w\|^2_{L^2(\Omega)} + \|w\|^6_{L^2(\Omega)}), \tag{11}$$

with $C_3$, $C_4$ and $C_5$ are constants.

As $\Omega \in R^2$ is bounded, by Sobolev embedding theorem, we have $\forall\xi > 0$, there exists $C_\xi > 0$ such that

$$\|w(t,\cdot)\|^6_{L^6(\Omega)} \le C_\xi\|w(t,\cdot)\|^6_{L^2(\Omega)} + \xi\|w(t,\cdot)\|^6_{H^{\frac{3}{4}}(\Omega)}, \tag{12}$$

By submitting the above result into (12) and using Grönwall inequality, we have

$$\sup_{0\ge t\le T}\|u(t)\|^2_{L^2\Omega} + \|\Delta u\|^2_{L^2(0,T;L^2\Omega)} \le C(T)(\|u_0\|^2_{L^2(\Omega)} + \|f\|^2_{L^2(0,T;L^2\Omega)} + \|w\|^2_{L^2(0,T;L^2\Omega)} + C_\xi\|w\|^6_{L^6(0,T;L^2\Omega)} + \xi\|w\|_{L^6(0,T;H^{\frac{3}{4}}(\Omega))}), \tag{13}$$

Then from the interpolation inequality Wang. X.-P., Wang. Y.-G, 2007 P.18, we have

$$\int_0^T \|w(t,\cdot)\|^6_{H^{\frac{3}{4}}(\Omega)}\,dt \le C_0\left(\int_0^T \|w(t,\cdot)\|^{18}_{L^2(\Omega)}\,dt\right)^{\frac{1}{12}}\left(\int_0^T \|w(t,\cdot)\|^2_{H^1(\Omega)}\,dt\right)^{\frac{9}{4}}$$

Submitting the above equation into (14), for properly chosen $\xi > 0$, $T \in (0, T_0]$ sufficiently small, such that when $w \in W(0,T)$, and

$$\|w\|_{L^\infty(0,T;L^2(\Omega))} + \|w\|_{L^2(0,T;H^2_0(\Omega))} \le C(T)(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))})$$

then we have

$$\|u\|_{L^\infty(0,T;L^2(\Omega))} + \|u\|_{L^2(0,T;H^2_0(\Omega))} \le C(T)(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))})$$

where $C(T)$ is a positive constant that only depends on $T$, $F$, and $\varphi$. From the above equation, then we have

$$\partial_t u \in L^2\left(0,T;\left(H^2_0(\Omega)\right)'\right)$$

From the above estimates we introduce the subspace $W_0$ of $W$ defined by

$$W_0 = \{w \in W(0,T)|\|w\|_{L^\infty(0,T;L^2(\Omega))} + \|w\|_{L^2(0,T;H^2_0(\Omega))})\} \le C(T)(\|u_0\|_{L^2(\Omega)} + \|f\|_{L^2(0,T;L^2(\Omega))})$$

From the above analysis, $U$ is a mapping from $W_0$ into $W_0$. In order to use the Schauder theorem, we need to prove that the mapping $u$ is weakly continuous from $W_0$ into $W_0$.

Let $\{w_i\}$ be a sequence in $W_0$ which converges weakly to some $w$ in $W_0$ and $u_i = U(w_i)$

From the above estimates, the sequence $\{u_i\}$ of $W_0$ contains asubsequence $\{u_i\}$ such that

$$u_i \to u \text{ weakly in } L^2(0,T;H^2_0(\Omega))$$

$$\frac{\partial}{\partial t}u_i \to \frac{\partial}{\partial t}u \text{ weakly in } L^2\left(0,\ T;\ (H^2_0(\Omega))'\right)$$

In $H^1(\Omega)'$ we have $u_i(0) \to u_0$ $i \to \infty$, therefore, we have

$$u_i \to u \text{ in } L^2(0,\ T;\ L^2(\Omega))$$

$$w_i \to w \text{ in } L^2(0,\ T;\ L^2(\Omega))$$

$$F(|\nabla w_i * G_\sigma|^2) \to F(|\nabla w * G_\sigma|^2) \text{ in } L^2(0,T;L^2(\Omega)$$

$$\varphi(w_i)' \to \varphi(w)' \text{ weakly in } L^2(0,T;L^2(\Omega))$$

Then we can pass to the limit in the relation $(E_{w_i})$, which yields $u = U(w)$ and it is weakly continuous. From Schauder theorem there exists a fixed point $u \in W_0(0,T)$ for mapping $u = U(w)$. Therefore, the proposed bounded function class has a solution

$$u \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; H_0^2(\Omega)).$$

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