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Strongly Multiplicative Labeling in the Context of Arbitrary Supersubdivision

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Abstract

We investigate some new results for strongly multiplicative labeling of graph. We prove that the graph obtained by arbitrary supersubdivision of tree T, grid graph $P_n \times P_m$, complete bipartite graph $K_{m,n}$, $C_n \odot P_m$ and one-point union of m cycle of length n are strongly multiplicative.

Keywords: Strongly multiplicative labeling, Strongly multiplicative graphs, Arbitrary supersubdivision

1. Introduction

We begin with simple, finite, undirected and connected graph $G = (V, E)$. In the present work T, $P_n \times P_m$ and $K_{m,n}$ denote the tree, grid graph, and complete bipartite graph respectively. $C_n \odot P_m$ is the graph obtained by identifying an end point of P_m with every vertex of cycle C_n. One point union of m cycles of length n denoted as $C_n^{(m)}$ is the graph obtained by identifying one vertex of each cycles. If V_1 and V_2 are two partitions correspond to complete bipartite graph $K_{m,n}$ then V_1 is called m-vertices part and V_2 is called n-vertices part of $K_{m,n}$. In the graph G eccentricity of a vertex u is $\max_{v \in V(G)} d(u, v)$. For all other terminology and notations we refer to (Harary, F., 1972). We will give brief summary of definitions and other information which are useful for the present investigations.

Definition 1.1 Let G be a graph with q edges. A graph H is called a supersubdivision of G if H is obtained from G by replacing every edge e_i of G by a complete bipartite graph K_{2,m_i} for some m_i, $1 \leq i \leq q$ in such a way that the end vertices of each e_i are merged with the two vertices of 2-vertices part of K_{2,m_i} after removing the edge e_i from graph G.

A supersubdivision H of G is said to be an arbitrary supersubdivision of G if every edge of G is replaced by an arbitrary $K_{2,m}$ (m may vary for each edge arbitrarily). Arbitrary supersubdivision of G is denoted by $SS(G)$.

Definition 1.2 If the vertices of the graph are assigned values subject to certain conditions then it is known as graph labeling.

Most interesting graph labeling problems have following three important characteristics.

1. a set of numbers from which the labels are chosen;

2. a rule that assigns a value to each edges;

3. a condition that these values must satisfy.

For detail survey on graph labeling one can refer to (Gallian, J., 2009). Vast amount of literature is available on different types of graph labeling. According to (Beineke, L., 2001, p.63-75) graph labeling serves as a frontier between number theory and structure of graphs.

Labeled graph have variety of applications in coding theory, particularly for missile guidance codes, design of good radar type codes and convolution codes with optimal autocorrelation properties. Labeled graph plays vital role in the study of X-ray crystallography, communication network and to determine optimal circuit layouts. A systematic study on applications of graph labeling is reported in (Bloom, G., 1977, p. 562-570).

Definition 1.3 A graph $G = (V, E)$ with p vertices is said to be multiplicative if the vertices of G can be labeled with p distinct positive integers such that label induced on the edges by the product of labels of end vertices are all distinct.

Multiplicative labeling was introduced in (Beineke, L., 2001, p.63-75) where it is shown that every graph G admits multiplicative labeling and strongly multiplicative labeling is defined as follows.

Definition 1.4 A graph $G = (V, E)$ with p vertices is said to be strongly multiplicative if the vertices of G can be labeled with p distinct integers $1, 2, ...p$ such that label induced on the edges by the product of labels of the end vertices are all distinct.

In the present investigations we prove that the graphs obtained by arbitrary supersubdivision of tree T, grid graph $P_n \times P_m$, complete bipartite graph $K_{m,n}$, $C_n \odot P_m$ and $C_n^{(m)}$ are strongly multiplicative for all n and m.

2. Main Results

Theorem-2.1: Arbitrary supersubdivisions of tree T are strongly multiplicative.

