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Non-uniform Bound on Normal Approximation of Latin Hypercube Sampling

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Abstract

Latin hypercube sampling(LHS)(McKay, M.D., 1979) is a method of sampling that can be used to estimate the value of multidimensional integration. Loh(Loh, W.L., 1996b) and Neammanee and Rattanawong(Neammanee, K., 2009) gave a uniform bound in normal approximation for LHS. In this paper, we give a non-uniform bound of this approximation by using Stein's method.

Keywords: Non-uniform bound, Latin hypercube sampling, Stein's method

1. Introduction and Main Results

Latin hypercube sampling(McKay, M.D., 1979) is a method of sampling that can be used to produce input values for estimation of integrals over multidimensional domains. The main feature of Latin hypercube sampling(LHS) is that, contrast to simple random sampling, it stratifies on all input dimensions simultaneously. More precisely, for positive integers d and n, $d \geq 2$, let:

1. $\pi_k,\ 1 \leq k \leq d$, be random permutations of $\{1, ..., n\}$ each uniformly distributed over all the $n!$ possible permutations;

2. $U_{i_1,...,i_d,j}$ $1 \leq i_1, ..., i_d \leq n,\ 1 \leq j \leq d$, be $[0, 1]$ uniform random variables;

3. The $U_{i_1,...,i_d,j}$'s and π_k's all be stochastically independent.

A Latin hypercube sample of size n (taken from the d-dimensional hypercube $[0, 1]^d$) is defined to be $\{X(\pi_1(i), \pi_2(i), ..., \pi_d(i)) : 1 \leq i \leq n\}$, where for all $1 \leq i_1, ..., i_d \leq n$,

$$X_j(i_1, ..., i_d) = (i_j - U_{i_1,...,i_d,j})/n \qquad \forall 1 \leq j \leq d,$$
$$X(i_1, ..., i_d) = (X_1(i_1, ..., i_d), ..., X_d(i_1, ..., i_d)).$$

Let X be a random vector uniformly distributed on $[0, 1]^d$ and f be a measurable function from $[0, 1]^d$ to $\mathbb{R}$. In many problem, we would like to estimate $\mu = \int_{[0,1]^d} f(x)dx$. We note that $E(f \circ X) = \displaystyle\int_{[0,1]^d} f(x)dx$. The estimator for μ that based on a Latin hypercube sampling is

$$\hat{\mu}_n = \frac{1}{n}\sum_{k=1}^{n} f \circ X(\pi_1(k), \pi_2(k), ..., \pi_d(k)).$$

Then $\hat{\mu}_n$ is an unbiased estimator for μ. McKay Beckman and Conover(McKay, M.D., 1979) showed that in a great number of instances, the varaince of $\hat{\mu}_n$ is substantially smaller than that of the estimator based on simple random sampling. Stein(Stein, M.L., 1987) further proved that the asymptotic variance of $\hat{\mu}_n$ is less than the asymtotic variance of an analogous estimator based on an independently and identically distributed sample. Owen(Owen,A.B., 1992) showed that the multivariate central limit theorem holds for $\hat{\mu}_n$ when f is a bounded function. Assume that $Var(\hat{\mu}_n) > 0$, we define

$$W = \frac{\hat{\mu}_n - \mu}{\sqrt{Var(\hat{\mu}_n)}}.$$

In 1996, Loh(Loh, W.L., 1996) gave a uniform bound on the normal approximation of W but he yield the convergence rate $\frac{C}{\sqrt{n}}$ under the finiteness of third moments without the value of C. Neammanee and Rattanawong(Neammanee, K., 2009) give a constant C by using Stein's method. Their result described in Theorem 1.1.

Theorem 1.1 For each $i_1, ..., i_d \in \{1, ..., n\}$, let

$$\mu(i_1, ..., i_d) = Ef \circ X(i_1, ..., i_d),$$
$$U_k(i_1, ..., i_d) = \frac{(-1)^k}{n^k} \sum_{1\le j_1<j_2<...<j_k\le d} \sum_{q_{j_1}=1}^{n} ... \sum_{q_{j_k}=1}^{n} \mu(l_1, ..., l_d),$$

where

$$l_p = \begin{cases} q_p & \text{if } p = j_1, ..., j_k, \\ i_p & \text{otherwise,} \end{cases}$$

and

$$Y(i_1, ..., i_d) = \frac{1}{n\sqrt{var(\hat{\mu}_n)}}[f \circ X(i_1, ..., i_d) + \sum_{k=1}^{d-1} U_k(i_1, ..., i_d) + (-1)^d \mu]. \tag{1.1}$$

Suppose that $E(f \circ X(i_1, ..., i_d))^4 < \infty$, $1 \le i_1, ..., i_d \le n$. Then for $n \ge 6^d + 3$,

$$\sup_{z\in\mathbb{R}} |P(W \le z) - \Phi(z)| \le \frac{\sqrt{d}\delta_4^{\frac{1}{4}}}{n^{\frac{3}{8}}} + \frac{11.68}{\sqrt{n}} + 8.845d^{\frac{1}{4}}\delta_4 + 12.513\delta_4 + \frac{5.014d\delta_4^{\frac{3}{4}}}{n^{\frac{1}{8}}} + \frac{2\sqrt{2\pi}\delta_4^{\frac{3}{4}}}{n^{\frac{1}{8}}}$$

where Φ is the standard normal distribution, and

$$\delta_k = \frac{1}{n^{d-\frac{k-1}{2}}} \sum_{i_1=1}^{n} ... \sum_{i_d=1}^{n} E|Y(i_1, ..., i_d)|^k.$$

In this paper, we give a non-uniform bound on the normal approximation of W. Here is our result.

Theorem 1.2 Suppose $E(f \circ X(i_1, ..., i_d))^6 < \infty$. Then for $z \in \mathbb{R}$,

$$\left|P(W \le z) - \Phi(z)\right| \le \frac{C}{1+|z|}\{\frac{\delta_4^{\frac{1}{4}}}{n^{\frac{3}{8}}} + n^{\frac{5}{4}}\delta_4^{\frac{7}{2}} + \delta_3 + \delta_6^{\frac{1}{2}}(\delta_4^{\frac{1}{2}} + \delta_4)\}.$$

Furthermore, if $\delta_k \sim n^{-1/2} (k = 3, 4, 6)$,

$$\left|P(W \le z) - \Phi(z)\right| \le \frac{C}{(1+|z|)\sqrt{n}}.$$

To obtain a non-uniform bound for orthogonal array, Laipaporn and Neammanee(Laipaporn, K., 2007) assume 30^{th} moment. But in Theorem 1.2, we assume 6^{th} moment which is better than their result.

This paper is organized as follows. In section 2, we give a number of lemmas that are needed in proving our main theorem(Theorem 1.2). A proof of Theorem 1.2 is given in section 3.

2. Auxiliary Results

In this section, we shall give auxiliary results for proving our main theorem.

Neammanee and Rattanawong (Neammanee, K., 2009) showed that there exist random permutations $\rho_1, ..., \rho_{d-1}$ on $\{1, ..., n\}$ such that

$$W = \sum_{i=1}^{n} Y(i, \rho_1(i), ..., \rho_{d-1}(i)).$$

For fixed $z \geq 0$, we also define

$$Y_z(i_1, ..., i_d) = Y(i_1, ..., i_d)\mathbb{I}(|Y(i_1, ..., i_d)| > 1 + z),$$
$$\widehat{Y}_z(i_1, ..., i_d) = Y(i_1, ..., i_d)\mathbb{I}(|Y(i_1, ..., i_d)| \leq 1 + z),$$

where$\mathbb{I}$is the indicator function and

$$\widehat{Y}(\rho) = \sum_{i=1}^{n} \widehat{Y}_z(i, \rho_1(i), ..., \rho_{d-1}(i)).$$

To define a coupling pair $\widetilde{Y}(\rho)$ of $\widehat{Y}(\rho)$, let I and K be uniformly distributed random variables on $\{1, ..., n\}$, (I, K) uniformly distributed on $\{(i, k)|i, k = 1, ..., n, i \neq k\}$ and assume that they are independent of $\rho_1, ..., \rho_{d-1}$ and $Y(i_1, ..., i_d)$'s.

