Simple Γ-Seminearrings

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Abstract

The aim of this paper is to introduce Γ-seminearrings and (left/right) simple Γ-seminearrings. Moreover, some related properties of those are investigated. Finally, necessary and sufficient conditions for being (left/right) simple Γ-seminearrings are provided.

Keywords: Γ-seminearring, simple Γ-seminearring

1. Introduction

The concept of seminearrings (or nearsemirings in another terminology) was introduced by B. V. Rootselaar in 1962. It is known that seminearrings are common generalization of nearrings and semirings. In fact, seminearrings appear in a natural way in computer science. Some papers concerning theory of seminearrings are as follows: W. G. V. Hoorn (1970), H. J. Weinert (1976) and, recently, M. Shabir and I. Ahmed (2007). The notion of Γ-rings were first introduced by N. Nobusawa (Nobusawa, N., 1964, p. 81-89) as a generalization of rings. The followings are examples of papers regarding theory of Γ-rings: W. E. Barnes (1966), S. Kyuno (1978) and S. M. Hong and Y. B. Jun (1995). It is found that Bh. Satyanarayana (Satyanarayana, Bh., 1984) defined and studied Γ-nearrings. Unsurprisingly, Γ-nearrings are common generalization of nearrings and Γ-rings. Many results in Γ-nearrings have been developed:– see G. L. Booth (1988), Y. B. Jun, M. Sapanci and M. A. Öztürk (1998) and, lately, Bh. Satyanarayana and K. P. Syam (2005). M. K. Rao (Rao, M. K., 1995, p. 49-54) set up Γ-semirings as generalization of semirings and Γ-rings. Further research involving Γ-semirings are found in T. K. Dutta and S. K. Sardar (2002) and R. Chinram (2008).

In this paper, we first define Γ-seminearrings which are generalization of seminearrings, Γ-nearrings and Γ-semirings and then Γ-rings. Next, we investigate some general properties of Γ-seminearrings, especially, (left/right) simple Γ-seminearrings. At the end, necessary and sufficient conditions for being (left/right) simple Γ-seminearrings are provided.

2. Preliminaries

Definition 1. Let R be an additive semigroup and Γ a nonempty set. Then R is called a **(right) Γ-seminearring** if there exists a mapping $R \times \Gamma \times R \to R$ (sending $(a, \alpha, b) \mapsto a\alpha b$) satisfying the following conditions:

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$

(ii) $(a\alpha b)\beta c = a\alpha(b\beta c)$

for all $a, b, c \in R$ and $\alpha, \beta \in \Gamma$. Conveniently, the condition (i) is called the **right distributivity** and the condition (ii) the **associativity**.

Let R be a Γ-seminearring. If the additive semigroup R has the additive identity, then R is called a **Γ-seminearring with identity**. In this paper, 0 denotes the additive identity of the semigroup R.

We can see from Definition 1 that a right Γ-seminearring satisfies the right distributive law while the left distributive law is not necessary.

Remark 2. Let R be a Γ-seminearring. Then R is also a semigroup. Thus, whenever we would like to emphasize that

R is considered as a semigroup, it will be stated so. If *A* is a subsemigroup of the semigroup *R*, then we say that *A* is **a subsemigroup of** *R*.

It is obvious that for a seminearring $(R, +, \cdot)$, the semigroup *R* under the addition $+$ is an *R*-seminearring under the mapping $R \times R \times R \to R$ defined by $(a, \alpha, b) \mapsto a \cdot \alpha \cdot b$ for all $a, b, \alpha \in R$. Moreover, any Γ-nearrings and Γ-semirings are Γ-seminearrings directly from the definitions. Furthermore, Γ-semirings are generalization of Γ-rings (Dutta, T. K. & Sardar, S. K., 2002, p. 203-213). Consequently, Γ-seminearrings are generalization of seminearrings, Γ-nearrings and Γ-semirings and then of Γ-rings.

