Journal of Mathematics Research
http: www.ccsenet.org/jmr
Vol. 1, No. 2
September 2009

An Improvement of a Non-uniform Concentration Inequality for Randomized Orthogonal Array Sampling Designs

Kittipong Laipaporn (Corresponding author)
Department of Mathematics, School of Science
Walailak University, Nakhon Si Thammarat 80160, Thailand
E-mail: lkittipo@wu.ac.th

Kritsada Sungkamongkol
Department of Mathematics, Faculty of Science
Chulalongkorn University, Bangkok 10330, Thailand
E-mail: k_man2002@hotmail.com

Abstract

Let f be an integrable function from $\mathbb{R}^3$ to $\mathbb{R}$ and $\mu = \int_{[0,1]^3} f(x)dx$. A simple estimator of μ is $\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} f \circ X_i$ where $X_1, X_2, ...X_n$ are independent random vectors and uniformly distributed on $[0, 1]^3$. In 2006, Neammanee and Laipaporn used the orthogonal array to choose the points X_i's and established a non-uniform concentration inequality. In this article, we improve their result.

Keywords: Computer experiment, Orthogonal array, Non-uniform concentration inequality

1. Introduction

Let $f : [0, 1]^k \to \mathbb{R}$ be a measurable function and let X be a uniform random vector on $[0, 1]^k$. In computer experiments, for examples, the behavior of nuclear weapon, climate change, the electrical circuit and the fluid flow problems, we simply refer the collection of input as x, the simulator of each circumstance as f, and the output of experiment as $f(x)$. The expected value,

$$\mu = E(f \circ X) = \int_{[0,1]^k} f(x)dx,$$

is cosidered as the representative of output tendency. Sometimes the function f is not easily computed or the dimension k is very high. It is widely acceptable that Monte Carlo method is competitive for high-dimensional problem ((Davis, P. J.,1984) and (Evans, M., 2000)). That is, we randomly choose n points from $[0, 1]^k$,say $X_1, X_2, ...X_n$, and define

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} f \circ X_i$$

to estimate the expected value $E(f \circ X)$.

Alternatively, the selection of the n points $X_1, X_2, ..., X_n$ are presented by many mathematicians such as lattice point sampling (Patterson, H. D.,1954), Latin hypercube sampling (Owen, A. B., 1992a), orthogonal array sampling (Owen, A. B., 1992b) and scrambled net sampling (Owen, A. B., 1997) and (Tang, B., 1993). In this article, we use the orthogonal array sampling which provide many applications ((Davis, J., 2007), (Lazic, L., 2008), (Lee, K. H., 2003) and (Lin, K. P., 2008)) and the definition is stated as follow.

Let d, n and t be positive integers with $t \le d$. An orthogonal array of strength t is a matrix of n rows and d columns with elements taken from the set $\{0, 1, \ldots, q-1\}$ such that in any $n \times t$ submatrix, each of the q^t possible rows occurs the same number of times. The class of all such arrays is denoted by $OA(n, d, q, t)$. In 1996, Loh considered the function $f : [0, 1]^3 \to \mathbb{R}$ and used $A = (a_{i,j}) \in OA(q^2, 3, q, 2)$ to approximate μ by the estimator

$$\hat{\mu} = \frac{1}{q^2}\sum_{i=1}^{q^2} f \circ X(\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3}))$$

where

(a) π_1, π_2, π_3 are random permutations of $\{0, 1, ..., q-1\}$,

(b) for any $i_1, i_2, i_3 \in \{0, 1, ..., q-1\}$ and $j \in \{1, 2, 3\}$

$$X(i_1, i_2, i_3) = (X_1(i_1, i_2, i_3), X_2(i_1, i_2, i_3), X_3(i_1, i_2, i_3)),$$

$$X_j(i_1, i_2, i_3) = \frac{i_j + U_{i_1,i_2,i_3,j}}{q} \text{ and}$$

$U_{i_1,i_2,i_3,j}$'s are uniform random variables on [0,1] and

(c) $U_{i_1,i_2,i_3,j}$'s and π_k's are all stochastically independent.

