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Nonaffine Primitive Solvable Subgroups in General Liner Group $GL\left(3, p^k\right)$, Where p Is an Odd Prime, $k \geq 1$ Is an Integer

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Abstract

We know that if Γ is the automorphisem group of primitive tournament $T = (X, U)$, then $\Gamma = H \cdot G$ where $H = (X, +)$, G is irreducible solvable subgroup in $GL\left(n, p^k\right)$ with an odd order. But there are three kinds of irreducible solvable subgroups in $GL\left(n, p^k\right)$: imprimitive groups, affine primitive groups, and nonaffine primitive groups.

In this paper we will study the structure of nonaffine primitive solvable subgroups in $GL\left(3, p^k\right)$, find the order of these subgroups, and show that all nonaffine primitive solvable subgroups in $GL\left(3, p^k\right)$ have an even order, and then there are no primitive tournaments as automorphisem group $\Gamma = H \cdot G$ where G is nonaffine primitive solvable subgroups in $GL\left(3, p^k\right)$.

Keywords: primitive group, affine group, tournaments

1. Introduction

Let G be a group, Aut(G) is the automorphisem group of G, and H is a subgroup in Aut(G). A subgroup B in G is H-invariant if $h(B) = B$, $\forall h \in H$. We say that H “irreducible” if G and $\{e\}$ are the only H-invariant subgroups in G, other than H is reducible.

We call group G is a semi-direct product of the tow subgroups H and K if: 1)$G = HK$, 2)$H \trianglelefteq G$, 3)$H \cap K = 1$, and we write $G = H \rtimes N$. If $K \trianglelefteq G$ then G is the direct product $G = H \times N$.

We also say that group G is the central product $G = H_1 \top \ldots \top H_n$ of subgroups $H_1, \ldots, H_n$, if:

1) $H_i \trianglelefteq G$: $1 \leq i \leq n$.

2) $G = H_1 \ldots H_n$.

3) $H_i \cap H_j \subseteq Z(G)$: $i \neq j$, where $Z(G)$ is the central of G.

Let p be a prime number. We say an Abelian p-group G is elementary Abelian, if $x^p = 1$ for all $x \in G$ (i.e $G = C_p \times C_p \times \ldots C_p$, where C_p is a p-subgroup in G). And a group G is extra-special if it is finite p-group and $|G'| = p, Z(G) = G'$ and G/G' is elementary Abelian, where G' is the commutator group of G. The Fitting subgroup of a group G which denoted by $Fit(G)$ is product of all nilpotent normal subgroups of G.

We denote to the order of group G by $|G|$. And $[x, y]$ is the commutator x, y.

In this paper we exclude the nonaffine primitive solvable subgroups in $GL\left(3, p^k\right)$ from the set of all subgroups which take in structure of the automorphism group of primitive tournament, and I calculate the order of nonaffine primitive solvable subgroups in $GL\left(3, p^k\right)$.

2. Methods

We depend on methods in Short, Dold, and Eckmann (1992) to study the nonaffine primitive solvable subgroups in $GL\left(3, p^k\right)$, where we consider p as an odd prime and n, k as positive integers. A (round-robin) tournament T of order n consists of n nodes $a_1, a_2, \ldots, a_n$ such that each pair of distinct nodes a_i and a_j is joined by one and only one of the oriented arcs $\overrightarrow{a_i a_j}$ or $\overrightarrow{a_j a_i}$ and denote to it by $T = (X, U)$, where X is the set of vertex and U is the set of arcs. The set of all permutations $\propto$ on X which $(\propto (a_i), \propto (a_j)) \in U$ if and only if $(a_i, a_j) \in U$ forms a group called the automorphism group $Aut(T)$ of T. If $Aut(T)$ is primitive on X then T is a primitive tournament, and $X = GF(p^{nk})$, $Aut(T) = H \cdot G$, where H is the additive group of Galois field $GF(p^{nk})$, G is irreducible subgroup in $GL(n, p^k)$ of odd order. We know from Short, Dold, and Eckmann (1992) that if G is irreducible solvable subgroup in $GL(n, p^k)$ three cases are possible:

1) G is imprimitive.

