Journal of Mathematics Research
http: www.ccsenet.org/jmr
Vol. 1, No. 2
September 2009

# Characteristic Ratio and Its Properties

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**Abstract**

Similar to the definition of cross ratio in high geometry, we propose a new definition of characteristic ratio. This paper mainly discuss some properties of characteristic ratio. Moreover, we find this definition plays an important role in researching the intrinsic of algebraic curve in projective plane.

**Keywords:** Characteristic ratio, Projective plane, Projective coordinate

## 1. Introduction

As well known, cross ratio is the basic invariant in projective geometry, which is defined by collinear four points. It can explain the projective mapping of two planes. In paper [Luo, 2008], the authors find an equivalence relation between the study of the intrinsic properties of plane algebraic curves and the study of singularity of bivariate spline spaces when studying computational geometry. From that, they proposed the definitions of characteristic ratio and characteristic mapping.

First, we list the definition of characteristic ratio and characteristic mapping. In what follows, we have used $u =< a, b >$ for the intersection point of lines $a$ and $b$, $a = (u, v)$ for the line determined by the points $u$ and $v$.

**Definition 1**(Characteristic Ratio) Let $u, v \in P^2$ be two points(or lines), $l_1, l_2, \cdots, l_k$ be distinct points(or lines) on the line determined by $u, v$ and $l_i = a_i u + b_i v, i = 1, 2, \cdots, k$. The ratio

$$[u, v; l_1, l_2, \cdots, l_k] = \frac{b_1 b_2 \cdots b_k}{a_1 a_2 \cdots a_k}$$

is called characteristic ratio of $l_1, l_2, \cdots .l_k$ to the basic points (or basic lines )$u, v$.

Especially, in projective plane, suppose $P_1, P_2, P_3, P_4$ are four different points in a line, and $P_3 = \lambda_1 P_1 + \lambda_2 P_2, P_4 = \mu_1 P_1 + \mu_2 P_2$. Denoted

$$[P_1, P_2; P_3, P_4] = \frac{\mu_2 \lambda_2}{\mu_1 \lambda_1},$$

then $[P_1, P_2; P_3, P_4]$ is the characteristic ratio of basic points $P_1, P_2$.

**Theorem 2** In projective plane, suppose $P_1, P_2, P_3, P_4$ are four different points, and $P_i = P + \lambda_i Q (i = 1, 2, 3, 4)$, where $P, Q$ are two defferent points in this line, then

$$[P_1, P_2; P_3, P_4] = \frac{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)}.$$

**Proof**: Let $P + \lambda_3 Q, P + \lambda_4 Q$ indicate the linear combination of $P + \lambda_1 Q, P + \lambda_2 Q$. The result can be proven from the definition of characteristic ratio.

In Theorem 2, the characteristic ratio of points $P_1, P_2, P_3, P_4$ has nothing to the selection of $P, Q$. In fact, if choose points $P'$ and $Q'$ in this line and suppose $P_i = P' + \lambda'_i Q' (i = 1, 2, 3, 4)$, then we can prove

$$\frac{(\lambda_1 - \lambda_3)(\lambda_1 - \lambda_4)}{(\lambda_2 - \lambda_3)(\lambda_2 - \lambda_4)} = \frac{(\lambda'_1 - \lambda'_3)(\lambda'_1 - \lambda'_4)}{(\lambda'_2 - \lambda'_3)(\lambda'_2 - \lambda'_4)}.$$

**Definition 3** (Characteristic mapping)

Let $P_1, P_2, P_3, P_4 \in P^2$ be collinear points, the mapping $\chi_{(P_1,P_2)} : P_3 \longmapsto P_4$ is called a characteristic mapping if $[P_1, P_2; P_3, P_4] = 1$ holds, and the characteristic is denoted by $P_4 = \chi_{(P_1,P_2)}(P_3)$.

It can be seen that if $P_3$ is the characteristic mapping of $P_4$, then $P_4$ is is the characteristic mapping of $P_3$ as well. That is, the characteristic mapping is reflexive mapping. i.e. $\chi_{(P_1,P_2)} \cdot \chi_{(P_1,P_2)} = I$(Identity mapping).

**2. Properties of characteristic ratio**

**Theorem 4** In projective plane, the characteristic ratio of four different points in a line remain unchanged under center projective.

**Proof** In projective plane, suppose $A, B, C, D$ in line $l$ turn into $A', B', C', D'$ in line $l'$ under center projective $S$ respectively(See Figure 1). Without loss generality, we regard coordinate system $[A', B', A; B]$, and let $A'(1,0,0)$, $B'(0,1,0)$,$A(0,0,1)$, $B(1,1$ We have $l_{AA'} = (0,1,0)$, $l_{BB'} = (1,0,-1)$, $S = AA' \times BB' = (1,0,1)$ , where $AA' \times BB'$ denotes the points of intersection of the lines $AA'$ and $BB'$.