Proof: Let T be the tree with n vertices. Arbitrary supersubdivision SS(T) of tree T obtained by replacing every edge of tree with K_{2,m_i} and we denote such graph by G. Let $K = \sum m_i$ $(1 \leq i \leq n-1)$. Let v_j $(1 \leq j \leq K+n)$ be the vertices of G. Denote the vertex with minimum eccentricity as v_1. Then v_2 will be the vertex which is at 1- distance apart from v_1. If there are more than one such vertices then throughout the work we will follow one of the direction (clockwise or anticlockwise) and denote them as $v_3, v_4, \ldots$. Next consider the vertices which are at 2- distance apart from v_1, 3- distance apart from v_1 and so on. (e.g. if there are seven vertices and two vertices are at distance 1- apart, one vertex is at distance 2- apart and three vertices are at distance 3- apart respectively form v_1. In this situation the vertices which are at 1- distance apart from v_1 will be identified as v_2 and v_3, the vertex which is at distance 2- apart will be identified as v_4 and the vertices which are at distance 3- apart will be identified as v_5, v_6 and v_7.) We define vertex labeling $f : V(G) \rightarrow \{1, 2 \ldots K+n\}$ as follows.

For any $1 \leq i \leq n+K$ define

$$f(v_i) = i$$

Then the graph G under consideration admits strongly multiplicative labeling.

Illustration 2.2: In *Fig.2* strongly multiplicative labeling of SS(T) corresponding to tree T of *Fig.1* is shown where $n = 13$ and $K = 26$.

Theorem 2.3: Arbitrary supersubdivisions of complete bipartite graph $K_{m,n}$ are strongly multiplicative.
Proof: Let $v_1, v_2, v_3, \ldots v_m$ be the vertices of m-vertices part and $v_{m+1}, v_{m+2}, v_{m+3}, \ldots v_{m+n}$ be the vertices of n-vertices part of $K_{m,n}$. Arbitrary supersubdivision SS$(K_{m,n})$ of $K_{m,n}$ obtained by replacing every edge of $K_{m,n}$ with K_{2,m_i} and we denote such graph by G. Let $K = \sum m_i$ $(1 \leq i \leq mn)$. Let u_j be the vertices which are used for arbitrary supersubdivision, where $1 \leq j \leq K$. We denote vertices by u_j which are used for supersubdivision of edges $v_1v_{m+1}, v_1v_{m+2}, \ldots v_1v_{m+n}, v_2v_{m+1}, \ldots v_nv_{m+n}$. Let p_o be the highest prime less than $K+m+n$. We define vertex labeling $f : V(G) \rightarrow \{1, 2 \ldots K+m+n\}$ as follows.

If $p_o \leq K+m$

$$\begin{aligned}
f(v_i) &= \begin{cases} i; \quad if \quad 1 \leq i \leq m, \\ k+i; \quad if \quad m+2 \leq i \leq m+n \end{cases} \\
f(v_{m+1}) &= p_o; \\
f(u_j) &= \begin{cases} m+j; \quad if \quad 1 \leq j < p_o, \\ m+j+1; \quad if \quad p_o \leq j \leq K \end{cases}
\end{aligned}$$

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If $p_o > K + m$

$$\begin{aligned}
f(v_i) &= \begin{cases} i; \quad if \quad 1 \leq i \leq m, \\ k+i-1; \quad if \quad m+2 \leq i < p_o, \\ k+i; \quad if \quad p_o \leq i \leq m+n \end{cases} \\
f(v_{m+1}) &= p_o; \\
f(u_j) &= m+j; \quad where \quad 1 \leq j \leq K
\end{aligned}$$

Then in each possibilities described above the graph G under consideration admits strongly multiplicative labeling.

Illustration 2.4: Consider $SS(K_{2,3})$. Here $m = 2$, $n = 3$ and $K = 14$. The strongly multiplicative labeling is as shown in *Fig.3*.

Theorem 2.5: Arbitrary supersubdivisions of grid graph $P_n \times P_m$ are strongly multiplicative.