Definition 3. Let *R* be a Γ-seminearring under the mapping $f : R \times \Gamma \times R \to R$. A subsemigroup *A* of *R* is called a **sub Γ-seminearring** of *R* if *A* is a Γ-seminearring under the restriction of f to $A \times \Gamma \times A$.

Proposition 4. Let *A* be a nonempty subset of a Γ-seminearring *R*. Then *A* is a sub Γ-seminearring of *R* if and only if $a + b \in A$ and $a\alpha b \in A$ for all $a, b \in A$ and $\alpha \in \Gamma$.

Proof. It is enough to show only the reverse direction. Assume that $a + b \in A$ and $a\alpha b \in A$ for all $a, b \in A$ and $\alpha \in \Gamma$. Then *A* ia an additive subsemigroup of *R* and $A \times \Gamma \times A \subseteq A$. Since $A \subseteq R$, it follows that $(a + b)\alpha c = a\alpha c + b\alpha c$ and $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a, b \in A$ and $\alpha, \beta \in \Gamma$. Therefore, *A* is a sub Γ-seminearring of *R*.

Let *R* be a Γ-seminearring. If *A* and *B* are nonempty subsets of *R*, we denote by $A\Gamma B$ the subset of *R* consisting of all finite sums of the form $\sum a_i \alpha_i b_i$ where $a_i \in A$, $b_i \in B$ and $\alpha_i \in \Gamma$, i.e.,

$$A\Gamma B = \left\{ \sum_{i=1}^{m} a_i \alpha_i b_i \ \middle| \ m \in \mathbb{N}, a_i \in A, b_i \in B, \alpha_i \in \Gamma \text{ for all } i \right\};$$

moreover, for each $x \in R$, we write $x\Gamma B$ and $A\Gamma x$ instead of $\{x\}\Gamma B$ and $A\Gamma\{x\}$, respectively. Similarly, we write $A\alpha B$ instead of $A\{\alpha\}B$ for each $\alpha \in \Gamma$.

In particular, by the right distributivity, $R\alpha x = \{r\alpha x \mid r \in R\}$ for all $x \in R$ and $\alpha \in \Gamma$. However, it is not true in general that $x\alpha R = \{x\alpha r \mid r \in R\}$ where $x \in R$ and $\alpha \in \Gamma$.

Proposition 5. Let *A* and *B* be nonempty subsets of a Γ-seminearring *R*. Then $A\Gamma B$ is a subsemigroup of *R*.

Proof. Since *A*, Γ and *B* are nonempty sets, $A\Gamma B \neq \emptyset$. For each $x = \sum_{i=1}^{m} r_i \alpha_i s_i$, $y = \sum_{j=1}^{n} u_j \beta_j v_j \in A\Gamma B$, we see that $x + y = \sum_{i=1}^{m} r_i \alpha_i s_i + \sum_{j=1}^{n} u_j \beta_j v_j \in A\Gamma B$ which implies that $x + y \in A\Gamma B$. Therefore, $A\Gamma B$ is a subsemigroup of *R*.

Theorem 6. Let *R* be a Γ-seminearring. If *A*, *B* and *C* are nonempty subsets of *R*, then $(A\Gamma B)\Gamma C \subseteq A\Gamma(B\Gamma C)$.

Proof. Let $x \in (A\Gamma B)\Gamma C$. Then $x = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_j \beta_j b_j \right) \alpha_i c_i$ where $m, n \in \mathbb{N}$, $a_j \in A$, $b_j \in B$, $c_i \in C$ and $\alpha_i, \beta_j \in \Gamma$ for all i, j. This implies that

$$
\begin{aligned}
x &= \sum_{i=1}^{n} \left(\sum_{j=1}^{m} a_j \beta_j b_j \right) \alpha_i c_i \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \left(\left(a_j \beta_j b_j \right) \alpha_i c_i \right) && \text{because of the right distributivity} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} \left(a_j \beta_j \left(b_j \alpha_i c_i \right) \right) && \text{because of the associativity} \\
&\in A\Gamma(B\Gamma C) && \text{since } a_j \beta_j \left(b_j \alpha_i c_i \right) \in A\Gamma(B\Gamma C) \text{ for all } i, j.
\end{aligned}
$$

