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Remarks about Two Theorems in Principles of Mathematical Analysis

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Abstract

The first chapter of the classic book Principles of Mathematical Analysis (WALTER RUDIN, Third Edition) is the real and complex number systems. Theorem 1.20 of the chapter is extracted from the construction of real number R, and it provides a good illustration of what one can do with the least-upper-bound property. Besides, theorem 1.21 proves the existence of nth roots of positive real numbers. Remarks were given because of the importance of the two theorems. Particularly, the proof of "Hence there is an integer m (with$-m_2 \leq nx \leq m_1$) such that $m-1 \leq nx < m$. "which is not mentioned in theorem 1.20 was given in 2.2. A loophole "For every real$x > 0$ and every integer$n > 0$ there is one and only one real$y$ such that $y^n = x$." and a flaw of "If $t = \frac{x}{1+x}$ then $0 < t < 1$. Hence $t^n < t < x$." of theorem 1.21 were indicated in 3.1 and 3.2.

Keywords: Upper bound, Lower bound, Least-upper-bound property

1. Remarks about theorem 1.20 (a)

Theorem 1.20

(a) If$x \in R, y \in R$, and $x > 0$, then there is a positive integer n such that $nx > y$.

(b) If $x \in R, y \in R$, and $x < y$, then there exists a $p \in Q$ such that $x < p < y$.

Theorem 1.21

For every real $x > 0$ and every integer $n > 0$ there is one and only one real y such that $y^n = x$.

These two theorems have been proofed previously. (Rudin, 1976, p. 9-11).

1.1 Put $A = \{nx|x > 0, n \in Z^+\}$. If (a)were false, that is to say, "If $x \in R, y \in R$, and $x > 0$, then there is not a positive integer n such that $nx > y$." Namely, for $\forall n \in Z^+, nx \leq y$. Then y would be an upper bound of $A, A \subset R, A \neq \phi$ (For example $1 \in A$),and A is bounded above, then, according to the definition of least-upper-bound property(see[1],page4),$supA$ exists in R . Namely, A has a least upper bound in R . Put $\alpha = supA$.

Since $x > 0, -x < 0, \alpha - x < 0 + \alpha, \alpha - x < \alpha, \alpha$ is the least upper bound of $A, \alpha - x$ is smaller than α, and $\alpha - x$ is not an upper bound of A .

Hence $\alpha - x < mx$ for some positive integer m .(Assume this is not correct, then $\forall m \in Z^+$, such that $\alpha - x \geq mx$, which is contradictory to $\alpha - x$ is not an upper bound of A .)

But $\alpha < mx + x, \alpha < (m+1)x \in A, (m \in Z^+, m+1 \in Z^+$).Which is impossible, since α is an upper bound of A . This is a contradiction, the assumption which (a) were false is not correct.

The proof is complete.

1.2 Sometimes, the proposition below is satisfied.

If $x \in R, y \in R$, and $x > 0$, then there is a negative integer n such that $nx > y$.

For example, if $x = 1, y = -2$, there exists $n = -1$ such that $nx = -1 > -2 = y$.

But it is not correct for all the situations, for example, if $x \in R_{>0}, y \in R_{>0}$, for all negative integer n , the multiplication of n and x is smaller than the positive number$y$ in R.

2. Remarks about theorem 1.20 (b)

2.1 Since $x < y$,we have $y - x > 0, 1 \in R$, and (a) furnishes a positive integer n such that $n(y - x) > 1$.

Since $n \in Z^+, x \in R, 1 \in R \Rightarrow nx \in R, -nx \in R$, apply (a) again, then there is a positive integer m_1 such that $m_1 \cdot 1 > nx$; there is also a positive integer m_2 such that $m_2 \cdot 1 > -nx$.

We have$-m_2 \leq nx \leq m_1$.

2.2 Here we will give the proof of "Hence there is an integer m (with $-m_2 \leq nx \leq m_1$) such that $m - 1 \leq nx < m$."