In 1996, Loh was the first who considered the normal approximation of

$$W = \frac{\hat{\mu} - \mu}{\sqrt{Var(\hat{\mu})}}$$

where $Var(\hat{\mu}) > 0$. He gave a uniform bound of orthogonal array sampling designs and Laipaporn and Neammanee improved the bound to the rate $O(q^{-\frac{1}{2}})$ in (Laipaporn, K., 2008). Besides the normal approximation, they also investigated a non-uniform concentration inequality of W in Theorem 1(Laipaporn, K., 2006).

Theorem 1 Assume that $E(f \circ X)^4 < \infty$. Then, there exists a constant C such that

$$P(z \leq W \leq w) \leq \frac{C}{1+z}(w-z) + \frac{1}{1+z}O(\frac{1}{\sqrt{q}}), \ \ as \ \ q \to \infty,$$

for any real number $0 < z \leq w$.

In this paper we will improve the bound of Theorem 1 as in Theorem 2.

Theorem 2 Assume that $E(f \circ X)^4 < \infty$. Then, there exists a constant C such that

$$P(z \leq W \leq w) \leq \frac{C}{(1+z)^3}(w-z) + \frac{1}{(1+z)^2}O(\frac{1}{\sqrt{q}}), \ \ as \ \ q \to \infty,$$

for any real number $0 < z \leq w$.

2. Auxiliary Results

To prove Theorem 2, we need the following lemmas and some notations.

In 1996, Loh introduced a random function $\rho_\pi : \{0, 1, ..., q-1\}^2 \to \{0, 1, ..., q-1\}$

which is defined by

$$(i_1, i_2, \rho_\pi(i_1, i_2)) = (\pi_1(a_{i,1}), \pi_2(a_{i,2}), \pi_3(a_{i,3})) \text{ for some } i \in \{1, 2, .., q^2\}.$$

For each i, j and $k \in \{0, 1, ..., q-1\}$, and $z > 0$, we let I and K be uniformly distributed random variables on $\{0, 1, ..., q-1\}$, (I, K) uniformly distributed on $\{(i, k)|i, k = 0, 1, ..., q-1, i \neq k\}$.

Let

$$Y_z(i, j, k) = Y(i, j, k)\mathbb{I}(\left|Y(i, j, k)\right| > 1 + z),$$

$$\widehat{Y}_z(i, j, k) = Y(i, j, k)\mathbb{I}(\left|Y(i, j, k)\right| \leq 1 + z),$$

$$\widehat{Y} = \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\widehat{Y}_z(i, j, \rho_\pi(i, j))$$

and

$$\widetilde{Y} = \widehat{Y} - \widehat{S}_{1,z} - \widehat{S}_{2,z} + \widehat{S}_{3,z} + \widehat{S}_{4,z}$$

where

$$\widehat{S}_{1,z} = \sum_{j=0}^{q-1}\widehat{Y}_z(I, j, \rho_\pi(I, j)), \quad \widehat{S}_{2,z} = \sum_{j=0}^{q-1}\widehat{Y}_z(K, j, \rho_\pi(K, j)),$$

$$\widehat{S}_{3,z} = \sum_{j=0}^{q-1}\widehat{Y}_z(I, j, \rho_\pi(K, j)), \quad \widehat{S}_{4,z} = \sum_{j=0}^{q-1}\widehat{Y}_z(K, j, \rho_\pi(I, j)).$$

Lemma 3 *((Laipaporn, K., 2006) and (Loh, W.L., 1996))* Assume that $E(f \circ X)^r < \infty$ for some positive even integer r. Then

1. $E|\widetilde{Y} - \widehat{Y}|^r = O(\frac{1}{q^{\frac{r}{2}}})$, as $q \to \infty$,
2. $\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} EY^r(i, j, k) = O(q^{3-r})$, $as\ \ q \to \infty$,
3. For any positive integer n and t such that $n + t$ is an even number and $n + t \leq r$,
$$\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} E|Y_z^n(i, j, k)| = \frac{O(q^{3-n-t})}{(1+z)^t}, \text{ as } q \to \infty.$$