Proof: Arbitrary supersubdivision SS($P_n \times P_m$) of $P_n \times P_m$ obtained by replacing every edge of grid graph with K_{2,m_i} and we denote such graph by G. Let $K = \sum m_i$ $(1 \leq i \leq mn)$. Let v_i $(1 \leq i \leq mn + K)$ be the vertices of G. Denote the vertex of left upper corner with v_1. Here we designate vertices by v_i $(2 \leq i \leq mn + K)$ according to the procedure described in Theorem 2.1. We define vertex labeling $f : V(G) \to \{1, 2, \ldots, mn + K\}$

$$f(v_i) = i; \quad where \quad 1 \leq i \leq mn + K$$

Then the graph G under consideration admits strongly multiplicative labeling.

Illustration 2.6: Consider SS($P_4 \times P_3$). Here $n = 4$, $m = 3$ and $K = 41$. The corresponding strongly multiplicative labeling is shown in *Fig.4*.

Theorem 2.7: Arbitrary supersubdivisions of $C_n \odot P_m$ are strongly multiplicative.

Proof: Arbitrary supersubdivision SS($C_n \odot P_m$) of $C_n \odot P_m$ obtained by replacing every edge of $C_n \odot P_m$ with K_{2,m_i} and we denote such graph by G. Let $K = \sum m_i$ $(1 \leq i \leq mn)$. Let v_i $(1 \leq i \leq mn + K)$ be the vertices of G. Designate arbitrary vertex of C_n as v_1 and employing the scheme used in Theorem 2.1 the remaining vertices will receive labels $v_2, v_3, \ldots, v_{mn+K}$. We define vertex labeling $f : V(G) \to \{1, 2, \ldots, mn + K\}$ as follows.

$$f(v_i) = i; \quad where \quad 1 \leq i \leq mn + K$$

Then the graph G under consideration admits strongly multiplicative labeling.

Illustration 2.8: Consider $SS(C_5 \odot P_3)$. Here $n = 5$, $m = 3$ and $K = 37$. The corresponding strongly multiplicative labeling is as shown in *Fig.5*.

Theorem 2.9: Arbitrary supersubdivisions of $C_n^{(m)}$ are strongly multiplicative.

Proof: Arbitrary supersubdivision of $C_n^{(m)}$ is obtained by replacing every edge of $C_n^{(m)}$ with K_{2,m_i} and we denote this graph by G. Let $K = \sum m_i$. Let $v_i(1 \leq i \leq m(n-1) + K + 1$ be the vertices of G. Denote the common vertex of cycles by v_1. According to the procedure followed in previous results the remaining vertices will be designated as v_i $(2 \leq i \leq m(n-1) + K + 1)$. We define vertex labeling $f : V(G) \to \{1, 2, \ldots, m(n-1) + K + 1\}$ as follows.

For any $1 \leq i \leq m(n-1) + K + 1$ we define

$$f(v_i) = i;$$

Then the graph G under consideration admits strongly multiplicative labeling.

Illustration 2.10: Consider SS($C_4^{(3)}$). Here $n = 4$, $m = 3$ and $K = 26$. The strongly multiplicative labeling is as shown in *Fig.6*.

3. Concluding Remarks And Open Problem

Labeled graph is the topic of current interest for many researchers as it has diversified applications. It is also very interesting to investigate graph or families of graph which admits particular type of labeling. In (Sethuraman, G., 2001 p.1059-1064) and (Kathiresan, K., 2004 p.81-84) graceful labeling in the context of arbitrary supersubdivision is discussed while we discuss here strongly multiplicative labeling in the context of arbitrary supersubdivision. We consider five different graph families and investigate their strongly multiplicative labeling. This work is a nice combination of combinatorial number theory and graph theory which will provide enough motivation to any researcher.

Open Problems:

• Similar investigations are possible for other graph families.

• Parallel results can be investigated corresponding to other graph labeling techniques.