Proof (hints: If we can proof $R = \cup_{m\in Z}[m - 1, m)$, then $nx \in R = \cup_{m\in Z}[m - 1, m)$, so, there exists $m_i \in Z$ such that $nx \in [m_i - 1, m_i)$ .Then the proposition above is satisfied.)

Step1 Let M be a nonempty set of real numbers which is bounded below, let $-M$ be the set of all numbers $-m$, where $m \in M$. We will prove that $inf(M)$ exists and $inf(M) = -sup(-M)$.$(*)$

Solution: First note that $-M$ is nonempty and bounded above. Indeed, M contains some element m , and then $-m \in -M$; moreover, M has a lower bound $\alpha, \ni \alpha \in R, \forall m \in M, m \geq \alpha$.

Then $\ni -\alpha \in R, \forall -m \in -M, -m \leq -\alpha$, and $-\alpha$ is an upper bound for $-M$.

We now know that $-M \neq \phi$ and $-\alpha$ is an upper bound for $-M$, hence, $sup(-M)$ exist, according to the least-upper-bound property. Denote $\beta = sup(-M)$. And $-\beta$ is a lower bound for M is immediate from the fact that β is an upper bound for $-M$. Next we show that $-\beta$ is the greatest lower bound, we let $\gamma > -\beta$ and prove that γ is not a lower bound for M. Now $-\gamma < \beta$, hence, $-\gamma$ is not an upper bound for $-M$, so there exists $m \in -M$ such that $m > -\gamma$. Then $-m \in M$ and $-m < \gamma$.

Hence,γ is not a lower bound for M . Hence, $-\beta = inf(M)$.

We have $-\beta = inf(M) = -sup(-M)$.(Rudin,2004,p.7).

$(*)$ is proved.

Let $M = \{m|m \in Z^+, m > nx\}$. According to 2.1, there is a positive integer m_1 such that $m_1 \cdot 1 > nx$ .We know $M \neq \phi$, and nx is a lower bound of M. So, $inf(M)$ exists, according to$(*)$.

Besides, $inf(M) = min\{m|m \in Z^+, m > nx\}$.

To complete the proof, another proposition$(**)$ is needed:

$(**)$ For any $x \in R$, if $0 \leq x < m$ then there is a positive integer k, such that $k - 1 \leq x < k$.

Solution:

(1) When $m = 1, 0 \leq x < 1$, then there is a positive integer $k = 1$, such that $0 = 1 - 1 \leq x < 1$.

(2) Assume $0 \leq x < m$, then there is a positive integer k, such that $k - 1 \leq x < k$.

We consider$0 \leq x < m + 1$, there are two cases:

(i)If $0 \leq x < m$, according to the assumption, there is a positive integer k, such that $k - 1 \leq x < k$.

(ii)If $m \leq x < m + 1$, there exists a positive integer $k = m + 1$, such that $k - 1 = (m + 1) - 1 \leq x < m + 1 = k$, namely, $k - 1 \leq x < k$.

It is proved that $[0, m) = \cup_{k=1}^{m}[k - 1, k)$.$(k \in Z^+)$.

Next we will show $R = \cup_{k\in Z}[k - 1, k)$.

On the one hand, $k \in Z, [k - 1, k) \subseteq R$, then we have $R \supseteq \cup_{k\in Z}[k - 1, k)$.

On the other hand, for any $y \in R$, in the case of $y \geq 0, 1 > 0$, according to the theorem 1.20 (a), there is a positive integer n, such that $n \cdot 1 > y$. According to $(**)$, there is a positive integer k, such that $k - 1 \leq y < k$.

In the case of $y < 0, -y > 0$, according to$(**)$, there is a positive integer k, such that $k - 1 \leq y < k$.

If$-y = k - 1$, so, $y = -(k - 1) = -k + 1 \in [-k + 1, -k + 2)$.