Lemma 4 *(Laipaporn, K., 2006)* Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous and piecewise continuously differentiable function, then

$$E\widehat{Y}g(\widehat{Y}) = E\int_{-\infty}^{\infty} g'(\widehat{Y} + t)K(t)dt + \widetilde{\Delta}g(\widehat{Y})$$

where

$$K(t) = \frac{q-1}{4}\Big(\widetilde{Y} - Y\Big)\Big(\mathbb{I}(0 \leq t \leq \widetilde{Y} - Y) - \mathbb{I}(\widetilde{Y} - Y \leq t < 0)\Big),$$

and

$$\widetilde{\Delta}g(\widehat{Y}) = \frac{1}{q}Eg(\widehat{Y})\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1}\widehat{Y}_z(i, j, k).$$

Lemma 5 *(Laipaporn, K., 2006)* If $E(f \circ X)^2 < \infty$, then $E(\widehat{Y} - \widetilde{Y})^2 = \frac{4}{q} + O(\frac{1}{q^2})$, as $q \to \infty$.

Lemma 6 *(Laipaporn, K., 2007)* If $E(f \circ X)^4 < \infty$, then

1. $E\left(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y(i, j, k)\right)^4 \leq O(q^2)$, as $q \to \infty$,
2. $E\left(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y_z(i, j, k)\right)^4 \leq \frac{1}{(1+z)^2}O(\frac{1}{q^2})$, as $q \to \infty$.

Lemma 7 Assume that $E(f \circ X)^4 < \infty$. Let $\gamma = \max\left(\frac{q(q-1)}{4(q-4)}E|\widehat{Y} - \widetilde{Y}|^3,\ \frac{1}{\sqrt{q}}\right)$ and

$$U_\gamma = \sum_{i \neq k}\Big|\sum_{j=0}^{q-1}\{\widehat{Y}_z(i, j, \rho_\pi(i, j)) + \widehat{Y}_z(k, j, \rho_\pi(k, j)) - \widehat{Y}_z(i, j, \rho_\pi(k, j)) - \widehat{Y}_z(k, j, \rho_\pi(i, j))\}\Big|$$

$$\times \min\Big(\gamma, \sum_{i \neq k}\Big|\sum_{j=0}^{q-1}\{\widehat{Y}_z(i, j, \rho_\pi(i, j)) + \widehat{Y}_z(k, j, \rho_\pi(k, j)) - \widehat{Y}_z(i, j, \rho_\pi(k, j)) - \widehat{Y}_z(k, j, \rho_\pi(i, j))\}\Big|\Big).$$

Then

1. $EU_\gamma \geq 3q + O(1)$, as $q \to \infty$,
2. $Var(U_\gamma) \leq \gamma^2 O(q^2)$.

Proof. (1.) From the fact that $\min(a, b) \geq b - \frac{b^2}{4a}$ for any $a, b > 0$,

$$\begin{aligned}E(U_\gamma) &= q(q-1)E(\widehat{Y}-\widetilde{Y})\min(\gamma, |\widehat{Y}-\widetilde{Y}|)\\ &\geq q(q-1)E(\widehat{Y}-\widetilde{Y})^2 - \frac{q(q-1)}{4\gamma}E|\widehat{Y}-\widetilde{Y}|^3\\ &\geq q(q-1)E(\widehat{Y}-\widetilde{Y})^2 - \frac{q(q-1)}{4\left(\frac{q(q-1)}{4(q-4)}E|\widehat{Y}-\widetilde{Y}|^3\right)}E|\widehat{Y}-\widetilde{Y}|^3.\end{aligned}$$

By Lemma 5, we have

$$\begin{aligned}E(U_\gamma) &\geq q(q-1)\left(\frac{4}{q} + O(\frac{1}{q^2})\right) - (q-4)\\ &= 3q + O(1).\end{aligned}$$

(2.) By using the same argument of Lemma 2.5(Laipaporn, K., 2006), we have

$$Var(U_\gamma) \leq \frac{q(q-1)}{4}\left\{\gamma^2 O(\frac{1}{q}) + \frac{\gamma^2}{(1+z)^2}O(\frac{1}{q^2})\right\} + \frac{1}{q(q-1)}\left\{\gamma^2(q^4)\right\} = \gamma^2 O(q^2).$$

Lemma 8. If $E(f \circ X)^4 < \infty$, then $E\widehat{Y}^4 = O(1)$, as $q \to \infty$.