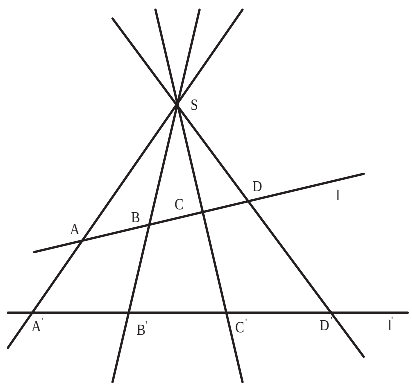


Figure 1 Center Projective

Suppose

$$C = A + \lambda B = \begin{pmatrix}0\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}\lambda\\\lambda\\1+\lambda\end{pmatrix},$$
$$D = A + \mu B = \begin{pmatrix}0\\0\\1\end{pmatrix} + \mu\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}\mu\\\mu\\1+\mu\end{pmatrix}, \tag{1}$$

then $[A, B; C, D] = \lambda\mu$.

Further, suppose

$$C' = A' + \lambda' B' = \begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda'\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\\lambda'\\0\end{pmatrix},$$
$$D' = A' + \mu' B' = \begin{pmatrix}1\\\mu'\\0,\end{pmatrix}, \tag{2}$$

Since

$$C' = C + kS = \begin{pmatrix}\lambda\\\lambda\\1+\lambda\end{pmatrix} + k\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}\lambda+k\\\lambda\\1+\lambda+k\end{pmatrix}. \tag{3}$$

The direct result by comparing (2) and (3) is

$$\begin{cases}\lambda + k = \rho,\\ \lambda = \rho\lambda' (\rho \neq 0), \\ 1 + \lambda + k = 0\end{cases},$$

So $\lambda' = -\lambda$. Similarly $\mu' = -\mu$.

Therefore $[A, B; C, D] = \lambda\mu = (-\lambda')(-\mu') = [A', B'; C', D']$.

**Theorem 5** In projective plane, suppose $P_1, P_2, P_3, P_4$ are four different points in a line, then the properties of their characteristic ratio are as follows:

(1) Exchange two basic points, the characteristic ratio is reciprocal of the original one, i.e. $[P_2, P_1; P_3, P_4] = \frac{1}{[P_1,P_2;P_3,P_4]}$;

(2) Exchange two distinguished points, the characteristic ratio is unchanged, i.e. $[P_1, P_2; P_4, P_3] = [P_1, P_2; P_3, P_4]$;

(3) Exchange two basic points and two distinguished points simultaneously, the characteristic ratio is reciprocal of the original one, i.e. $[P_2, P_1; P_4, P_3] = \frac{1}{[P_1,P_2;P_3,P_4]}$.

**Proof:**

(1) Suppose $P_3 = \lambda_1 P_1 + \lambda_2 P_2, P_4 = \mu_1 P_1 + \mu_2 P_2$,

then

$$[P_1, P_2; P_3, P_4] = \frac{\lambda_2\mu_2}{\lambda_1\mu_1} = \alpha, [P_2, P_1; P_3, P_4] = \frac{\lambda_1\mu_1}{\lambda_2\mu_2} = \frac{1}{\alpha}.$$

(2) Suppose $P_3 = \lambda_1 P_1 + \lambda_2 P_2, P_4 = \mu_1 P_1 + \mu_2 P_2$,

then

$$[P_1, P_2; P_3, P_4] = \frac{\lambda_2\mu_2}{\lambda_1\mu_1} = \alpha, [P_2, P_1; P_3, P_4] = \frac{\lambda_2\mu_2}{\lambda_1\mu_1} = \alpha$$

(3) From (1) and (2), it can be easily proven.

**Theorem 6** Suppose three points of the four different points in a line and their characteristic ratio are known, the fourth point can be determined.

**3. Application of Characteristic Ratio in Algebraic Curves**

The following two theorem explain the importance of characteristic ratio in algebraic curve.

**Theorem 7** Suppose $P, Q, R \in \mathbf{P}^2$ be three different points, if $P, Q, R$ are collinear, then

$$[P, Q; R] \cdot [Q, R; P] \cdot [R, P; Q] = 1.$$

**Proof:** Since $P, Q, R$ are collinear, without loss of generality, suppose $P = aQ + bR$, So

$$Q = -\frac{b}{a}R + \frac{1}{a}P, R = \frac{1}{b}P - \frac{a}{b}Q.$$

and

$$[P, Q; R] \cdot [Q, R; P] \cdot [R, P; Q] = \frac{-\frac{a}{b}}{\frac{1}{b}} \cdot \frac{b}{a} \cdot \frac{\frac{1}{a}}{-\frac{b}{a}} = 1.$$