If $-y \neq k - 1$, so, $k - 1 < -y < k, -k < y < -k + 1, y \in (-k, -k + 1) \subseteq [-k, -k + 1)$.

Then $R \subseteq \cup_{k\in Z}[k - 1, k)$.

We have proved $R \supseteq \cup_{k\in Z}[k - 1, k)$ and $R \subseteq \cup_{k\in Z}[k - 1, k)$. So, $R = \cup_{k\in Z}[k - 1, k)$.

According to the hints at the beginning of 2.2, $nx \in R$, there exists an integer m (with$-m_2 \leq nx \leq m_1$) such that $m - 1 \leq nx < m$.

3. Remarks about theorem 1.21

3.1 There is a loophole in the content of theorem 1.21."For every real $x > 0$ and every integer $n > 0$ there is one and only one real y such that $y^n = x$.[1] "

For example, if $y^2 = 2$, so,$y = \pm 2^{\frac{1}{2}}$ in R.(Heinz,1991,p.30-31)

Hence, exactly, the theorem should be described like this: "For every real $x > 0$ and every integer $n > 0$ there is one and only one positive real y such that $y^n = x$. "

3.2 If $t = \frac{x}{1+x}$, then $x > 0 \Rightarrow 1 + x > x > 0 \Rightarrow \frac{1}{1+x} > 0 \Rightarrow \frac{1}{1+x} \cdot (1 + x) > \frac{1}{1+x} \cdot x > \frac{1}{1+x} \cdot 0 \Rightarrow 1 > \frac{x}{1+x} > 0, 1 > t > 0$. And $t^n \leq t.(***)$

If $t > 1 + x, x > 0$,then $t > 1 + x > 1, t > 1 + x > x \Rightarrow t^n \geq t.(****)$

Two"="s are omitted in [1] which is a flaw because "="s are satisfied when $n = 1 > 0$ in $(***)$ and$(****)$.

3.3 Assume $y^n < x$. Let $q = \frac{x-y^n}{n(y+1)^{n-1}}$ since $x - y^n > 0, n(y + 1)^{n-1} > 0$, then $q > 0$. There are two cases to choose $h.(0 < h < 1)$.

If $1 \geq q > 0, 1 \geq q > h > 0$, for example, we may choose $h = \frac{q}{2}$.

If $q > 1, 1 > h > 0$, for example, we may choose $h = \frac{1}{2}$.

Summarize these two cases, $h = min\{\frac{1}{2}, \frac{q}{2}\}$.

Choose h like above so that $0 < h < 1$ and $h < \frac{x-y^n}{n(y+1)^{n-1}}$.

Put $a = y, b = y + h$. Then $(y + h)^n - y^n < hn(y + h)^{n-1} \leq hn(y + 1)^{n-1} < x - y^n$.

3.4 Assume $y^n > x$. Let $k = \frac{y^n-x}{ny^{n-1}}$, then $y^n - x > 0, ny^{n-1} > 0, y > 0 \Rightarrow k = \frac{y^n-x}{ny^{n-1}} > 0$.

And$x > 0, (n - 1)y^n > 0(n = 1, 2, 3, \cdots), \Rightarrow x > 0 > -(n - 1)y^n \Rightarrow y^n - ny^n = (1 - n)y^n = -(n - 1)y^n > x \Rightarrow ny^n > y^n - x \Rightarrow y > k = \frac{y^n-x}{ny^{n-1}}$. We obtain $y > k > 0$.

If $t \geq y - k > 0$, we conclude that:

$t^n \geq (y - k)^n, -t^n \leq -(y - k)^n, y^n - t^n \leq y^n - (y - k)^n$,

And $y^n - t^n \leq y^n - (y - k)^n < (y - y + k)ny^{n-1} = kny^{n-1} = y^n - x$.

Thus $y^n - t^n < y^n - x$, we have $t^n > x$.

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