Proof. First we note that

$$\begin{aligned}E\widehat{Y}^4 &= E\left(\sum_{i,j}\widehat{Y}_z(i,j,\rho_\pi(i,j))\right)^4\\ &= E\left(\sum_{i,j}Y(i,j,\rho_\pi(i,j)) - \sum_{i,j}Y_z(i,j,\rho_\pi(i,j))\right)^4\\ &\leq CEW^4 + CE\left(\sum_{i,j}Y_z(i,j,\rho_\pi(i,j))\right)^4.\end{aligned}$$

In (Laipaporn, K., 2009), we proved that $EW^4 = O(1)$, so it remains to show that $E\left(\sum_{i,j}Y_z(i,j,\rho_\pi(i,j))\right)^4 = O(1)$.

Note that $$E\left(\sum_{i,j}Y_z(i,j,\rho_\pi(i,j))\right)^4 = B_1 + B_2 + B_3 + B_4 + B_5$$

where

$$\begin{aligned}B_1 &= \sum_{i,j} EY_z^4(i,j,\rho_\pi(i,j))\\ B_2 &= \sum_{i_1,j_1}\sum_{\substack{i_2,j_2\\(i_2,j_2)\neq(i_1,j_1)}} EY_z^3(i_1,j_1,\rho_\pi(i_1,j_1))Y_z(i_2,j_2,\rho_\pi(i_2,j_2))\\ B_3 &= \sum_{i_1,j_1}\sum_{\substack{i_2,j_2\\(i_2,j_2)\neq(i_1,j_1)}} EY_z^2(i_1,j_1,\rho_\pi(i_1,j_1))Y_z^2(i_2,j_2,\rho_\pi(i_2,j_2))\\ B_4 &= \sum_{i_1,j_1}\sum_{\substack{i_2,j_2\\(i_2,j_2)\neq(i_1,j_1)}}\sum_{\substack{i_3,j_3\\(i_3,j_3)\neq(i_1,j_1)\\(i_3,j_3)\neq(i_2,j_2)}} EY_z^2(i_1,j_1,\rho_\pi(i_1,j_1))Y_z(i_2,j_2,\rho_\pi(i_2,j_2))Y_z(i_3,j_3,\rho_\pi(i_3,j_3))\\ B_5 &= \sum_{i_1,j_1}\sum_{\substack{i_2,j_2\\(i_2,j_2)\neq(i_1,j_1)}}\sum_{\substack{i_3,j_3\\(i_3,j_3)\neq(i_1,j_1)\\(i_3,j_3)\neq(i_2,j_2)}}\sum_{\substack{i_4,j_4\\(i_4,j_4)\neq(i_1,j_1)\\(i_4,j_4)\neq(i_2,j_2)\\(i_4,j_4)\neq(i_3,j_3)}}\\ &\quad EY_z(i_1,j_1,\rho_\pi(i_1,j_1))Y_z(i_2,j_2,\rho_\pi(i_2,j_2))Y_z(i_3,j_3,\rho_\pi(i_3,j_3))Y_z(i_4,j_4,\rho_\pi(i_4,j_4)).\end{aligned}$$

