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Note on Askey-Wilson q-Contour Integral Formula

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Abstract

We use Askey-Wilson q-contour integral formula and the q-Saalschütz's summation to derive a new recurring q-contour integral formula in this paper. Using this formula, we present a simple proof of the Sears' transformation of terminating balanced ${}_4\Phi_3$ series.

Keywords: q-contour integral, Askey and Wilson q-contour integral, q-Saalschütz's summation formula, Sears' terminating ${}_4\Phi_3$ transformation formula

1. Introduction and Main Result

We adopt the notations used by Gasper and Rahman (2004). Throughout this paper, it is supposed that $0 < |q| < 1$. For any complex parameter a, the q-shifted factorial are defined as

$$(a;q)_0 = 1,\ (a;q)_n = \prod_{k=0}^{n-1}(1-aq^k),\ \ (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k), n = 1,2,\cdots. \tag{1}$$

The following compact notation for the multiple q-shifted factorial is used:

$$(a_1,a_2,\cdots,a_m;q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_m;q)_n, m,n = 0,1,2,\cdots. \tag{2}$$

The basic hypergeometric series ${}_s\Phi_t$ is given by:

$${}_s\Phi_t\left(\begin{matrix} a_1, & a_2, & \cdots, & a_s \\ b_1, & b_2, & \cdots, & b_t\end{matrix} ;\ q,\quad x\right) = \sum_{k=0}^{\infty}\frac{(a_1,a_2\cdots,a_s;q)_k}{(q,b_1,\cdots,b_t;q)_k}\left[(-1)^k q^{\binom{k}{2}}\right]^{1+t-s} x^k, \tag{3}$$

where $s,t = 0,1,2,\cdots$.

For any complex numbers a and b, the polynomial $P_n(a;b)$ is defined as:

$$P_0(a;b) = 1, P_n(a;b) = (a-b)(a-bq)\cdots(a-bq^{n-1}), n \geq 1. \tag{4}$$

For $a \neq 0$, we define

$$F_n(a;b) = P_n(a;b)P_n(1/a;b). \tag{5}$$

Since q-integral was introduced by Jackson (1910), it has been studied by many researchers who produced numerous literatures about it (see Andrews & Askey, 1981; Askey, 1981; Askey & Wilson, 1985; Costas-Santos & Sánchez-Lara, 2013; Ismail & Masson, 1994, 1995; Jackson, 1910; Liu, 2003; Wang, 2010). Inspired by Wang (2010), a new of q-contour integral formula have been derived in Fang (2014) from the following elegant Askey-Wilson integral formula (consult Askey & Wilson, 1985, Theorem 2.1)

$$\frac{1}{2\pi i}\int_C \frac{(z^2,z^{-2};q)_\infty}{(az,a/z,bz,b/z,cz,c/z,dz,d/z;q)_\infty}\frac{dz}{z} = \frac{2(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}, \tag{6}$$

where the contour C is a deformation of unit circle so that the poles of $1/(az,bz,cz,dz;q)_\infty$ lie outside the contour and the origin and poles of $1/(a/z,b/z,c/z,d/z;q)_\infty$ lie inside the contour.

In this paper, we continue to use this method to give a new recurring q-contour integral formula. Using this new recurring formula and the symmetry of the function, we present a simple proof of the famous Sears' transformation of terminating balanced ${}_4\Phi_3$ series. And we also present two extension transformation formulas of terminating ${}_4\Phi_3$ series which are different from the extension presented in Fang (2007). The main result of this paper is stated as follows:

Theorem 1 *Assume the pairwise products of* $\{a, b, c, d\}$ *as a multiset (i.e. both* a^2 *and ab are considered among the products) do not belong to the set of* $\{q^i, i = 0, -1, -2, \cdots\}$*, and assume that there are no zero factors in the denominator, then*

$$\begin{aligned}
&\int_C \frac{(z^2, z^{-2}; q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty} \prod_{j=1}^{m+1} F_{n_j}(z; d_j) \frac{dz}{z} \\
=\;& (d_{m+1}a, d_{m+1}/a; q)_{n_{m+1}} \sum_{k=0}^{n_{m+1}} \frac{(q^{-n_{m+1}}; q)_k q^k}{(q, d_{m+1}a, q^{1-n_{m+1}}a/d_{m+1}; q)_k} \\
&\times \int_C \frac{(z^2, z^{-2}; q)_\infty}{(q^k az, q^k a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty} \prod_{j=1}^{m} F_{n_j}(z; d_j) \frac{dz}{z},
\end{aligned} \tag{7}$$

where m, n_j *are nonnegative integer numbers, and* $j = 1, 2, \cdots, m+1$.

We can derive the formula (7) from (6) and the following q-Saalschütz's summation (see Gasper & Rahman, 2004, p. 17, Equation (1.7.2)):

$${}_3\Phi_2\left(\begin{matrix} q^{-n}, & a, & f \\ & h, & afq^{1-n}/h \end{matrix}\; ; \; q, \; q\right) = \frac{(h/a, h/f; q)_n}{(h, h/af; q)_n}. \tag{8}$$

Rewriting (8) as follows:

$$\sum_{k=0}^{n} \frac{(q^{-n}; q)_k q^k}{(q, h, q^{1-n}af/h; q)_k} \frac{1}{(q^k a, q^k f; q)_\infty} = \frac{(h/a, h/f; q)_n}{(h, h/af; q)_n} \frac{1}{(a, f; q)_\infty}. \tag{9}$$

In (9), replacing (a, f) by $(az, a/z)$ respectively, we get

$$\sum_{k=0}^{n} \frac{(q^{-n}; q)_k q^k}{(q, h, q^{1-n}a^2/h; q)_k} \frac{1}{(q^k az, q^k a/z; q)_\infty} = \frac{1}{(h, h/a^2; q)_n} \frac{F_n(z; h/a)}{(az, a/z; q)_\infty}. \tag{10}$$

Both sides of (10) multiply by

$$\frac{(z^2, z^{-2}; q)_\infty}{(bz, b/z, cz, c/z, dz, d/z; q)_\infty} \prod_{j=1}^{m} F_{n_j}(z; d_j) \frac{1}{z}, \tag{11}$$

then we find that

$$\begin{aligned}
&\sum_{k=0}^{n} \frac{(q^{-n}; q)_k q^k}{(q, h, q^{1-n}a^2/h; q)_k} \frac{(z^2, z^{-2}; q)_\infty}{(q^k az, q^k a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty} \frac{\prod_{j=1}^{m} F_{n_j}(z; d_j)}{z} \\
=\;& \frac{1}{(h, h/a^2; q)_n} \frac{(z^2, z^{-2}; q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty} \prod_{j=1}^{m} F_{n_j}(z; d_j) \frac{F_n(z; h/a)}{z}.
\end{aligned} \tag{12}$$

Taking the contour integral on both sides of (12) with respect to variable z, denoting $n = n_{m+1}, h = d_{m+1}a$, we have the desired result. □

Remark 1 The zero is single pole of the function $F_{n_j}(z; d_j)(j = 1, 2, \cdots, m+1)$ in Unit Cirle. So we can use the similar method appeared in Askey and Wilson (1985) (or Costas-Santos & Sánchez-Lara, 2013) to deform the Unit Circle. In the context of this paper, convergence of series is no issue at all because they are the terminating series.

2. Some Applications

Next, we will give some applications of (7). Since we assume the integrals are the same established condition as Theorem, we omit the condition in the following.