Let

$$\widetilde{Y}(\rho) = \widehat{Y}(\rho) - \widehat{S}_{1,z} - \widehat{S}_{2,z} + \widehat{S}_{3,z} + \widehat{S}_{4,z}$$

where

$$\widehat{S}_{1,z} = \widehat{Y}_z(I, \rho_1(I), ..., \rho_{d-1}(I)),\ \widehat{S}_{2,z} = \widehat{Y}_z(K, \rho_1(K), ..., \rho_{d-1}(K)),$$
$$\widehat{S}_{3,z} = \widehat{Y}_z(I, \rho_1(K), .., \rho_{d-1}(K)),\ \widehat{S}_{4,z} = \widehat{Y}_z(K, \rho_1(I), ..., \rho_{d-1}(I)).$$

Note that, by the same argument as in Neammanee and Suntornchost(Neammanee, K., 2005), $\widehat{Y}(\rho)$ and $\widetilde{Y}(\rho)$ is an exchangeable pair and $\widehat{S}_{i,z}$ for $i = 1, 2, 3, 4$ are identically distributed.

Lemma 2.1 Let g be a continuous and piecewise continuously differential function. Then,

$$E\widehat{Y}(\rho)g(\widehat{Y}(\rho)) = E\int_{-\infty}^{\infty} g'(\widehat{Y}(\rho) + t)K(t)dt + \Delta g(\widehat{Y}(\rho))$$

and

$$|\Delta g(\widehat{Y}(\rho))| \leq C\Big(\frac{\delta_4^{\frac{1}{4}}}{n^{\frac{3}{8}}} + \delta_4\Big)\Big(Eg^2(\widehat{Y}(\rho))\Big)^{\frac{1}{2}}$$

where

$$K(t) = \frac{n-1}{4}(\widetilde{Y}(\rho) - \widehat{Y}(\rho))(\mathbb{I}(0 \leq t \leq \widetilde{Y}(\rho) - \widehat{Y}(\rho)) - \mathbb{I}(\widetilde{Y}(\rho) - \widehat{Y}(\rho) \leq t < 0)).$$

proof Let $\mathcal{A}$ be the σ-algebra generated by

$$\{\widehat{Y}_z(i, \rho_1(i), ..., \rho_{d-1}(i)) : 1 \leq i \leq n\}.$$

By the same argument as in Neammanee and Suntornchost(Neammanee, K., 2005), we can show that

$$2E\{g(\widehat{Y}(\rho))E^{\mathcal{A}}(\widetilde{Y}(\rho) - \widehat{Y}(\rho))\} + E(\widetilde{Y}(\rho) - \widehat{Y}(\rho))[g(\widetilde{Y})(\rho) - g(\widehat{Y}(\rho))] = 0.$$

From this fact and the fact that

$$\begin{aligned}
&E^{\mathcal{A}}[\widetilde{Y}(\rho) - \widehat{Y}(\rho)] \\
&= E^{\mathcal{A}}[-\widehat{S}_{1,z} - \widehat{S}_{2,z} + \widehat{S}_{3,z} + \widehat{S}_{4,z}] \\
&= -E^{\mathcal{A}}\widehat{Y}_z(I, \rho_1(I), ..., \rho_{d-1}(I)) - E^{\mathcal{A}}\widehat{Y}_z(K, \rho_1(K), ..., \rho_{d-1}(K)) \\
&\quad + E^{\mathcal{A}}\widehat{Y}_z(I, \rho_1(K), ..., \rho_{d-1}(K)) + E^{\mathcal{A}}\widehat{Y}_z(K, \rho_1(I), ..., \rho_{d-1}(I)) \\
&= -\frac{2}{n}\sum_{i=1}^{n}\widehat{Y}_z(i, \rho_1(i), ..., \rho_{d-1}(i)) + \frac{2}{n(n-1)}E^{\mathcal{A}}\sum_{i=1}^{n}\sum_{\substack{k \\ k\neq i}}\widehat{Y}_z(i, \rho_1(k), ..., \rho_{d-1}(k)) \\
&= -\frac{2}{n}\widehat{Y}(\rho) + \frac{2}{n(n-1)}E^{\mathcal{A}}\sum_{i=1}^{n}\{\sum_{k=1}^{n}\widehat{Y}_z(i, \rho_1(k), ..., \rho_{d-1}(k)) \\
&\quad - \widehat{Y}_z(i, \rho_1(i), ..., \rho_{d-1}(i))\} \\
&= -\frac{2}{n-1}\widehat{Y}(\rho) + \frac{2}{n(n-1)}E^{\mathcal{A}}\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_z(i, \rho_1(k), ..., \rho_{d-1}(k)),
\end{aligned}$$

we have

$$
\begin{aligned}
E\widehat{Y}(\rho)g(\widehat{Y}(\rho)) &= \frac{n-1}{4}E(\widetilde{Y}(\rho)-\widehat{Y}(\rho))[g(\widetilde{Y}(\rho))-g(\widehat{Y}(\rho))] \\
&\quad + \frac{1}{n}Eg(\widehat{Y}(\rho))\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_z(i,\rho_1(k),...,\rho_{d-1}(k)) \\
&= \frac{n-1}{4}E(\widetilde{Y}(\rho)-\widehat{Y}(\rho))[g(\widetilde{Y}(\rho))-g(\widehat{Y}(\rho))] + \Delta g(\widehat{Y}(\rho)) \\
&= \frac{n-1}{4}E(\widetilde{Y}(\rho)-\widehat{Y}(\rho))\int_0^{\widetilde{Y}(\rho)-\widehat{Y}(\rho)} g'(\widehat{Y}(\rho)+t)dt + \Delta g(\widehat{Y}(\rho)) \\
&= E\int_{-\infty}^{\infty} g'(\widehat{Y}(\rho)+t)K(t)dt + \Delta g(\widehat{Y}(\rho))
\end{aligned}
$$

where

$$
\begin{aligned}
|\Delta g(\widehat{Y}(\rho))| &= \frac{1}{n}E|g(\widehat{Y}(\rho))\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_z(i,\rho_1(k),...,\rho_{d-1}(k))| \\
&\le \frac{1}{n}\{Eg^2(\widehat{Y}(\rho))\}^{\frac{1}{2}}\{E[\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_z(i,\rho_1(k),...,\rho_{d-1}(k))]^2\}^{\frac{1}{2}}.
\end{aligned}
$$

The proof will be completed, if we can show that

$$
E[\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_z(i,\rho_1(k),...,\rho_{d-1}(k))]^2 \le C\Big(n^{\frac{5}{4}}\delta_4^{\frac{1}{2}} + n^2\delta_4^2\Big).
$$