By Lemma 3(3) we have

$$|B_1| \le \frac{1}{q}\sum_{i,j,k} E|Y_z^4(i,j,k)| = \frac{1}{q}O(q^{3-4}) = O(\frac{1}{q^2}),$$

$$\begin{aligned}|B_2| &\le \frac{1}{q^2}\left[\sum_{i_1,j_1,k_1} E|Y_z^3(i_1,j_1,k_1)|\right]\left[\sum_{i_2,j_2,k_2} E|Y_z(i_2,j_2,k_2)|\right]\\ &= \frac{1}{q^2(1+z)}O(q^{3-3-1})(\frac{1}{1+z})O(q^{3-1-1})\\ &= \frac{1}{(1+z)^2}O(\frac{1}{q^2}),\end{aligned}$$

$$\begin{aligned}|B_3| &\le \frac{1}{q^2}\left[\sum_{i,j,k} E|Y_z^2(i,j,k)|\right]^2\\ &= \frac{1}{q^2}\left(\frac{1}{(1+z)^2}O(q^{3-2-2})\right)^2\\ &= \frac{1}{(1+z)^4}O(\frac{1}{q^4}),\\ |B_4| &\le \frac{1}{q^3}\left[\sum_{i_1,j_1,k_1} E|Y_z^2(i_1,j_1,k_1)|\right]\left[\sum_{i_2,j_2,k_2} E|Y_z(i_2,j_2,k_2)|\right]^2\\ &= \frac{1}{q^3(1+z)^2}O(q^{3-2-2})\left(\frac{1}{(1+z)}O(q^{3-1-1})\right)^2\\ &= \frac{1}{(1+z)^4}O(\frac{1}{q^2})\\ \text{and }\ |B_5| &\le \frac{1}{q^4}\left[\sum_{i,j,k} E|Y_z(i,j,k)|\right]^4\\ &= (\frac{1}{q^4})\left(\frac{1}{(1+z)}O(q^{3-1-1})\right)^4\\ &= \frac{1}{(1+z)^4}O(1).\end{aligned}$$

Hence $E\left(\sum_{i,j} Y_z(i,j,\rho_\pi(i,j))\right)^4 = O(1)$. Therefore $E\widehat{Y}^4 = O(1)$.

3. Proof of Theorem 2

Proof. Note that $P(z \le W \le w) \le P(W \ne \widehat{Y}) + P(z \le \widehat{Y} \le w)$
and by Lemma 3(2),

$$\begin{aligned}P(W \ne \widehat{Y}) &= P\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1} \mathbb{I}(|Y(i,j,\rho_\pi(i,j))| > 1+z) \ge 1\Big)\\ &\le E\Big(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1} \mathbb{I}(|Y(i,j,\rho_\pi(i,j))| > 1+z)\Big)\\ &= \frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} E\mathbb{I}(|Y(i,j,k)| > 1+z)\\ &\le \frac{1}{q}\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \frac{E|Y(i,j,k)|^4}{(1+z)^4}\\ &\le \frac{1}{q(1+z)^4}O(q^{3-4})\\ &= \frac{1}{(1+z)^4}O(\frac{1}{q^2}).\end{aligned}$$

If we can show that $P(z \le \widehat{Y} \le w) \le \dfrac{C}{(1+z)^3}(w-z) + \dfrac{1}{(1+z)^2}O(\dfrac{1}{\sqrt{q}})$. Then we have Theorem 2. Let γ be defined as in Lemma 7.

Case 1 $(1+z)^2\gamma \ge 1$.

By Lemma 8 and the fact that $\gamma \ge \dfrac{1}{(1+z)^2}$, we have

$$\begin{aligned} P(z \le \widehat{Y} \le w) \le P(z \le \widehat{Y}) = P(1+z \le 1+\widehat{Y}) &\le \frac{E|\widehat{Y}+1|^4}{(1+z)^4} \\ &\le C\frac{E|\widehat{Y}|^4}{(1+z)^4} + \frac{C}{(1+z)^4} \\ &\le \frac{C}{(1+z)^4} \\ &\le \frac{C\gamma}{(1+z)^2}. \end{aligned}$$

By Hölder's inequality and Lemma 3(1), $E|\widetilde{Y} - \widehat{Y}|^3 = O(\dfrac{1}{q\sqrt{q}})$. Then $\gamma = O(\dfrac{1}{\sqrt{q}})$ and we have $P(z \le W \le w) \le \dfrac{1}{(1+z)^2}O(\dfrac{1}{\sqrt{q}})$.