In general, if *A*, *B* and *C* are nonempty subsets of a Γ-seminearring *R*, then it is not necessary that $A\Gamma(B\Gamma C)$ is contained in $(A\Gamma B)\Gamma C$ because it may not be true that $\sum_{i=1}^{m} a_i \alpha_i (\sum_{j=1}^{n} b_j \beta_j c_j) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_i \alpha_i (b_j \beta_j c_j)$ for any element $\sum_{i=1}^{m} a_i \alpha_i (\sum_{j=1}^{n} b_j \beta_j c_j) \in A\Gamma(B\Gamma C)$.

Definition 7. A subset *I* of a Γ-seminearring *R* is called a **left (right) ideal** of *R* if *I* is a subsemigroup of *R* and $r\alpha x \in I$ $(x\alpha r \in I)$ for all $r \in R$, $x \in I$ and $\alpha \in \Gamma$. If *I* is both a left and a right ideal of *R*, then *I* is called an **ideal** of *R*.

We see that if *I* is an ideal of a Γ-seminearring *R*, then *I* is a sub Γ-seminearring of *R*.

Proposition 8. Let *I* be a subsemigroup of a Γ-seminearring *R*. Then *I* is a left (right) ideal of *R* if and only if $R\Gamma I \subseteq I$ $(I\Gamma R \subseteq I)$.

Proof. It is clear that if *I* is a left (right) ideal of *R*, then $R\Gamma I \subseteq I$ $(I\Gamma R \subseteq I)$.

Next, assume that $R\Gamma I \subseteq I$. For each $r \in R$, $x \in I$ and $\alpha \in \Gamma$, we see that $r\alpha x \in R\Gamma I \subseteq I$. Thus *I* is a left ideal of *R*. The proof for the case of right ideals is obtained similarly.

Theorem 9. Let *R* be a Γ-seminearring.

(i) For each $a \in R$ and $\alpha \in \Gamma$, $R\alpha a$ ($a\alpha R$) is a left (right) ideal of R.

(ii) If A is a nonempty subset of R and B is a right ideal of R, then $A\Gamma B$ is a right ideal of R.

(iii) If A and B are left (right) ideals of R such that $A \cap B \neq \emptyset$, then $A \cap B$ is a left (right) ideal of R.

Proof. (i) Let $a \in R$ and $\alpha \in \Gamma$. Then $R\alpha a$ and $a\alpha R$ are subsemigroups of R. Obviously, if $x, r \in R$ and $\beta \in \Gamma$, then $r\beta(x\alpha a) = (r\beta x)\alpha a \in R\alpha a$. Thus $R\alpha a$ is a left ideal of R.

To show that $a\alpha R$ is a right ideal of R, let $r \in R$, $x \in a\alpha R$ and $\beta \in \Gamma$. Then $x = \sum a\alpha r_i$ where the sum is finite and $r_i \in R$ for all i. Then $x\beta r = (\sum a\alpha r_i)\beta r = \sum a\alpha(r_i\beta r) \in a\alpha R$. Thus $a\alpha R$ is a right ideal of R.

(ii) Let A be a nonempty subset of R and B a right ideal of R. Then $A\Gamma B$ is a subsemigroup of R. Let $r \in R$, $x \in A\Gamma B$ and $\beta \in \Gamma$. Then $x = \sum a_i\alpha_i b_i$ where the sum is finite and $a_i \in A$, $b_i \in B$ and $\alpha_i \in \Gamma$ for all i. So $x\beta r = (\sum a_i\alpha_i b_i)\beta r = \sum a_i\alpha_i(b_i\beta r) \in A\Gamma B$. Thus $A\Gamma B$ is a right ideal of R.