By the fact that

$$
\sum_{i_j}^{n} EY(i_1,...,i_d) = 0, \tag{2.1}
$$

for each $j \in \{1,...,d\}$(Neammanee, K., 2009), we have

$$
\begin{aligned}
&\sum_{i,k}\sum_{\substack{l,m \\ (l,m)\neq(i,k)}} EY(i,\rho_1(k),...,\rho_{d-1}(k))Y(l,\rho_1(m),...,\rho_{d-1}(m)) \\
&= \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{\substack{l \\ l\neq i}}\sum_{\substack{m \\ m\neq k}} EY(i,\rho_1(k),...,\rho_{d-1}(k))Y(l,\rho_1(m),...,\rho_{d-1}(m)) \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{\substack{l \\ l\neq i}} EY(i,\rho_1(k),...,\rho_{d-1}(k))Y(l,\rho_1(k),...,\rho_{d-1}(k)) \\
&\quad + \sum_{i=1}^{n}\sum_{k=1}^{n}\sum_{\substack{m \\ m\neq k}} EY(i,\rho_1(k),...,\rho_{d-1}(k))Y(i,\rho_1(m),...,\rho_{d-1}(m)) \\
&= \frac{1}{(n(n-1))^{(d-2)}}\sum_{k_1=1}^{n}...\sum_{k_d=1}^{n} EY(k_1,...,k_d)\sum_{\substack{m_1 \\ m_1\neq k_1}}...\sum_{\substack{m_d \\ m_d\neq k_d}} EY(m_1,...,m_d) \\
&\quad + \frac{1}{n^{(d-2)}}\sum_{k_1=1}^{n}...\sum_{k_d=1}^{n} EY(k_1,...,k_d)\sum_{\substack{l \\ l\neq k_1}} EY(l,k_2,...,k_d) \\
&\quad + \frac{1}{(n(n-1))^{(d-2)}}\sum_{k_1=1}^{n}...\sum_{k_d=1}^{n} EY(k_1,...,k_d)\sum_{\substack{m_2 \\ m_2\neq k_2}}...\sum_{\substack{m_d \\ m_d\neq k_d}} EY(k_1,m_2,...,m_d) \\
&\le \frac{C}{n^{d-2}}\sum_{k_1=1}^{n}...\sum_{k_d=1}^{n}[EY(k_1,...,k_d)]^2 \\
&= Cn\sqrt{n}\delta_2.
\end{aligned}
$$

Thus

$$E[\sum_{i=1}^{n}\sum_{k=1}^{n}Y(i,\rho_1(k),...,\rho_{d-1}(k))]^2 \le \sum_{i=1}^{n}\sum_{k=1}^{n}EY^2(i,\rho_1(k),...,\rho_{d-1}(k)) + Cn\sqrt{n}\delta_2$$
$$= Cn\sqrt{n}\delta_2. \tag{2.2}$$

Using the fact that $2ab \le a^2+b^2$ and

$$\begin{aligned} &E|Y(i_1,...,i_d)|^m|Y_z(i_1,...,i_d)|^n \\ &\le E|Y(i_1,...,i_d)|^m|Y_z(i_1,...,i_d)|^n(\frac{|Y_z(i_1,...,i_d)|}{1+z})^t \\ &\le \frac{E|Y(i_1,...,i_d)|^{m+n+t}}{(1+z)^t} \end{aligned} \tag{2.3}$$

for any integers m,n and t which $m\ge 0, n,t>0$,
we have

$$\begin{aligned} &\sum_{i,k}\sum_{\substack{l,m\\(l,m)\neq(i,k)}} EY_z(i,\rho_1(k),...,\rho_{d-1}(k))Y_z(l,\rho_1(m),...,\rho_{d-1}(m)) \\ &= \frac{1}{(n(n-1))^{(d-2)}}\sum_{k_1=1}^{n}...\sum_{k_d=1}^{n}EY_z(k_1,...,k_d)\sum_{\substack{m_1\\m_1\neq k_1}}...\sum_{\substack{m_d\\m_d\neq k_d}}EY_z(m_1,...,m_d) \\ &+ \frac{1}{n^{(d-2)}}\sum_{k_1=1}^{n}...\sum_{k_d=1}^{n}EY_z(k_1,...,k_d)\sum_{\substack{l\\l\neq k_1}}EY_z(l,k_2,...,k_d) \\ &+ \frac{1}{(n(n-1))^{(d-2)}}\sum_{k_1=1}^{n}...\sum_{k_d=1}^{n}EY_z(k_1,...,k_d)\sum_{\substack{m_2\\m_2\neq k_2}}...\sum_{\substack{m_d\\m_d\neq k_d}}EY_z(k_1,m_2,...,m_d) \\ &\le \frac{C}{n^{2(d-2)}}[\sum_{k_1=1}^{n}...\sum_{k_d=1}^{n}E|Y_z(k_1,...,k_d)|]^2 + \frac{C}{n^{d-3}}\sum_{k_1=1}^{n}...\sum_{k_d=1}^{n}EY_z^2(k_1,...,k_d) \\ &\le \frac{C}{n^{2(d-2)}}[\sum_{k_1=1}^{n}...\sum_{k_d=1}^{n}\frac{EY^4(k_1,...,k_d)}{(1+z)^3}]^2 + \frac{C}{n^{d-3}}\sum_{k_1=1}^{n}...\sum_{k_d=1}^{n}\frac{EY^4(k_1,...,k_d)}{(1+z)^2} \\ &\le C\Big(n\delta_4^2 + n\sqrt{n}\delta_4\Big). \end{aligned}$$

Thus

$$\begin{aligned} &E[\sum_{i=1}^{n}\sum_{k=1}^{n}Y_z(i,\rho_1(k),...,\rho_{d-1}(k))]^2 \\ &\le \sum_{i=1}^{n}\sum_{k=1}^{n}EY_z^2(i,\rho_1(k),...,\rho_{d-1}(k)) + C\Big(n\delta_4^2+n\sqrt{n}\delta_4\Big) \\ &= C\Big(n\sqrt{n}\delta_2 + n\delta_4^2 + n\sqrt{n}\delta_4\Big). \end{aligned} \tag{2.4}$$

Now, we conclude from (2.2) and (2.4) that

$$\begin{aligned} &E[\sum_{i=1}^{n}\sum_{k=1}^{n}\widehat{Y}_z(i,\rho_1(k),...,\rho_{d-1}(k))]^2 \\ &= E[\sum_{i=1}^{n}\sum_{k=1}^{n}Y(i,\rho_1(k),...,\rho_{d-1}(k)) - \sum_{i=1}^{n}\sum_{k=1}^{n}Y_z(i,\rho_1(k),...,\rho_{d-1}(k))]^2 \\ &\le 2E[\sum_{i=1}^{n}\sum_{k=1}^{n}Y(i,\rho_1(k),...,\rho_{d-1}(k))]^2 + 2E[\sum_{i=1}^{n}\sum_{k=1}^{n}Y_z(i,\rho_1(k),...,\rho_{d-1}(k))]^2 \\ &\le C\Big(n\sqrt{n}\delta_2 + n\delta_4^2 + n\sqrt{n}\delta_4\Big) \\ &\le C\Big(n^{\frac{5}{4}}\delta_4^{\frac{1}{2}} + n\delta_4^2 + n\sqrt{n}\delta_4\Big) \\ &\le C\Big(n^{\frac{5}{4}}\delta_4^{\frac{1}{2}} + n^2\delta_4^2\Big) \end{aligned}$$

where we have used the fact that

$$\delta_2^2 = \frac{1}{n^{2d-1}}[\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} EY^2(i_1,...,i_d)]^2 \le \frac{1}{n^{d-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} EY^4(i_1,...,i_d) = \frac{\delta_4}{\sqrt{n}} \tag{2.5}$$

in the third inequality and $n\sqrt{n}\delta_4 \le n^{\frac{5}{4}}\delta_4^{\frac{1}{2}} + n^2\delta_4^2$ in the last inequality.