2) G is primitive subgroup in affine group, which has the form:

$$\{x \to a\sigma(x) + b : a \neq 0, \ \ b \in GF(p^k), \ \ \sigma \in Aut(GF(p^k))\}$$

3) G is nonaffine primitive.

In this paper we study the structure of nonaffine primitive subgroup in $GL(3, p^k)$, and calculate it's order.

Theorem 1 (Dornhoff, 1971) Let G be a nonaffine primitive solvable subgroup in $GL(n, p^k)$, then:

1) $Fit(G)$ is irreducible.

2) $Fit(G)/Z(G) \cong E$, where E is extra-special group.

3) $G/Fit(G) \cong D$, where D is an irreducible subgroup of $SL(2, n)$.

4) $p \neq n$.

5) n divides $|Z(G)|$.

Thus by this theorem, a nonaffine primitive solvable subgroup G in $GL(n, p^k)$ has a maximal abelian normal subgroup $Z(G)$ with n divides $|Z(G)|$, and has the form $G = (Z(G) \top E) \rtimes D$, where E is extra-special group, D is an irreducible subgroup of $Sl(2, n)$. If G is maximal in $GL(n, p^k)$ it has the form $G = (C_{p^k - 1} \top E) \rtimes SL(2, n)$.

Now let $n = q^l m$, where $l > 0$, q is a prime number divides $p^{mk} - 1$. And let $\bar{z}, \bar{a} \in GL(m, p^k)$, where $\bar{z}$ be an element of order $p^{mk} - 1$, $\bar{a}$ be another element of order m, such that $(\bar{a})^{-1}\overline{za} = (\bar{z})^{p^k}$. Let a and z be the $n \times n$ block diagonal matrices defined by:

$$z = \begin{bmatrix} \bar{z} & 0 & \dots & 0 \\ 0 & \bar{z} & \dots & 0 \\ 0 & 0 & \dots & \bar{z} \end{bmatrix}, \quad a = \begin{bmatrix} \bar{a} & 0 & \dots & 0 \\ 0 & \bar{a} & \dots & 0 \\ 0 & 0 & \dots & \bar{a} \end{bmatrix}$$

Then $Fit(G)$ has the form $F = Fit(G) = < z > \top E$, thus from Short, Dold, and Eckmann (1992) the generators of G come in pairs (u_i, v_i), where each pair generating an extra-special q-group of order q^3, where:

$$u_i^q = v_i^q = I_{q^l}$$

$$[u_j, u_i] = [v_j, v_i] = I_{q^l}$$

$$[v_j, u_i] = \begin{cases} I_{q^l} & : j \neq i \\ \varepsilon I_{q^l} & : j = i \end{cases}$$

And ε is a primitive root of unity of order q in $GF(p^{mk})$. To construct matrices u_i, and v_j satisfying the above relations we use Jordan's method (Dieudonne, 1961). Let u and v be the $q \times q$ matrices defined by:

$$u = \begin{bmatrix} 0 & 1 \\ I_q^l & 0 \end{bmatrix}, \quad v = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \varepsilon & 0 & \dots & 0 \\ 0 & 0 & \varepsilon^2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & \varepsilon^{q-1} \end{bmatrix}$$

Consequently, $[v, u] = \varepsilon I_q$ and for $1 \leq i \leq l$ define u_i and v_i by: $u_i = I_{q^{l-i}} \bigotimes u \bigotimes I_{q^{i-1}}$, $v_i = I_{q^{l-i}} \bigotimes v \bigotimes I_{q^{i-1}}$, where $\bigotimes$ Kronecker products.