(iii) Assume that A and B are left ideals of R with $A \cap B \neq \emptyset$. Clearly, $A \cap B$ is a subsemigroup of R. Let $r \in R$, $x \in A \cap B$ and $\alpha \in \Gamma$. Then $r\alpha x \in A$ and $r\alpha x \in B$ since A and B are left ideals of R so that $r\alpha x \in A \cap B$. Hence $A \cap B$ is a left ideal of R.

The proof for the case right ideal is obtained similarly.

Corollary 10. Let R be a Γ-seminearring and $a \in R$. Then $a\Gamma R$, $(a\Gamma R)\Gamma R$ and $(R\Gamma a)\Gamma R$ are right ideals of R.

Proof. This follows directly from Theorem 9.

Let R be a Γ-seminearring and $a \in R$. Generally, $R\Gamma a$ need not be a left ideal of R. Nevertheless, if R satisfies the left distributivity, then $R\Gamma a$ is, definitely, a left ideal of R.

However, we can weaken the condition that R satisfies the left distributivity and still obtain the same result.

Definition 11. Let R be a Γ-seminearring under the mapping from $R \times \Gamma \times R$ into R, say f, and D be the set of all distributive elements of R, i.e., $D = \{d \in R \mid d\alpha(a + b) = d\alpha a + d\alpha b$ for all $a, b \in R$ and $\alpha \in \Gamma\}$. Then R is called **distributively generated** (d.g. for short) if the set D is a nonempty subset of R which $f_{|D \times \Gamma \times D} : D \times \Gamma \times D \to D$ and $(\langle D \rangle, +) = (R, +)$ where $\langle D \rangle = \{\sum_{i=1}^m \alpha_i d_i \mid m, \alpha_i \in \mathbb{N}$ and $d_i \in D$ for all $i\}$.

In fact, $\langle D \rangle = \{\sum_{i=1}^n d_i \mid n \in \mathbb{N}$ and $d_i \in D\}$ where all d_i's in $\sum d_i$ may not be distinct. In addition, $(\langle D \rangle, +) = (R, +)$ means that every element in R can be written as a finite sum of distributive elements.

Theorem 12. Let R be a distibutively generated Γ-seminearring.

(i) If A is a left ideal of R and B is a nonempty subset of R, then $A\Gamma B$ is a left ideal of R.

(ii) If A is a left ideal and B is a right ideal of R, then $A\Gamma B$ is an ideal of R.

Proof. (i) Let A be a left ideal of R and B be a nonempty subset of R. Then $A\Gamma B$ is a subsemigroup of R. Let $r \in R$, $x \in A\Gamma B$ and $\beta \in \Gamma$. Then $x = \sum_i a_i\alpha_i b_i$ where the sum is finite and $a_i \in A$, $b_i \in B$ and $\alpha_i \in \Gamma$ for all i. Since R is d.g., we have $r = \sum_k d_k$ where the sum is finite and d_k is a distributive element for all k. So $r\beta x = r\beta(\sum_i a_i\alpha_i b_i) = (\sum_k d_k)\beta(\sum_i a_i\alpha_i b_i) = \sum_k (d_k\beta(\sum_i a_i\alpha_i b_i)) = \sum_k \sum_i (d_k\beta(a_i\alpha_i b_i))$ because each d_k is a distributive element of R. Then $r\beta x = \sum_k \sum_i (d_k\beta(a_i\alpha_i b_i)) = \sum_k \sum_i ((d_k\beta a_i)\alpha_i b_i) \in A\Gamma B$ since A is a left ideal of R. Thus $A\Gamma B$ is a right ideal of R.

(ii) This is a consequence of (i) and Theorem 9.

Corollary 13. Let R be a distibutively generated Γ-seminearring and $a \in R$. Then $R\Gamma a$ and $R\Gamma(R\Gamma a)$ are left ideals of R. Furthermore, $R\Gamma R$, $(R\Gamma a)\Gamma R$ and $R\Gamma(a\Gamma R)$ are ideals of R.

Proof. This follows directly from Theorem 12 and Corollary 10.

The following theorem shows another importance of the distributively generated property on the associative property of a Γ-seminearring.