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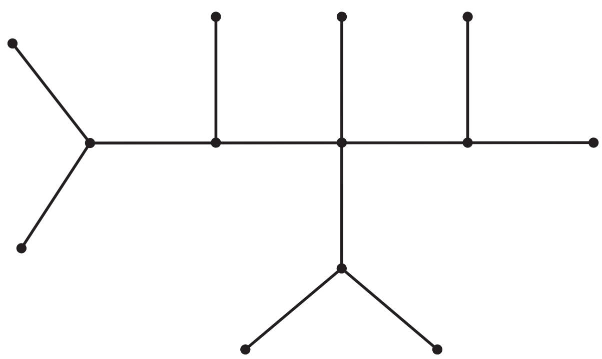

Figure 1. Tree T before arbitrary supersubdivision

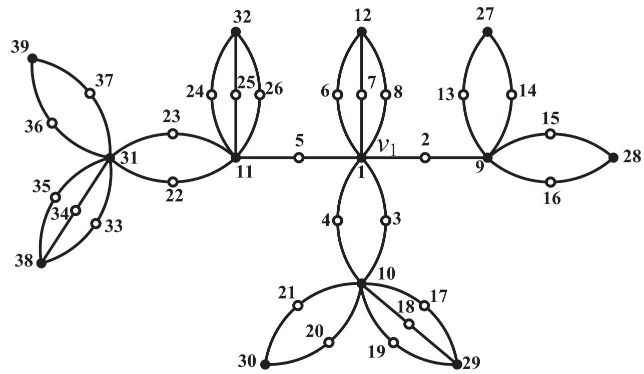

Figure 2. Strongly multiplicative labeling of SS(T)

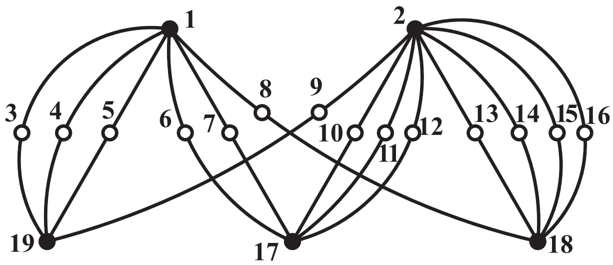

Figure 3. Strongly multiplicative labeling of $SS(K_{2,3})$

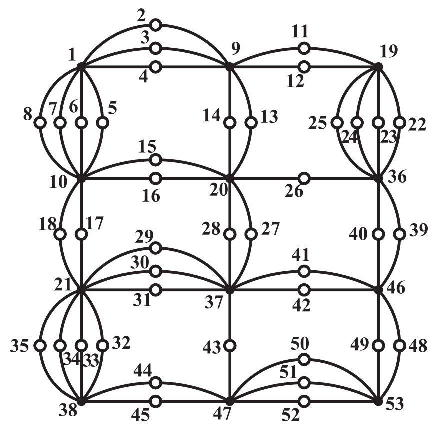

Figure 4. Strongly multiplicative labeling of SS($P_4 \times P_3$)

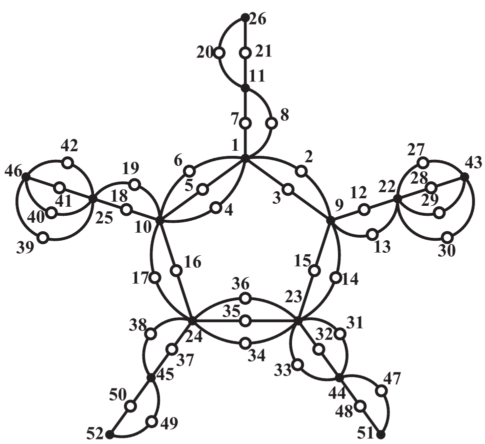

Figure 5. Strongly multiplicative labeling of $SS(C_5 \odot P_3)$

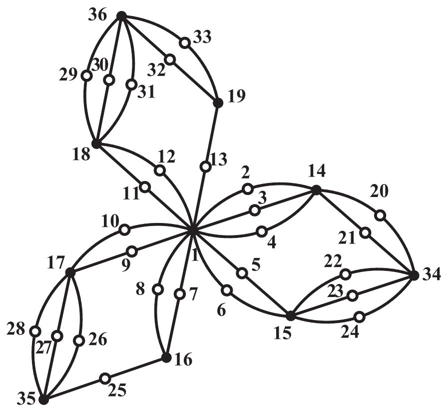

Figure 6. Strongly multiplicative labeling of SS($C_4^{(3)}$)