Now define the map ρ: $N_{GL(n,p^k)}(F) \longrightarrow GL(2l,q)$ by:

$$\rho(g) = \begin{bmatrix} \alpha_{1\,1} & \beta_{1\,1} & \dots & \alpha_{1\,l} & \beta_{1\,l} \\ \gamma_{1\,1} & \delta_{1\,1} & \dots & \gamma_{1\,l} & \delta_{1\,l} \\ \dots & \dots & \dots & \dots & ... \\ \alpha_{l\,1} & \beta_{l\,1} & \dots & \alpha_{l\,l} & \beta_{l\,l} \\ \gamma_{l\,1} & \delta_{l\,1} & \dots & \gamma_{l\,l} & \delta_{l\,l} \end{bmatrix}$$

where:

$$gu_ig^{-1} = u_1^{\alpha_{i1}}v_1^{\beta_{i1}}\dots u_l^{\alpha_{il}}v_l^{\beta_{il}}.\ z^{\lambda_i}$$

$$gv_ig^{-1} = u_1^{\gamma_{i1}}v_1^{\delta_{i1}}\dots u_l^{\gamma_{il}}v_l^{\delta_{il}}.\ z^{\mu_i}$$

$$gzg^{-1} = z^{\nu}$$

And the λ_i, μ_i and $\nu$are integers. Here we have the following:

Theorem 2 (Short, Dold, & Eckmann, 1992) Let D be an irreducible solvable subgroup in $SL(2l,\ q)$, Suppose D has generating set $\{d_1, d_2, \dots, d_r\}$. If d_i is the matrix:

$$\begin{bmatrix} \alpha_{1\,1} & \beta_{1\,1} & \dots & \alpha_{1\,l} & \beta_{1\,l} \\ \gamma_{1\,1} & \delta_{1\,1} & \dots & \gamma_{1\,l} & \delta_{1\,l} \\ \dots & \dots & \dots & \dots & ... \\ \alpha_{l\,1} & \beta_{l\,1} & \dots & \alpha_{l\,l} & \beta_{l\,l} \\ \gamma_{l\,1} & \delta_{l\,1} & \dots & \gamma_{l\,l} & \delta_{l\,l} \end{bmatrix}$$

Then there are matrixes g_i of $GL\left(n, p^k\right)$ satisfying:

$$g_iu_jg_i^{-1} = u_1^{\alpha_{j\,1}}v_1^{\beta_{j\,1}}\dots u_l^{\alpha_{j\,l}}v_l^{\beta_{j\,l}}.\ z^{\lambda_j}$$

$$g_iv_jg_i^{-1} = u_1^{\gamma_{j\,1}}v_1^{\delta_{j\,1}}\dots u_l^{\gamma_{j\,l}}v_l^{\delta_{j\,l}}.\ z^{\mu_j}$$

Where λ_j and μ_j are integers.

And if P is a subgroup of $GL\left(n, p^k\right)$ defined by $P =< C_{<a>}(v_1, \dots, v_l),\ \ g_1,\ \dots,\ \ g_r, v_1, \dots, v_l,\ \ z >$, then P is the complete inverse image of D under ρ, P is primitive and has a maximal Abelian normal subgroup of order divides $p^{mk} - 1$. If $\mathcal{D}$ is the set of conjugacy class representatives of the irreducible solvable subgroups of $SL(2l, q)$, $\mathcal{P}$ is the set of groups P obtained by the method above. One for each D, where $D \in \mathcal{D}$. Then no two elements of $\mathcal{P}$ are conjugate in $GL(n, p^k)$. If M is a primitive solvable subgroup of $GL\left(n,\ p^k\right)$ whose unique maximal Abelian normal subgroup has order that divides $p^{mk} - 1$, then M is conjugate to an element of $\mathcal{P}$.