Theorem 14. Let R be a distributively generated Γ-seminearring, and B, C be nonempty subsets of R. Then $R\Gamma(B\Gamma C) = (R\Gamma B)\Gamma C$.

Proof. It suffices to show only that $R\Gamma(B\Gamma C) \subseteq (R\Gamma B)\Gamma C$ as a result of Theorem 6. Let $x \in R\Gamma(B\Gamma C)$. Then $x = \sum_{i=1}^n r_i\alpha_i\left(\sum_{j=1}^m b_j\beta_j c_j\right)$ for some $m, n \in \mathbb{N}$, $r_i \in R$, $b_j \in B$, $c_j \in C$ and $\alpha_i, \beta_j \in \Gamma$ for all i, j. Since R is d.g., each r_i can be written as $\sum_{k=1}^{l_i} d_{ik}$ where all d_{ik}'s are distributive elements of R. Then

$$x = \sum_{i=1}^{n} \left(\sum_{k=1}^{l_i} d_{ik} \right) \alpha_i \left(\sum_{j=1}^{m} b_j \beta_j c_j \right)$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{l_i} \left(d_{ik} \alpha_i (\sum_{j=1}^{m} b_j \beta_j c_j) \right)$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{l_i} \sum_{j=1}^{m} \left(d_{ik} \alpha_i \left(b_j \beta_j c_j \right) \right) \qquad \text{since each } d_{ik} \text{ is a distributive element}$$

$$= \sum_{i=1}^{n} \sum_{k=1}^{l_i} \sum_{j=1}^{m} \left(\left(d_{ik} \alpha_i b_j \right) \beta_j c_j \right)$$

$$\in (R\Gamma B)\Gamma C \qquad \text{since } \left(d_{ik} \alpha_i b_j \right) \beta_j c_j \in (R\Gamma B)\Gamma C \text{ for all } i, j, k.$$

Definition 15. Let R be a Γ-seminearring. An element $x \in R$ is called a **left (right) zero** if $x\alpha y = x$ $(y\alpha x = x)$ for all $y \in R$ and $\alpha \in \Gamma$. Furthermore, if x is both a left and a right zero of R, then x is called a **zero** of R.

Moreover, if all elements of R are left (right) zeros, then R is called a **left (right) zero** Γ-seminearring.

Theorem 16. Let R be a Γ-seminearring.

(i) If R has a left zero and a right zero, then R has a zero.

(ii) If R has a zero, then that zero is unique.

Proof. (i) Let e and f be a left zero and a right zero of R, respectively. Fix an element $\alpha \in \Gamma$. Then $e = e\alpha f = f$ implies that R has a zero.

(ii) This is obvious from the proof of (i) that the zero is unique.

If R is a Γ-seminearring with additive identity 0, then, in general, 0 is not necessary a left zero and a right zero of R. However, if the semigroup R also satisfies either the left cancellation or the right cancellation, then 0 is a left zero of R but need not be a right zero of R.

Definition 17. A Γ-seminearring R with additive identity 0 is called **zero-symmetric** if $0\alpha x = 0 = x\alpha 0$ for all $x \in R$ and $\alpha \in \Gamma$.

If R is a zero-symmetric Γ-seminearring, then it follows directly from the definition that 0 is the zero of R.

Proposition 18. Let R be a Γ-seminearring. Then R is zero-symmetric if and only if $\{0\}$ is an ideal of R.

Proof. It is clear that $\{0\}$ is a subsemigroup of R. Moreover, $\{0\}$ is an ideal of R because 0 is the zero of R. On the other hand, it is obvious that if $\{0\}$ is an ideal of R, then R is zero-symmetric.

3. Main Results

Definition 19. A Γ-seminearring R is called **left (right) simple** if the only left (right) ideal of R is itself. Furthermore, R is called **simple** if the only ideal of R is itself.

It is clear that if a nonzero Γ-seminearring R is zero-symmetric, then R is neither left simple nor right simple.