From Definition 3 and Theorem 7, we have

**Corollary 8** Soppose $P, Q, R \in \mathbf{P}^2$ are three different points, if $P, Q, R$ are collinear, then

$$[P, Q; \chi_{(P,Q)}(R)] \cdot [Q, R; \chi_{(Q,R)}(P)] \cdot [R, P; \chi_{(R,P)}(Q)] = 1$$

**Theorem 9** Let $P_1, P_2, \cdots, P_6 \in \mathbf{P}^2$ be any distinct points and $a = (P_1, P_2), b = (P_3, P_4), c = (P_5, P_6)$. Denoted by $u =< a, c >, v =< b, c >, w =< a, b >$, then $P_1, P_2, \cdots, P_6$ lie on a conic curve if and only if

$$[u, w; P_1, P_2] \cdot [w, v; P_3, P_4] \cdot [v, u; P_5, P_6] = 1.$$

**Proof:** Without loss of generality, suppose that $u = (1, 0, 0), v = (0, 1, 0), w = (0, 0, 1)$, and then

$$a = w \times u = (0, 1, 0), b = v \times w = (0, 0, 1), c = u \times v = (1, 0, 0)$$

Let

$$P_1 = (a_1, 0, b_1), P_2 = (a_2, 0, b_2), P_3 = (0, a_3, b_3),$$
$$P_4 = (0, a_4, b_4), P_5 = (a_5, b_5, 0), P_6 = (a_6, b_6, 0).$$

By Pascal Theorem, the equivalent condition that $P_1, P_2, \cdots, P_6$ lie on a conic is that

$$(< (P_1, P_5), (P_2, P_6) >, < (P_3, P_6), (P_1, P_4) >, < (P_2, P_4), (P_3, P_5) >) = 0.$$

By computation,

$$\begin{aligned}&(<(P_1,P_5),(P_2,P_6)>,<(P_3,P_6),(P_1,P_4)>,<(P_2,P_4),(P_3,P_5)>)\\&=(a_6b_5-a_5b_6)(a_1b_2-a_2b_1)(a_3b_4-a_4b_3)(b_1b_2b_3b_4b_5b_6-a_1a_2a_3a_4a_5a_6).\end{aligned} \tag{4}$$

Since $P_1, P_2, P_3, P_4, P_5, P_6$ is different,

$$(a_6b_5-a_5b_6)(a_1b_2-a_2b_1)(a_3b_4-a_4b_3)\neq 0.$$

(4) is equivalent that

$$\frac{b_1b_2b_3b_4b_5b_6}{a_1a_2a_3a_4a_5a_6}=1. \tag{5}$$

Also

$$[u,w;P_1,P_2]\cdot[w,v;P_3,P_4]\cdot[v,u;P_5,P_6]=\frac{b_1b_2b_3b_4b_5b_6}{a_1a_2a_3a_4a_5a_6}.$$

From (5), this theorem is proven.

It is easily to verify from the above definition 3 and Theorem 9 that

**Corollary 10** Let $P_1, P_2, \cdots, P_6 \in \mathbf{P}^2$ be any distinct points and $a=(P_1,P_2), b=(P_3,P_4), c=(P_5,P_6)$. Denoted by $u=<a,c>, v=<b,c>, w=<a,b>$, then $P_1, P_2, \cdots, P_6$ lie on a conic curve if and only if

$$[u,w;\chi_{(u,w)}(P_1),\chi_{(u,w)}(P_2)]\cdot[w,v;\chi_{(w,v)}(P_3),\chi_{(w,v)}(P_4)]\cdot[v,u;\chi_{(v,u)}(P_5),\chi_{(v,u)}(P_6)]=1.$$

**Corollary 11** Let $P_1, P_2, \cdots, P_6 \in \mathbf{P}^2$ be any distinct points and $a=(P_1,P_2), b=(P_3,P_4), c=(P_5,P_6)$. Denoted by $u=<a,c>, v=<b,c>, w=<a,b>$, then $\chi_{(u,w)}(P_1), \chi_{(u,w)}(P_2), \chi_{(w,v)}(P_3), \chi_{(w,v)}(P_4), \chi_{(v,u)}(P_5), \chi_{(v,u)}(P_6)$ lie on a conic curve if and only if

$$[u,w;P_1,P_2]\cdot[w,v;P_3,P_4]\cdot[v,u;P_5,P_6]=1.$$

## 4. Conclusion

In this paper, we discuss the properties of characteristic ratio and find that the characteristic ratio plays an important role in intrinsic property of algebraic curve. Those conclusion can help us to study the corresponding relationship about point and line in projective plane. Also, the intrinsic property of algebraic curve. Moreover, we should further study the property of characteristic mapping, which can help us to research computational mathematics and high geometry.

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