Lemma 2.2 Suppose $EY^4(i_1,...,i_d) < \infty$, $1 \le i_1,...,i_d \le n$. Then

$$E\widehat{Y}^4(\rho) \le C\Big(\sqrt{n}\delta_4 + \delta_4^2\Big).$$

proof We observe that

$$E(\sum_{i=1}^{n} Y_z(i,\rho_1(i),...,\rho_{d-1}(i))^4 = M_1 + M_2 + M_3 + M_4 + M_5$$

where

$$\begin{aligned}
M_1 &= \sum_{i=1}^{n} EY_z^4(i,\rho_1(i),...,\rho_{d-1}(i)),\\
M_2 &= \sum_{i=1}^{n}\sum_{\substack{j\\ j\neq i}} EY_z(i,\rho_1(i),...,\rho_{d-1}(i))Y_z^3(j,\rho_1(j),...,\rho_{d-1}(j)),\\
M_3 &= \sum_{i=1}^{n}\sum_{\substack{j\\ j\neq i}} EY_z^2(i,\rho_1(i),...,\rho_{d-1}(i))Y_z^2(j,\rho_1(j),...,\rho_{d-1}(j)),\\
M_4 &= \sum_{i=1}^{n}\sum_{\substack{j\\ j\neq i}}\sum_{\substack{k\\ k\neq i,j}} EY_z^2(i,\rho_1(i),...,\rho_{d-1}(i))Y_z(j,\rho_1(j),...,\rho_{d-1}(j))\\
&\quad Y_z(k,\rho_1(k),...,\rho_{d-1}(k)) \quad \text{and}\\
M_5 &= \sum_{i=1}^{n}\sum_{\substack{j\\ j\neq i}}\sum_{\substack{k\\ k\neq i,j}}\sum_{\substack{l\\ l\neq i,j,k}} EY_z(i,\rho_1(i),...,\rho_{d-1}(i))Y_z(j,\rho_1(j),...,\rho_{d-1}(j))\\
&\quad Y_z(k,\rho_1(k),...,\rho_{d-1}(k))Y_z(l,\rho_1(l),...,\rho_{d-1}(l)).
\end{aligned}$$

We note that

$$|M_1| = \frac{1}{n^{d-1}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} EY_z^4(i_1,...,i_d) = \frac{\delta_4}{\sqrt{n}}.$$

It follows from (2.3) and (2.5)that

$$\begin{aligned}
|M_2| &\le \frac{C}{n^{2(d-1)}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} E|Y_z(i_1,...,i_d)|\sum_{j_1=1}^{n}\cdots\sum_{j_d=1}^{n} E|Y_z(j_1,...,j_d)|^3\\
&\le \frac{C}{n^{2(d-1)}}[\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n} EY^4(i_1,...,i_d)]^2\\
&= \frac{C\delta_4^2}{n},
\end{aligned}$$

$$
\begin{aligned}
|M_3| &\le \frac{C}{n^{2(d-1)}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}\sum_{j_1=1}^{n}\cdots\sum_{j_d=1}^{n}EY_z^2(i_1,...,i_d)EY_z^2(j_1,...,j_d)\\
&\le \frac{C}{n^{2(d-1)}}[\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}EY^4(i_1,...,i_d)]^2\\
&= \frac{C\delta_4^2}{n},\\
|M_4| &\le \frac{C}{n^{3(d-1)}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}EY_z^2(i_1,...,i_d)[\sum_{j_1=1}^{n}\cdots\sum_{j_d=1}^{n}E|Y_z(j_1,...,j_d)|]^2\\
&= \frac{C}{n^{3(d-1)}}\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}EY^4(i_1,...,i_d)[\sum_{j_1=1}^{n}\cdots\sum_{j_d=1}^{n}E|Y(j_1,...,j_d)|^2]^2\\
&= C\sqrt{n}\delta_4\delta_2^2\\
&\le C\delta_4^2 \text{ and}\\
|M_5| &\le \frac{C}{n^{4(d-1)}}[\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}E|Y_z(i_1,...,i_d)|]^4\\
&\le \frac{C}{n^{4(d-1)}}[\sum_{i_1=1}^{n}\cdots\sum_{i_d=1}^{n}EY^2(i_1,...,i_d)]^4\\
&= C\delta_2^4\\
&\le \frac{C\delta_4^2}{n}.
\end{aligned}
$$

Thus

$$E(\sum_{i=1}^{n}Y_z(i,\rho_1(i),...,\rho_{d-1}(i)))^4 \le C\Big(\frac{\delta_4}{\sqrt{n}}+\delta_4^2\Big).$$

By this fact and $EW^4 \le C\sqrt{n}\delta_4$(Neammanee, K., 2008),

$$
\begin{aligned}
E\widehat{Y}^4(\rho) &= E(\sum_{i=1}^{n}Y(i,\rho_1(i),...,\rho_{d-1}(i)) - \sum_{i=1}^{n}Y_z(i,\rho_1(i),...,\rho_{d-1}(i)))^4\\
&\le EW^4 + E(\sum_{i=1}^{n}Y_z(i,\rho_1(i),...,\rho_{d-1}(i)))^4\\
&\le C\Big(\sqrt{n}\delta_4+\delta_4^2\Big).
\end{aligned}
$$

Lemma 2.3

$$E(\widetilde{Y}(\rho)-\widehat{Y}(\rho))^2 = \frac{4}{n}+\Delta$$

where $|\Delta| \le C\Big(\frac{\delta_4^{\frac{1}{2}}}{n^{\frac{5}{4}}}+\frac{\delta_4}{n}\Big)$.

proof Let

$$
\begin{aligned}
S_1 &= Y(I,\rho_1(I),...,\rho_{d-1}(I)), & S_{1,z} &= Y_z(I,\rho_1(I),...,\rho_{d-1}(I)),\\
S_2 &= Y(K,\rho_1(K),...,\rho_{d-1}(K)), & S_{2,z} &= Y_z(K,\rho_1(K),...,\rho_{d-1}(K))\\
S_3 &= Y(I,\rho_1(K),...,\rho_{d-1}(K)), & S_{3,z} &= Y_z(I,\rho_1(K),...,\rho_{d-1}(K)), \text{ and}\\
S_4 &= Y(K,\rho_1(I),...,\rho_{d-1}(I)), & S_{4,z} &= Y_z(K,\rho_1(I),...,\rho_{d-1}(I)).
\end{aligned}
$$

We observe that

$$E(\widetilde{Y}(\rho)-\widehat{Y}(\rho))^2 = \sum_{k=1}^{4}ES_k^2+\Delta_1 \tag{2.6}$$

where

$$|\Delta_1| \le \sum_{1\le i<j\le 4} |ES_iS_j| + 4\sum_{i=1}^{4} ES_{i,z}^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} E|S_iS_{j,z}|$$

and, by (2.1),

$$\begin{aligned}
ES_1^2 &= \frac{1}{n}\sum_{i=1}^{n} EY^2(i,\rho_1(i),...,\rho_{d-1}(i))\\
&= \frac{1}{n}[EW^2 - \sum_{i=1}^{n}\sum_{\substack{j\\ j\ne i}} EY(i,\rho_1(i),...,\rho_{d-1}(i))Y(j,\rho_1(j),...,\rho_{d-1}(j))]\\
&= \frac{1}{n} - \frac{1}{n^d(n-1)^{d-1}}\sum_{i_1=1}^{n}...\sum_{i_d=1}^{n}\sum_{\substack{j_1\\ j_1\ne i_1}}...\sum_{\substack{j_d\\ j_d\ne i_d}} EY(i_1,...,i_d)EY(j_1,...,j_d)\\
&= \frac{1}{n} - \frac{(-1)^d}{n^d(n-1)^{d-1}}\sum_{i_1=1}^{n}...\sum_{i_d=1}^{n}[EY(i_1,...,i_d)]^2. \qquad (2.7)
\end{aligned}$$

Thus, from (2.6), (2.7) and the fact that S_1, S_2, S_3, S_4 have the same distribution, we have

$$E(\widetilde{Y}(\rho) - \widehat{Y}(\rho))^2 = \frac{4}{n} + \Delta_2$$

where

$$|\Delta_2| \le \frac{C\delta_2}{n\sqrt{n}} + \sum_{1\le i<j\le 4} |ES_iS_j| + 4\sum_{i=1}^{4} ES_{i,z}^2 + \sum_{i=1}^{n}\sum_{j=1}^{n} E|S_iS_{j,z}|. \qquad (2.8)$$