Definition 20. A zero-symmetric Γ-seminearring R with more that one element is called **left (right) 0-simple** if $R\Gamma R \neq \{0\}$ and R has no left (right) ideals other than $\{0\}$ and itself. Furthermore, R is called **0-simple** if $R\Gamma R \neq \{0\}$ and R has no ideals other than $\{0\}$ and itself.

Note that, in the above definition, the zero-symmetric property is compulsory otherwise $\{0\}$ may not be an ideal of R according to Proposition 18.

Theorem 21. Let R be a Γ-seminearring. If R is left (right) zero, then R is left (right) simple.

Proof. Let R be left zero, A a left ideal of R and $x \in R$. Fix $a \in A$ and $\alpha \in \Gamma$. Then $x = x\alpha a \in A$ since A is a left ideal of R. Thus $R = A$. As a result, R is left simple.

Theorem 22. Let R be a Γ-seminearring.

(i) If $R\Gamma x = R$ for all $x \in R$, then R is left simple.

(ii) $x\Gamma R = R$ for all $x \in R$ if and only if R is right simple.

(iii) If $(R\Gamma x)\Gamma R = R$ for all $x \in R$, then R is simple.

Proof. (i) Assume that $R\Gamma x = R$ for all $x \in R$. Let L be a left ideal of R and $a \in L$. Then $R = R\Gamma a \subseteq R\Gamma L \subseteq L$ so that

$R = L$.

(ii) First, assume that $x\Gamma R = R$ for all $x \in R$. Let A be a right ideal of R and $a \in A$. Then $R = a\Gamma R \subseteq A\Gamma R \subseteq A$ so that $R = A$.

Conversely, assume that R is right simple. For each $x \in R$, we know from Corollary 10 that $x\Gamma R$ is a right ideal of R so that $x\Gamma R = R$.

(iii) Assume that $(R\Gamma x)\Gamma R = R$ for all $x \in R$. Let I be an ideal of R and $a \in I$ Then $R = (R\Gamma a)\Gamma R \subseteq I\Gamma R \subseteq I$ so that $R = I$.

If a Γ-seminearring R is distributively generated, then the converse of (i) and (iii) in Theorem 22 hold.

Theorem 23. Let R be a distributively generated Γ-seminearring.

(i) $R\Gamma x = R$ for all $x \in R$ if and only if R is left simple.

(ii) $(R\Gamma x)\Gamma R = R$ for all $x \in R$ if and only if R is simple.

Proof. (i) It suffices to show the converse direction only. Assume that R is left simple. Since R is d.g., $R\Gamma x$ is a left ideal of R for all $x \in R$ by Corrollary 13. Thus $R\Gamma x = R$ for all $x \in R$.

The proof of (ii) is obtained similarly to that of (i).

Theorem 24. Let R be a zero-symmetric Γ-seminearring such that $|R| > 1$.

(i) If $R\Gamma x = R$ for all $x \in R \setminus \{0\}$, then R is left 0-simple.

(ii) $x\Gamma R = R$ for all $x \in R \setminus \{0\}$ if and only if R is right 0-simple.

(iii) If $(R\Gamma x)\Gamma R = R$ for all $x \in R \setminus \{0\}$, then R is 0-simple.

Proof. (i) Assume that $R\Gamma x = R$ for all $x \in R \setminus \{0\}$. Let $x \in R \setminus \{0\}$. Then $\{0\} \neq R = R\Gamma x \subseteq R\Gamma R$, i.e., $R\Gamma R \neq \{0\}$. Let L be a nonzero left ideal of R and $a \in L \setminus \{0\}$. Then $R = R\Gamma a \subseteq R\Gamma L \subseteq L$ so that $R = L$.

(ii) Note that R is right 0-simple provided that $x\Gamma R = R$ for all $x \in R \setminus \{0\}$ can be proved in the same way as the proof of (i).