3. Results and Discussion

3.1 The Nonaffine Primitive Soluble Subgroups in $GL\left(3,\ p^k\right)$

Let $n = 3$, then $l = 1$, $q = 3$, By Short, Dold, and Eckmann (1992), there is a unique type of maximal nonaffine primitive soluble subgroups in $GL\left(3,\ p^k\right)$ which have maximal Abelian normal subgroup of order $p^k - 1$. This type has the following form: $M = \left(C_{p^k-1} \top E\right) \rtimes SL(2,3)$, where E is extra-special of order 27 and exponent 3. Then $3 | \ p^k - 1$. If $GL\left(1,\ p^k\right) =< \overline{z}>$, $\varepsilon = (\overline{z})^{\frac{p^k-1}{3}}$, then ε is a 3-power order primitive root of unity in $GF(p^k)$ and

$$v = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon^2 \end{bmatrix}, u = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Here we have $u^3 = v^3 = I_3$. Consequently, $E =< u, v >$ is 3-extra-special of order 27 and exponent 3. If we consider G as irreducible soluble subgroup of M, then it has a maximal Abelian normal subgroup $Z(G)$ satisfying that 3 divides $|Z(G)|$, and is written as $G \cong (Z(G) \top E) \rtimes D$, where E is 3-extra-special of order 27 and exponent 3, D is an irreducible solvable subgroup of $SL(2,\ 3)$. Then there are three values of the group D, which are $C_4, Q_8, SL(2,\ 3)$. As 3 divides $|Z(G)|$, we can set $|Z(G)| = 3$, which is the first value of $|Z(G)|$. Then $Z(G) =< z >$, where:

$$z = \begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}.$$

Thus:

$$F = Fit(G) = Z(G) \top E = < u,\ v,\ z > .$$

And to obtain the group G we have three cases:

3.1.1 Case (1): $D = C_4$

We have $C_4 =< \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} >$. By Theorem 2, there is a matrix $g_1 \in GL(3,\ p^k)$, where $g_1 u g_1{}^{-1} = \eta_1 u^2 v$, $g_1 v g_1{}^{-1} = \mu_1 uv$ and $\eta_1,\ \mu_1 \in Z(G)$.

One of the solutions for g_1 is:

$$g_1 = \frac{1}{\varepsilon - \varepsilon^2} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \varepsilon & \varepsilon^2 \\ 1 & \varepsilon^2 & \varepsilon \end{bmatrix}$$

Here we can note that $g_1 \in SL\left(3,\ p^k\right), |g_1| = 4$. Then there are two types of group G:

$$G_1 =< g_1,\ u,\ v,\ z >, \qquad G_2 =< -g_1,\ u,\ v,\ z > .$$

If $GF(p^k)$ contains a square root ι of -1, then the group G conjugates with:

$$G_3 =< \iota g_1,\ u,\ v,\ z > .$$

And we find that $|G_1| = |G_2| = |G_3| = 36|Z(G)| = 108$.

3.1.2 Case (2) D=Q$_8$

We have $Q_8 =< \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} >$. By Theorem 2, in addition to the matrix g_1, there is a matrix $g_2 \in GL(3, p^k)$ satisfying: $g_2 u g_2^{-1} = \eta_2 uv$, $g_2 v g_2^{-1} = \mu_2 uv^2$ where $\eta_2,\ \mu_2 \in Z(G)$.

Setting $\eta_2 = 1,\ \mu_2 = \varepsilon^2$, we find that one of the solutions for c is:

$$g_2 = \frac{1}{\varepsilon - \varepsilon^2} \begin{bmatrix} 1 & \varepsilon & \varepsilon \\ \varepsilon^2 & \varepsilon & \varepsilon^2 \\ \varepsilon^2 & \varepsilon^2 & \varepsilon \end{bmatrix}$$

We can also note that $g_1^2 = g_2{}^2,\ \ g_2 g_1 g_2{}^{-1} = g_1{}^3,\ \ det(g_2) = 1$.

Consequently, there are two kinds of group G:

$$G_1 =< g_1,\ g_2,\ u,\ v,\ z >, \quad G_2 =< g_1,\ -g_2,\ u,\ v,\ z > .$$

And we find that $|G_1| = |G_2| = 72|Z(G)| = 216$.