To prove the lemma, it suffices to find appropriate bounds the right-hand side of (2.8). We note from (2.3) that

$$\begin{aligned}
ES_{1,z}^2 &= \frac{1}{n}\sum_{i=1}^{n} EY_z^2(i,\rho_1(i),...,\rho_{d-1}(i))\\
&= \frac{1}{n^d}\sum_{i_1=1}^{n}...\sum_{i_d=1}^{n} EY_z^2(i_1,...,i_d)\\
&\le \frac{C}{n^d}\sum_{i_1=1}^{n}...\sum_{i_d=1}^{n} EY^4(i_1,...,i_d)\\
&= \frac{C\delta_4}{n\sqrt{n}}. \qquad (2.9)
\end{aligned}$$

From this fact, (2.5) and the fact that $E|S_1|^k \le \frac{\delta_k}{n^{\frac{k-1}{2}}}$ for $k \in \mathbb{N}$, we have

$$E|S_iS_{j,z}| \le \{ES_i^2\}^{\frac12}\{ES_{j,z}^2\}^{\frac12} = \{ES_1^2\}^{\frac12}\{ES_{1,z}^2\}^{\frac12} \le \frac{C\delta_2^{\frac12}\delta_4^{\frac12}}{n} \le \frac{C\delta_4^{\frac34}}{n^{\frac98}} \qquad (2.10)$$

for $i, j = 1, 2, 3, 4$. Note from (2.1) that

$$\begin{aligned}
|ES_1S_2| &= |EY(I,\rho_1(I),...,\rho_{d-1}(I))Y(K,\rho_1(K),...,\rho_{d-1}(K))|\\
&= |\frac{1}{n(n-1)}\sum_{i=1}^{n}\sum_{\substack{k\\ k\ne i}} EY(i,\rho_1(i),...,\rho_{d-1}(i))Y(k,\rho_1(k),...,\rho_{d-1}(k))|\\
&= |\frac{1}{(n(n-1))^d}\sum_{i_1=1}^{n}...\sum_{i_d=1}^{n} EY(i_1,...,i_d)\sum_{\substack{k_1\\ k_1\ne i_1}}...\sum_{\substack{k_d\\ k_d\ne i_d}} EY(k_1,...,k_d)|\\
&= |\frac{1}{(n(n-1))^d}\sum_{i_1=1}^{n}...\sum_{i_d=1}^{n}(EY(i_1,...,i_d))^2|\\
&= \frac{C\delta_2}{n\sqrt{n}}.
\end{aligned}$$

By the same argument, we can show that

$$|ES_iS_j| \leq \frac{C\delta_2}{n\sqrt{n}} \tag{2.11}$$

for $1 \leq i < j \leq 4$. Now we conclude from (2.8) - (2.11) that

$$|\Delta| \leq C\Big(\frac{\delta_2}{n\sqrt{n}} + \frac{\delta_4^{\frac{3}{4}}}{n^{\frac{9}{8}}} + \frac{\delta_4}{n\sqrt{n}}\Big) \leq C\Big(\frac{\delta_4^{\frac{1}{2}}}{n^{\frac{7}{4}}} + \frac{\delta_4^{\frac{3}{4}}}{n^{\frac{9}{8}}} + \frac{\delta_4}{n\sqrt{n}}\Big) \leq C\Big(\frac{\delta_4^{\frac{1}{2}}}{n^{\frac{5}{4}}} + \frac{\delta_4}{n}\Big).$$

3. Proof of Theorem 1.2

To bound $|P(W \leq z) - \Phi(z)|$, it suffices to consider $z \geq 0$ as we have used the fact that $\Phi(z) = 1 - \Phi(-z)$ and apply the result to $-W$ when $z < 0$. So, from now on, we assume $z \geq 0$.
Note that

$$|P(W \leq z) - \Phi(z)| \leq P(W \neq \widehat{Y}(\rho)) + |P(\widehat{Y}(\rho) \leq z) - \Phi(z)| \tag{3.1}$$

and

$$\begin{aligned}
P(W \neq \widehat{Y}(\rho)) &= P\Big(\sum_{i=1}^{n} \mathbb{I}(|Y(i, \rho_1(i), ..., \rho_{d-1}(i))| > 1 + z) \geq 1\Big) \\
&\leq \sum_{i=1}^{n} E\mathbb{I}(|Y(i, \rho_1(i), ..., \rho_{d-1}(i))| > 1 + z) \\
&= \frac{1}{n^{d-1}} \sum_{i_1=1}^{n} ... \sum_{i_d=1}^{n} E\mathbb{I}(|Y(i_1, ..., i_d)| > 1 + z) \\
&= \frac{1}{n^{d-1}} \sum_{i_1=1}^{n} ... \sum_{i_d=1}^{n} P(|Y(i_1, ..., i_d)| > 1 + z) \\
&\leq \frac{1}{n^{d-1}} \sum_{i_1=1}^{n} ... \sum_{i_d=1}^{n} \frac{E|Y(i_1, ..., i_d)|^3}{(1 + z)^3} \\
&= \frac{\delta_3}{(1 + z)^3}.
\end{aligned} \tag{3.2}$$

Hence, it suffices to bound $|P(\widehat{Y}(\rho) \leq z) - \Phi(z)|$. To do this we will apply Stein's method and the idea from Neammanee and Rattanawong(Neammanee, K., 2008).

In 1972, Stein(Stein, C.M., 1972) introduced a powerful and general method for obtaining explicit bound for the error in the normal approximation to the distribution of a sum of dependent random variables. His technique was relied instead on the elementary differantial equation. The Stein's equation for normal distribution function is

$$g'(w) - wg(w) = \mathbb{I}(w \leq z) - \Phi(z) \quad \text{for } w \in \mathbb{R}. \tag{3.3}$$

It is well-known that the solution g_z of (3.3) is of the form

$$g_z(w) = \begin{cases} \sqrt{2\pi}e^{\frac{1}{2}w^2}\Phi(w)[1 - \Phi(z)] & \text{if } w \leq z, \\ \sqrt{2\pi}e^{\frac{1}{2}w^2}\Phi(z)[1 - \Phi(w)] & \text{if } w > z, \end{cases}$$

with $\quad 0 \leq g_z(w) \leq 1$ for all $w \in \mathbb{R}$,

$$0 < g_z(w) \leq min\Big(\frac{\sqrt{2\pi}}{4}, \frac{1}{|z|}\Big) \text{ for all } w \in \mathbb{R} \text{ and } z \neq 0, \tag{3.4}$$

$$|g_z'(w) - g_z'(v)| \leq 1, \text{ for all real } w, v, \tag{3.5}$$

$$\text{and} \quad |g_z'(w)| \leq 1 \text{ for all } w \in \mathbb{R} \tag{3.6}$$

(Stein(1972, pp.22-23)).
From (3.3),

$$|P(\widehat{Y}(\rho) \leq z) - \Phi(z)| = |Eg_z'(\widehat{Y}(\rho)) - E\widehat{Y}(\rho)g_z(\widehat{Y}(\rho))|. \tag{3.7}$$

To bound the right handside of (3.7), Neammanee and Rattanawong(2008) construct the random permutation $\tau_1, \tau_2, \ldots, \tau_{d-1}$ in the followings.