Conversely, assume that R is right 0-simple. Let $L = \{x \in R \mid x\alpha r = 0$ for all $r \in R$ and $\alpha \in \Gamma\}$. Since R is zero-symmetric, $0 \in L$ so that $L \neq \emptyset$. We show that L is a right ideal of R. Let $x, y \in L$. Then, for each $r \in R$ and $\alpha \in \Gamma$, we see that $x\alpha r = 0 = y\alpha r$ so that $(x + y)\alpha r = x\alpha r + y\alpha r = 0 + 0 = 0$ since 0 is the additive identity of R. Thus $x + y \in L$. Next, $(x\beta s)\alpha r = 0\alpha r = 0$ for all $r, s \in R$ and $\alpha, \beta \in \Gamma$ and then $x\beta s \in L$. This implies that L is a right ideal of R. If $L = R$, then $R\Gamma R = L\Gamma R = \{0\}$ which is a contradiction because R is right 0-simple. Thus $L = \{0\}$. Finally, let $x \in R \setminus \{0\}$. Then $x \notin L$. So $x\alpha r \neq 0$ for some $r \in R$ and $\alpha \in \Gamma$. This implies that $x\Gamma R \neq \{0\}$. Recall that $x\Gamma R$ is a right ideal of R. Hence $x\Gamma R = R$ because R is right 0-simple.

(iii) The proof is similar to the proof of (i).

The following Theorem shows that the converse of (i) and (iii) in Theorem 24 hold if the distributively generated property is given.

Theorem 25. Let a Γ-seminearring R be zero-symmetric and distributively generated with $|R| > 1$.

(i) $R\Gamma x = R$ for all $x \in R \setminus \{0\}$ if and only if R is left 0-simple.

(ii) $(R\Gamma x)\Gamma R = R$ for all $x \in R \setminus \{0\}$ if and only if R is 0-simple.

Proof. (ii) Assume that R is 0-simple. Frist, we show that $(R\Gamma R)\Gamma R = R$. Since $R\Gamma R$ is an ideal of R, this implies that $R\Gamma R \neq \{0\}$ and then $R\Gamma R = R$. Thus $(R\Gamma R)\Gamma R = R\Gamma R = R$. Next, let $I = \{x \in R \mid (r\alpha x)\beta s = 0$ for all $r, s \in R$ and $\alpha, \beta \in \Gamma\}$. Since R is zero-symmetric, $0 \in I$. We claim that I is an ideal of R. Let $r, s, t \in R$, $x, y \in I$ and $\alpha, \beta, \gamma \in \Gamma$. Then $r = \sum_{i=1}^{m} d_i$ where all d_i's are distributive elements of R. Then $(r\alpha(x+y))\beta s = \left(\left(\sum_{i=1}^{m} d_i\right)\alpha(x+y)\right)\beta s = \left(\sum_{i=1}^{m}(d_i\alpha x + d_i\alpha y)\right)\beta s = \sum_{i=1}^{m}\left((d_i\alpha x)\beta s + (d_i\alpha y)\beta s\right) = 0$. Thus $x + y \in I$. We then verify that $x\gamma t, t\gamma x \in I$. Since $x, y \in I$ and 0 is the zero of R, we obtain that $(r\alpha(x\gamma t))\beta s = ((r\alpha x)\gamma t)\beta s = 0\beta s = 0$ and $(r\alpha(t\gamma x))\beta s = r\alpha((t\gamma x)\beta s) = r\alpha 0 = 0$. Thus $x\gamma t, t\gamma x \in I$. Since R is 0-simple, $I = \{0\}$ or $I = R$. If $I = R$, then $R = (R\Gamma R)\Gamma R = \{0\}$ which is absurd. Thus $I = \{0\}$. Finally, let $x \in R \setminus \{0\}$. Then $x \notin I$. This implies that $(r\alpha x)\beta s \neq 0$ for some $r, s \in R$, $\alpha, \beta \in \Gamma$. Then $(R\Gamma x)\Gamma R \neq \{0\}$. Hence $(R\Gamma x)\Gamma R = R$ since $(R\Gamma x)\Gamma R$ is an ideal of R.

The proof of (i) is obtained similarly to that of (ii).

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