3.1.3 Case (3): $D = SL(2, 3)$

We have $SL(2,3) =< \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} >$ and by Theorem 2, in addition to the matrixes g_1, g_2, there is a matrix $g_3 \in GL(3, p^k)$ satisfying: $g_3 u g_3^{-1} = \eta_3 uv^2$, $g_3 v g_3^{-1} = \mu_3 v$, where $\eta_3,\ \mu_3 \in Z(G)$.

One solution for g_3 is:

$$g_3 = \begin{bmatrix} \beta\varepsilon^{-1} & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

where $\beta = \varepsilon^{\frac{2}{3}} = (\overline{z})^{\frac{2(p^k-1)}{9}}$.

Since $p^k - 1 = 3\alpha$, α is an integer, we have three cases:

1) $\alpha = 3t$, t is a positive integer, then $p^k - 1 = 9t$ and $p^k \equiv 1(mod\ 9)$. Then:

$$\beta = (\overline{z})^{\frac{2(p^k-1)}{9}} = (\overline{z})^{2t} \in GF\left(p^k\right),$$

and thus the group G conjugates with: $G_1 =< g_1, g_2, g_3, u, v, z>$.

2) $a = 3t + 1$, then $p^k - 1 = 9t + 3 \equiv 3(mod\ 9)$ and $p^k \equiv 4(mod\ 9)$. Then

$$\beta = (\overline{z})^{\frac{2(p^k-1)}{9}} = (\overline{z})^{\frac{2(9t+3)}{9}} = (\overline{z})^{2t+\frac{2}{3}} = (\overline{z})^{2t}(\overline{z})^{\frac{2}{3}}.$$

If we set $=(\overline{z})^{\frac{1}{3}}$, then $\beta \in GF\left(p^k\right)$, and thus the group G conjugates with G_2= $< g_1,\ g_2,\ g_3,\ u,\ v,\ z>$.

3) $\alpha = 3t + 2$, then $p^k - 1 = 9t + 6 \equiv 6(mod\ 9)$ and $p^k \equiv 7(mod\ 9)$. Consequently,

$$\beta = (\overline{z})^{\frac{2(p^k-1)}{9}} = (\overline{z})^{\frac{2(9t+6)}{9}} = (\overline{z})^{2t+\frac{4}{3}} = (\overline{z})^{2t+1}(\overline{z})^{\frac{1}{3}}, \quad \text{and} \quad \gamma^2\beta \in GF(p^k).$$

And the group G conjugates with G_3= $< g_1, g_2, \gamma^2 g_3, u, v, z>$.

Thus the group G conjugates with:

$$\begin{array}{ll} G_1 = < g_1,\ g_2,\ g_3,\ u,\ v,\ z>: & p^k \equiv 1(mod\ 9) \\ G_2 = < g_1,\ g_2,\ \gamma g_3,\ u,\ v,\ z>: & p^k \equiv 4(mod\ 9) \\ G_3 = < g_1,\ g_2,\ \gamma^2 g_3,\ u,\ v,\ z>: & p^k \equiv 7(mod\ 9) \end{array}$$

And we find that: $|G_1| = |G_2| = |G_3| = 216\ |z| = 216 \times 3 = 648$.

3.2 Results

1) We find that if G is nonaffine primitive solvable subgroup in $GL\left(3,\ p^k\right)$, then its order is: $36|Z(G)|$, $72|Z(G)|$ or $216|Z(G)|$, where 3 divids $|Z(G)|$ and $|Z(G)|$ divids $p^k - 1$.

2) We find that all the nonaffine primitive solvable subgroups in $GL\left(3,\ p^k\right)$ have an even order. Consequently, we can note that these subgroups do not take in the structure of the automorphism group of primitive tournament .

Hence, we strongly recommend a pmabah search for imprimitive subgroups in $GL\left(3,\ p^k\right)$, because the structure of the tournament is unknown when its automorphism group is imprimitive.

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