Let $I, K, L_1, ..., L_{d-1}, M_1, ..., M_{d-1}$ be random variables with uniformly distribution on $\{1, 2, ..., n\}$ and $\rho_1, ..., \rho_{d-1}, \tau_1, ..., \tau_{d-1}$ are random permutations of $\{1, 2, ..., n\}$. Assume that
$\{I, K, L_1, ..., L_{d-1}, M_1, ..., M_{d-1}, \rho_1, ..., \rho_{d-1}, \tau_1, ..., \tau_{d-1}\}$ is independent of $Y(i_1, ..., i_d)'$s,
$(I, K), (L_1, M_1), ..., (L_{d-1}, M_{d-1})$ are uniformly distributed on
$\{(i, k)|i, k = 1, 2, ..., n$ and $i \neq k$,
$(I, K), (L_1, M_1), ..., (L_{d-1}, M_{d-1})$ and$\tau_1, ..., \tau_{d-1}$ are mutually independent,
(I, K) and $\rho_1, ..., \rho_{d-1}$ are mutually independent, and

$$\rho_i(\alpha) = \begin{cases} \tau_i(\alpha) & \text{if } \alpha \neq I, K, \tau_i^{-1}(L_i), \tau_i^{-1}(M_i), \\ L_i & \text{if } \alpha = I, \\ M_i & \text{if } \alpha = K, \\ \tau_i(I) & \text{if } \alpha = \tau_i^{-1}(L_i), \\ \tau_i(K) & \text{if } \alpha = \tau_i^{-1}(M_i), \end{cases}$$

where $\rho_i(\rho_i^{-1}(\alpha)) = \rho_i^{-1}(\rho_i(\alpha)) = \alpha,$ for $i = 1, ..., d-1$.
Then, they showed that

$$\begin{aligned} |g_z'(\widehat{Y}(\rho)) - \widehat{Y}(\rho)g_z(\widehat{Y}(\rho))| \le\ & |Eg_z'(\widehat{Y}(\tau)) \int_{-\infty}^{\infty} K(t)dt - E\int_{-\infty}^{\infty} g_z'(\widehat{Y}(\rho)+t)K(t)dt| \\ & + |Eg_z'(\widehat{Y}(\tau))E\int_{-\infty}^{\infty} K(t)dt - Eg_z'(\widehat{Y}(\tau))\int_{-\infty}^{\infty} K(t)dt| \\ & + |Eg_z'(\widehat{Y}(\tau)) - Eg_z'(\widehat{Y}(\tau))E\int_{-\infty}^{\infty} K(t)dt| \\ & + |\Delta g_z(\widehat{Y}(\rho))| \\ :=\ & |T_1| + |T_2| + |T_3| + |T_4|. \end{aligned} \tag{3.8}$$

We observe from Lemma 2.1 and (3.4) that

$$|T_4| \le C\{\frac{\delta_4^{\frac{1}{4}}}{n^{\frac{3}{8}}} + \delta_4\}\{Eg_z^2(\widehat{Y}(\rho))\}^{\frac{1}{2}} \le \frac{C}{1+z}\{\frac{\delta_4^{\frac{1}{4}}}{n^{\frac{3}{8}}} + \delta_4\}. \tag{3.9}$$

By the fact that

$$|g_z'(w+s) - g_z'(w+t) - \int_t^s h(w+u)du| \le \mathbb{I}(z - max(s,t) < w < z - min(s,t))$$

(Chen, L.H.Y., 2001), we have

$$\begin{aligned} |T_1| &\le E\int_{-\infty}^{\infty} |g_z'(\widehat{Y}(\tau)) - g_z'(\widehat{Y}(\rho)+t)|K(t)dt \\ &= E\int_{-\infty}^{\infty} |g_z'(\widehat{Y}(\rho)+\Delta\widehat{Y}) - g_z'(\widehat{Y}(\rho)+t)|K(t)dt \\ &\le T_{11} + T_{12} + T_{13} \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} \Delta\widehat{Y} &= \widehat{Y}(\tau) - \widehat{Y}(\rho) \\ T_{11} &= E\mathbb{I}(\delta < \frac{z}{4})\int_{-\infty}^{\infty} \mathbb{I}(z - max(\Delta\widehat{Y}, t) < \widehat{Y}(\rho) < z - min(\Delta\widehat{Y}, t))K(t)dt, \\ T_{12} &= E\mathbb{I}(\delta < \frac{z}{4})\int_{-\infty}^{\infty}\int_t^{\Delta\widehat{Y}} h(\widehat{Y}(\rho)+u)K(t)dudt, \\ T_{13} &= E\mathbb{I}(\delta \ge \frac{z}{4})\int_{-\infty}^{\infty} |g_z'(\widehat{Y}(\rho)+\Delta\widehat{Y}) - g_z'(\widehat{Y}(\rho)+t)|K(t)dt, \\ \delta &= |\Delta\widehat{Y}| + |\widetilde{Y}(\rho) - \widehat{Y}(\rho)| \quad \text{and} \\ h(w) &= (wg_z(w))'. \end{aligned}$$

Hence, by (3.1), (3.2), (3.7)-(3.10),

$$|P(W \le z) - \Phi(z)| \le T_{11} + T_{12} + T_{13} + |T_2| + |T_3| + \frac{C}{1+z}(\frac{\delta_4^{\frac{1}{4}}}{n^{\frac{3}{8}}} + \delta_3 + \delta_4). \tag{3.11}$$

Step 1. $T_{11} \le \frac{C}{1+z}\Big(\frac{\delta_4^{\frac{1}{4}}}{n^{\frac{3}{8}}} + n\delta_4^{\frac{7}{2}}\Big)$.

We used the idea from Chen and Shao(Chen, L.H.Y., 2001) to define $f_\delta : \mathbb{R} \to \mathbb{R}$ by

$$f_\delta(t) = \begin{cases} 0 & \text{if } t < z - 2\delta, \\ (1+t+\delta)(t-z+2\delta) & \text{if } z-2\delta \le t \le z+2\delta, \\ 4\delta(1+t+\delta) & \text{if } t > z+2\delta. \end{cases}$$

Note that

$$|f_\delta(t)| \le 4\delta(1+|t|+\delta) \qquad \text{for all } t \tag{3.12}$$

and

$$f'_\delta(t) \ge \begin{cases} 1+z-\delta & \text{if } z-2\delta \le t \le z+2\delta, \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

By the same argument as in Lemma 2.1, we have

$$E\int_{-\infty}^{\infty} f'_\delta(\widehat{Y}(\rho)+t)K(t)dt \le E\widehat{Y}(\rho)f_\delta(\widehat{Y}(\rho)) + C\Big(\frac{\delta_4^{\frac{1}{4}}}{n^{\frac{3}{8}}} + \delta_4\Big)\Big(Ef_\delta^2(\widehat{Y}(\rho))\Big)^{\frac{1}{2}}. \tag{3.14}$$

Thus, by (3.13) and (3.14),

$$\begin{aligned} T_{11} &\le E\int_{|t|\le|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|} \mathbb{I}(\delta < \frac{z}{4})\mathbb{I}(z-(|\Delta\widehat{Y}|+|t|) < \widehat{Y}(\rho) < z+(|\Delta\widehat{Y}|+|t|))K(t)dt \\ &\le E\int_{|t|\le|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|} \mathbb{I}(\delta < \frac{z}{4})\mathbb{I}(z-\delta < \widehat{Y}(\rho) < z+\delta)K(t)dt \\ &\le \frac{C}{1+z}E\int_{|t|\le|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|} \mathbb{I}(\delta < \frac{z}{4})(1+z-\delta)\mathbb{I}(z-\delta < \widehat{Y}(\rho) < z+\delta)K(t)dt \\ &\le \frac{C}{1+z}E\int_{|t|\le|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|} f'_\delta(\widehat{Y}(\rho)+t)K(t)dt \\ &\le \frac{C}{1+z}[E\widehat{Y}(\rho)f_\delta(\widehat{Y}(\rho)) + (\frac{\delta_4^{\frac{1}{4}}}{n^{\frac{3}{8}}} + \delta_4)(Ef_\delta^2(\widehat{Y}(\rho)))^{\frac{1}{2}}]. \end{aligned} \tag{3.15}$$