Case 2 $(1+z)^2\gamma < 1$.

Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(t) = \begin{cases} 0 & \text{if} \quad t < z-\gamma, \\ (1+t+\gamma)^3(t-z+\gamma) & \text{if} \quad z-\gamma \le t \le w+\gamma, \\ (1+t+\gamma)^3(w-z+2\gamma) & \text{if} \quad t > w+\gamma. \end{cases}$$

Then f is a non decreasing function satisfying

$$f'(t) \ge \begin{cases} (1+z)^3 & \text{for} \quad z-\gamma < t < w+\gamma, \\ 0 & \text{otherwise .} \end{cases}$$

Since f is a continuous and piecewise continuously differentiable function, by Lemma 4 we have

$$E\widehat{Y}f(\widehat{Y}) = E\int_{-\infty}^{\infty} f'(\widehat{Y}+t)K(t)dt + \widetilde{\Delta}f(\widehat{Y}).$$

Let U_γ be defined as in Lemma 7, we observe that

$$
\begin{aligned}
& E\int_{-\infty}^{\infty} f'(\widehat{Y}+t)K(t)dt \\
& \geq (1+z)^3 E\mathbb{I}(z\leq \widehat{Y}\leq w)\int_{|t|\leq\gamma} K(t)dt \\
& = \frac{(q-1)(1+z)^3}{4}E\mathbb{I}(z\leq \widehat{Y}\leq w)|\widehat{Y}-\widetilde{Y}|\min(\gamma,|\widehat{Y}-\widetilde{Y}|) \\
& = \frac{(q-1)(1+z)^3}{4q(q-1)}E\mathbb{I}(z\leq \widehat{Y}\leq w)U_\gamma \\
& \geq \frac{(1+z)^3}{4q}E\mathbb{I}(z\leq \widehat{Y}\leq w)U_\gamma\mathbb{I}(U_\gamma\geq q) \\
& \geq \frac{(1+z)^3}{4}E\mathbb{I}(z\leq \widehat{Y}\leq w)\mathbb{I}(U_\gamma\geq q) \\
& = \frac{(1+z)^3}{4}E\{E\mathbb{I}(z\leq \widehat{Y}\leq w)-\mathbb{I}(z\leq \widehat{Y}\leq w, U_\gamma\leq q)\} \\
& \geq \frac{(1+z)^3}{4}\{P(z\leq \widehat{Y}\leq w)-P(U_\gamma\leq q)\}.
\end{aligned}
$$

By this fact, Lemma 7(1,2), Lemma 8 and $\gamma = O(\frac{1}{\sqrt{q}})$,

$$
\begin{aligned}
P(z\leq \widehat{Y}\leq w) & \leq \frac{4}{(1+z)^3}E\widehat{Y}f(\widehat{Y}) - \frac{4}{(1+z)^3}\widetilde{\Delta}f(\widehat{Y}) + P(U_\gamma\leq q) \\
& \leq \frac{4}{(1+z)^3}(w-z+2\gamma)E|\widehat{Y}||1+\gamma+\widehat{Y}|^3 \\
& \quad + \frac{4}{(1+z)^3}|\widetilde{\Delta}f(\widehat{Y})| + P(EU_\gamma - U_\gamma \geq 3q + O(1) - q)
\end{aligned}
$$

$$
\begin{aligned}
& \leq \frac{C}{(1+z)^3}(w-z+2\gamma)E|\widehat{Y}||(1+\gamma)^3+\widehat{Y}^3| \\
& \quad + \frac{4}{(1+z)^3}|\widetilde{\Delta}f(\widehat{Y})| + P(EU_\gamma - U_\gamma\geq q) \\
& \leq \frac{C}{(1+z)^3}(w-z+2\gamma)\Big\{E|\widehat{Y}| + E|\widehat{Y}|^4\Big\} \\
& \quad + \frac{4}{(1+z)^3}|\widetilde{\Delta}f(\widehat{Y})| + \frac{C}{q^2}E(U_\gamma - EU_\gamma)^2 \\
& \leq \frac{C}{(1+z)^3}(w-z) + \frac{4}{(1+z)^3}|\widetilde{\Delta}f(\widehat{Y})| + \frac{C}{q^2}\gamma^2 O(q^2) \\
& \leq \frac{C}{(1+z)^3}(w-z) + \frac{4}{(1+z)^3}|\widetilde{\Delta}f(\widehat{Y})| + \frac{1}{(1+z)^2}O(\frac{1}{\sqrt{q}}). \qquad (1)
\end{aligned}
$$