To bound $E|\widehat{Y}(\rho)f_\delta(\widehat{Y}(\rho))|$ and $Ef_\delta^2(\widehat{Y}(\rho))$, we note that

$$\begin{aligned} E\widehat{Y}^2(\rho) &= E[\sum_{i=1}^n \widehat{Y}_z(i,\rho_1(i),...,\rho_{d-1}(i))]^2 \\ &= E[\sum_{i=1}^n Y(i,\rho_1(i),...,\rho_{d-1}(i)) - \sum_{i=1}^n Y_z(i,\rho_1(i),...,\rho_{d-1}(i))]^2 \\ &\le E[\sum_{i=1}^n Y(i,\rho_1(i),...,\rho_{d-1}(i))]^2 + E[\sum_{i=1}^n Y_z(i,\rho_1(i),...,\rho_{d-1}(i))]^2 \\ &\le EW^2 + n\sum_{i=1}^n EY_z^2(i,\rho_1(i),...,\rho_{d-1}(i)) \\ &= 1 + \frac{1}{n^{(d-2)}}\sum_{i_1=1}^n \cdots \sum_{i_{d-1}=1}^n EY_z^2(i_1,...,i_d) \\ &\le 1 + \frac{1}{n^{(d-2)}}\sum_{i_1=1}^n \cdots \sum_{i_{d-1}=1}^n \frac{EY^4(i_1,...,i_d)}{(1+z)^2} \\ &= 1 + C\sqrt{n}\delta_4, \end{aligned} \tag{3.16}$$

$$E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^k \le C\sum_{i=1}^4 E|\widehat{S}_{i,z}|^k = CE|\widehat{S}_{1,z}|^k \le \frac{C\delta_k}{n^{\frac{k-1}{2}}}, \tag{3.17}$$

$$E|\Delta\widehat{Y}|^k \le \frac{C\delta_k}{n^{\frac{k-1}{2}}} \tag{3.18}$$

and

$$E\delta^k \le \frac{C\delta_k}{n^{\frac{k-1}{2}}} \quad \text{for all } \ k \in \mathbb{N}. \tag{3.19}$$

We conclude from Lemma 2.2, (2.5), (3.12), (3.16) - (3.19) and Cauchy - Schwarz inequality,

$$\begin{aligned} E|\widehat{Y}(\rho) f_\delta(\widehat{Y}(\rho))| &\le C\Big(E|\widehat{Y}(\rho)|\delta + E\widehat{Y}^2(\rho)\delta + E|\widehat{Y}(\rho)|\delta^2\Big) \\ &\le C\Big(\frac{\delta_4^{\frac{1}{4}}}{n^{\frac{3}{8}}} + \frac{\delta_4^{\frac{3}{4}}}{n^{\frac{1}{8}}} + \frac{\delta_4^{\frac{5}{4}}}{n^{\frac{3}{8}}} + \frac{\delta_4^{\frac{1}{2}}}{n^{\frac{3}{4}}} + \frac{\delta_4}{n^{\frac{1}{2}}}\Big) \\ &\le C\Big(\frac{\delta_4^{\frac{1}{4}}}{n^{\frac{3}{8}}} + n^{\frac{1}{8}}\delta_4^{\frac{5}{4}}\Big) \end{aligned}$$

and

$$E f_\delta^2(\widehat{Y}(\rho)) \le CE(\delta^2 + \delta^2\widehat{Y}^2(\rho) + \delta^4) \le \frac{\delta_4^{\frac{1}{2}}}{n^{\frac{3}{4}}} + \frac{\delta_4^{\frac{3}{2}}}{n^{\frac{1}{4}}}.$$

From this fact and (3.15), $T_{11} \le \frac{C}{1+z}\Big(\frac{\delta_4^{\frac{1}{4}}}{n^{\frac{3}{8}}} + n\delta_4^{\frac{7}{2}}\Big)$.

Step 2. $T_{12} + T_{13} \le \frac{C}{1+z}(\delta_3 + \delta_4^{\frac{1}{2}}\delta_6^{\frac{1}{2}} + \delta_4\delta_6^{\frac{1}{2}})$.

Since

$$\begin{aligned} E\int_{-\infty}^{\infty}\int_t^{\Delta\widehat{Y}} K(t)dudt &\le E\int_{-\infty}^{\infty}(|\Delta\widehat{Y}| + |t|)K(t)dt \\ &\le Cn(E|\Delta\widehat{Y}|(\widetilde{Y}(\rho) - \widehat{Y}(\rho))^2 + E|\widetilde{Y}(\rho) - \widehat{Y}(\rho)|^3) \\ &\le C\delta_3, \end{aligned}$$

and

$$h(w) \le \begin{cases} C(1+z) & \text{if } \ \frac{z}{2} < w \le z, \\ \dfrac{C}{(1+z)^2} & \text{if } \ w \le \frac{z}{2} \text{ or } w > z \end{cases}$$

(Laipaporn, K., 2007),

$$\begin{aligned} &T_{12} \\ &\le E\int_{-\infty}^{\infty}\int_t^{\Delta\widehat{Y}} h(\widehat{Y}(\rho)+u)K(t)\mathbb{I}(\widehat{Y}(\rho)+u \le \frac{z}{2} \text{ or } \widehat{Y}(\rho)+u > z)dudt \\ &\quad + E\mathbb{I}(\delta < \frac{z}{4})\int_{-\infty}^{\infty}\int_t^{\Delta\widehat{Y}} h(\widehat{Y}(\rho)+u)K(t)\mathbb{I}(\frac{z}{2} < \widehat{Y}(\rho)+u \le z)dudt \\ &\le \frac{C\delta_3}{(1+z)^2} + C(1+z)E\mathbb{I}(\delta < \frac{z}{4})\int_{-\infty}^{\infty}\int_t^{\Delta\widehat{Y}} K(t)\mathbb{I}(\widehat{Y}(\rho)+u > \frac{z}{2})dudt. \end{aligned} \tag{3.20}$$

It remains to bound the second term on the right hand side of (3.20). By the fact that $K(t) = 0$ for $|t| > |\widehat{Y}(\rho) - \widetilde{Y}(\rho)|$, Lemma 2.2, (3.17) and (3.18),

$$\begin{aligned} &E\mathbb{I}(\delta < \frac{z}{4})\int_{-\infty}^{\infty}\int_t^{\Delta\widehat{Y}} K(t)\mathbb{I}(\widehat{Y}(\rho)+u > \frac{z}{2})dudt \\ &\le E\mathbb{I}(\delta < \frac{z}{4})\int_{-\infty}^{\infty}\int_t^{\Delta\widehat{Y}} K(t)\mathbb{I}(\widehat{Y}(\rho)+\delta > \frac{z}{2})dudt \\ &= E\int_{-\infty}^{\infty}\int_t^{\Delta\widehat{Y}} K(t)\mathbb{I}(\widehat{Y}(\rho) > \frac{z}{4})dudt \\ &\le E\mathbb{I}(\widehat{Y}(\rho) > \frac{z}{4})\int_{-\infty}^{\infty}(|\Delta\widehat{Y}| + |t|)K(t)dt \\ &\le \frac{n}{4}\Big\{P(\widehat{Y}(\rho) > \frac{z}{4})\Big\}^{\frac{1}{2}}\Big\{\Big(E(\Delta\widehat{Y})^2(\widetilde{Y}(\rho) - \widehat{Y}(\rho))^4\Big)^{\frac{1}{2}} + \Big(E(\widetilde{Y}(\rho) - \widehat{Y}(\rho))^6\Big)^{\frac{1}{2}}\Big\} \end{aligned}$$

$$\leq \frac{C}{z^2}\Big(E\widehat{Y}^4(\rho)\Big)^{\frac{1}{2}}\frac{\delta_6^{\frac{1}{2}}}{n^{\frac{1}{4}}}$$
$$\leq \frac{C}{(1+z)^2}(\delta_4^{\frac{1}{2}}\delta_6^{\frac{1}{2}}+\delta_4\delta_6^{\frac{1}{2}}).$$

From this fact and (3.20) we have $T_{12} \leq \frac{C}{1+z}(\delta_3+\delta_4^{\frac{1}{2}}\delta_6^{\frac{1}{2}}+\delta_4\delta_6^{\frac{1}{2}})$.

By (3.5), (3.17), and (3.19),

$$\begin{aligned} T_{13} &\leq CnE\mathbb{I}(\delta \geq \frac{z}{4})(\widetilde{Y}(\rho)-\widehat{Y}(\rho))^2 \\ &\leq \frac{Cn}{z}\{E\delta^3\}^{\frac{1}{3}}\{E|\widetilde{Y}(\rho)-\widehat{Y}(\rho)|^3\}^{\frac{2}{3}} \\ &\leq \frac{C\delta_3}{1+z}. \end{aligned}$$

Hence, $T_{12}+T_{13} \leq \frac{C}{1+z}(\delta_3+\delta_4^{\frac{1}{2}}\delta_6^{\frac{1}{2}}+\delta_4\delta_6^{\frac{1}{2}})$.

Step 3. $|T_2| \leq \frac{C}{(1+z)}(\frac{\delta_4^{\frac{1}{2}}}{n^{\frac{1}{4}}}+\delta_4)$.

We will bound T_2 by using the technique from Lemma 9 and Lemma 10 of Ho and Chen(Ho, S.T., 1978). Let

$$\begin{aligned} G = &\widehat{Y}_z(I, M_1, ..., M_{d-1}) + \widehat{Y}_z(K, L_1, ..., L_{d-1}) \\ &- \widehat{Y}_z(I, L_1, ..., L_{d-1}) - \widehat{Y}_z(K, M_1, ..., M_{d-1}) \end{aligned}$$

and

$$A = \{\tau_i(I) \neq L_i,\ \tau_i(K) \neq M_i,\ \tau_i(I) \neq M_i,\ \tau_i(K) \neq L_i; 1 \leq i \leq d-1\}.$$

By the same argument as Neammanee and Rattanawong(Neammanee, K., 2009),

$$|T_2| \leq n|Eg_z'(\widehat{Y}(\tau))[E^{\tau_1,...,\tau_{d-1}}G^2\mathbb{I}(A^c) - G^2\mathbb{I}(A^c)]|. \tag{3.21}$$

Let $\mathcal{B}$ be the σ-algebra generated by

$$\{I, K, L_1, ..., L_{d-1}, M_1, ..., M_{d-1}, Y(i_1, ..., i_d) : 1 \leq i_1, ..., i_d \leq n\}.$$

By the fact that $E^{\mathcal{B}}\mathbb{I}(A^c) \leq \frac{C}{n}$ (Neammanee, K., 2008)
we have

$$E|G|^k\mathbb{I}(A^c) = E|G|^kE^{\mathcal{B}}\mathbb{I}(A^c) = \frac{C}{n}E|G|^k = \frac{C\delta_k}{n^{\frac{k+1}{2}}} \quad \text{for } k \in \mathbb{N}. \tag{3.22}$$

We observe that for $w \leq \frac{z}{2}$,

$$\begin{aligned} |g_z'(w)| &= |[1-\Phi(z)][1+\sqrt{2\pi}we^{\frac{1}{2}w^2}\Phi(w)]| \\ &\leq \{\frac{e^{-\frac{z^2}{2}}}{z}\}[1+\sqrt{2\pi}(\frac{z}{2})e^{\frac{z^2}{8}}\Phi(\frac{z}{2})] \\ &\leq \frac{\sqrt{2\pi}e^{-\frac{3z^2}{8}}}{2}+\frac{e^{-\frac{z^2}{2}}}{z} \\ &\leq \frac{C}{1+z} \end{aligned}$$

where we have used the fact that $1-\Phi(z) \leq \frac{e^{-\frac{z^2}{2}}}{z}$ for $z > 0$(Barbour, A.D., 2005) in the second inequality.

From this fact, (2.5), (3.6), (3.21), (3.22), we have

$$\begin{aligned}
|T_2| &\le n\Big|Eg_z'(\widehat{Y}(\tau))\mathbb{I}(\widehat{Y}(\tau) \le \frac{z}{2})\Big[E^{\tau_1,\tau_2,\cdots,\tau_{d-1}}G^2\mathbb{I}(A^c) - G^2\mathbb{I}(A^c)\Big]\Big| \\
&\quad + n\Big|Eg_z'(\widehat{Y}(\tau))\mathbb{I}(\widehat{Y}(\tau) > \frac{z}{2})\Big[E^{\tau_1,\tau_2,\cdots,\tau_{d-1}}G^2\mathbb{I}(A^c) - G^2\mathbb{I}(A^c)\Big]\Big| \\
&\le \frac{Cn}{1+z}EG^2\mathbb{I}(A^c) + Cn\Big|E\mathbb{I}(\widehat{Y}(\tau) > \frac{z}{2})\Big[E^{\tau_1,\tau_2,\cdots,\tau_{d-1}}G^2\mathbb{I}(A^c) - G^2\mathbb{I}(A^c)\Big]\Big| \\
&\le \frac{C\delta_2}{(1+z)\sqrt{n}} + Cn\Big(E\mathbb{I}(\widehat{Y}(\tau) > \frac{z}{2})E(E^{\tau_1,\tau_2,\cdots,\tau_{d-1}}G^2\mathbb{I}(A^c) - G^2\mathbb{I}(A^c))^2\Big)^{\frac{1}{2}} \\
&\le \frac{C\delta_2}{(1+z)\sqrt{n}} + \frac{Cn}{(1+z)}\Big(E\widehat{Y}^2(\tau)(EG^2\mathbb{I}(A^c)EG^2 + EG^4\mathbb{I}(A^c))\Big)^{\frac{1}{2}} \\
&\le \frac{C\delta_2}{(1+z)\sqrt{n}} + \frac{Cn}{(1+z)}\Big((1+\sqrt{n}\delta_4)(\frac{\delta_4}{n^2\sqrt{n}})\Big)^{\frac{1}{2}} \\
&\le \frac{C}{(1+z)}(\frac{\delta_4^{\frac{1}{2}}}{n^{\frac{1}{4}}} + \delta_4).
\end{aligned}$$

Step 4. $|T_3| \le \frac{C}{(1+z)}\Big(\frac{\delta_4^{\frac{1}{2}}}{n^{\frac{1}{4}}} + \sqrt{n}\delta_4^2\Big).$

By the same argument as Chen and Shao(Chen, L.H.Y., 2001), we can show that $E|g_z'(\widehat{Y}(\tau))| \le \frac{C}{(1+z)^2}(1+\sqrt{n}\delta_4)$, for $z > 0$. From this fact and Lemma 2.3, we have

$$\begin{aligned}
|T_3| &= |Eg_z'(\widehat{Y}(\tau)) - Eg_z'(\widehat{Y}(\tau))E\int_{-\infty}^{\infty}K(t)dt| \\
&= |Eg_z'(\widehat{Y}(\tau))||1 - E\int_{-\infty}^{\infty}K(t)dt| \\
&\le \frac{C(1+\sqrt{n}\delta_4)}{(1+z)^2}|1 - \frac{n-1}{4}E(\widetilde{Y}(\rho) - \widehat{Y}(\rho))^2| \\
&\le \frac{C}{(1+z)^2}(1+\sqrt{n}\delta_4)(\frac{\delta_4^{\frac{1}{2}}}{n^{\frac{1}{4}}} + \delta_4) \\
&\le \frac{C}{(1+z)}\Big(\frac{\delta_4^{\frac{1}{2}}}{n^{\frac{1}{4}}} + \sqrt{n}\delta_4^2\Big).
\end{aligned}$$

From (3.11) and step 1 - step 4, we have the theorem.

□

Remark There are various alternative ways to select the points X_i's for estimation of integrals over multidimensional domains. For examples, simple random sampling, lattice sampling(Patterson, H.D., 1954), the orthogonal arrays((Laipaporn, K., 2007), (Loh, W.L., 1996a), (Owen, A.B., 1992) and (Tang, B., 1993)), scrambled net((Owen, A.B., 1997a) and (Owen, A.B., 1997b)).

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