From, Lemma 6(1,2), Lemma 8 and the fact that $\gamma = O(\frac{1}{\sqrt{q}})$,

$$\begin{aligned}
|\widetilde{\Delta} f(\widehat{Y})| &= \Big|\frac{1}{q} E f(\widehat{Y}) \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \widehat{Y}_z(i,j,k)\Big| \\
&\leq \frac{1}{q}(w-z+2\gamma)E\Big|(1+\gamma+\widehat{Y})^3 \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \widehat{Y}_z(i,j,k)\Big| \\
&\leq \frac{C}{q}(w-z+2\gamma)E\Big|(1+\gamma)^3 \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \widehat{Y}_z(i,j,k)\Big| \\
&\quad + \frac{C}{q}(w-z+2\gamma)E\Big|\widehat{Y}^3 \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} \widehat{Y}_z(i,j,k)\Big| \\
&\leq C(w-z+2\gamma)(1+\gamma^3)\frac{1}{q}\sum_{k=0}^{q-1} E\Big|\sum_{i=0}^{q-1}\sum_{j=0}^{q-1} \widehat{Y}_z(i,j,k)\Big| \\
&\quad + \frac{C}{q}(w-z+2\gamma)E\Big|\widehat{Y}^3 (\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y(i,j,k) - \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y_z(i,j,k))\Big| \\
&\leq C(w-z+2\gamma)(1+\gamma^3)E\Big|\sum_{i=0}^{q-1}\sum_{j=0}^{q-1} \widehat{Y}_z(i,j,\rho_\pi(i,j))\Big| \\
&\quad + \frac{C}{q}(w-z+2\gamma)E\Big|\widehat{Y}^3 \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y(i,j,k)\Big| \\
&\quad + \frac{C}{q}(w-z+2\gamma)E\Big|\widehat{Y}^3 \sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y_z(i,j,k)\Big| \\
&\leq C(w-z+2\gamma)(1+\gamma^3)E|\widehat{Y}| \\
&\quad + \frac{C}{q}(w-z+2\gamma)\{E\widehat{Y}^4\}^{\frac{3}{4}}\{E(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y(i,j,k))^4\}^{\frac{1}{4}} \\
&\quad + \frac{C}{q}(w-z+2\gamma)\{E\widehat{Y}^4\}^{\frac{3}{4}}\{E(\sum_{i=0}^{q-1}\sum_{j=0}^{q-1}\sum_{k=0}^{q-1} Y_z(i,j,k))^4\}^{\frac{1}{4}} \\
&\leq C(w-z+2\gamma)(1+\gamma^3) + \frac{C}{q}(w-z+2\gamma)O(q^{\frac{1}{2}}) \\
&\quad + \frac{C}{q}(w-z+2\gamma)\frac{1}{(1+z)^{\frac{1}{2}}}O(q^{-\frac{1}{2}}) \\
&\leq C(w-z+2\gamma)\left(1+O(\frac{1}{\sqrt{q}})\right) \\
&\leq C(w-z)+O(\frac{1}{\sqrt{q}}).
\end{aligned}$$

From this fact and (1), we can conclude that

$$P(z \leq \widehat{Y} \leq w) \leq \frac{C}{(1+z)^3}(w-z) + \frac{1}{(1+z)^2}O(\frac{1}{\sqrt{q}}).$$

Acknowledgment

The authors would like to thank Prof. Kritsana Neammanee who give an idea for improving the boundedness.

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