Setting $m = 0$ in (7), then employing (6), we get

Corollary 1 *We have*

$$\frac{1}{2\pi i}\int_C \frac{(z^2, z^{-2}; q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty}\frac{F_{n_1}(z; d_1)}{z}dz$$
$$= \frac{2(d_1a, d_1/a; q)_{n_1}(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}\sum_{k=0}^{n_1}\frac{(q^{-n_1}, ab, ac, ad; q)_k q^k}{(q, d_1a, q^{1-n_1}a/d_1, abcd; q)_k}. \tag{13}$$

Interchanging a and b in the above identity, we have

$$\frac{1}{2\pi i}\int_C \frac{(z^2, z^{-2}; q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty}\frac{F_{n_1}(z; d_1)}{z}dz$$
$$= \frac{2(d_1b, d_1/b; q)_{n_1}(abcd; q)_\infty}{(q, ab, bc, bd, ac, ad, cd; q)_\infty}\sum_{k=0}^{n_1}\frac{(q^{-n_1}, ab, bc, bd; q)_k q^k}{(q, d_1b, q^{1-n_1}b/d_1, abcd; q)_k}. \tag{14}$$

Combining (14) and (13), then replacing $(ab, ac, ad, d_1a, q^{1-n_1}a/d_1, abcd)$ by (a, d, c, d, e, f) respectively, giving the famous Sears' transformation of terminating balanced ${}_4\Phi_3$ series:

Proposition 1 *We have*

$${}_4\Phi_3\left(\begin{matrix} q^{-n_1}, & a, & b, & c \\ & d, & e, & f \end{matrix}; q, \quad q\right) = \frac{(df/bc, d/a; q)_{n_1}}{(d, df/abc; q)_{n_1}}\,{}_4\Phi_3\left(\begin{matrix} q^{-n_1}, & a, & f/c, & f/b \\ & df/bc, & ef/bc, & f \end{matrix}; q, \quad q\right), \tag{15}$$

provided that $def = q^{1-n_1}abc$.

Remark 2 Interchanging d and f in the above identity, noting that $def = q^{1-n_1}abc$, we have $df/bc = q^{1-n_1}a/e, de/bc = q^{1-n_1}a/f, df/abc = q^{1-n_1}/e$. If we substitute them into (15), we can conclude that Sears' transformation of terminating balanced ${}_4\Phi_3$ series (see Gasper & Rahman, 2004, p. 360, Equation III. 15) after some simplification:

$${}_4\Phi_3\left(\begin{matrix} q^{-n_1}, & a, & b, & c \\ & d, & e, & f \end{matrix}; q, \quad q\right) = \frac{(e/a, f/a; q)_{n_1}}{(e, f; q)_{n_1}}\,{}_4\Phi_3\left(\begin{matrix} q^{-n_1}, & a, & d/c, & d/b \\ & d, & q^{1-n_1}a/e, & q^{1-n_1}a/f \end{matrix}; q, \quad q\right), \tag{16}$$

where $def = q^{1-n_1}abc$.

Letting $m = 1$ in (7), then applying (13), we have

Corollary 2 *We have*

$$\frac{1}{2\pi i}\int_C \frac{(z^2, z^{-2}; q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty}\frac{F_{n_2}(z; d_2)F_{n_1}(z; d_1)}{z}dz$$
$$= \frac{2(d_2a, d_2/a; q)_{n_2}(abcd; q)_\infty}{(q, ab, ac, ad, bc, bd, cd; q)_\infty}\sum_{k=0}^{n_2}\frac{(q^{-n_2}, ab, ac, ad; q)_k q^k (q^k d_1a, d_1/q^k a; q)_{n_1}}{(q, d_2a, q^{1-n_2}a/d_2, abcd; q)_k}$$
$$\times \sum_{k_1=0}^{n_1}\frac{(q^{-n_1}, q^k ab, q^k ac, q^k ad; q)_{k_1} q^{k_1}}{(q, q^k d_1a, q^{1-n_1+k}a/d_1, q^k abcd; q)_{k_1}}. \tag{17}$$

Interchanging a with b, n_1 with n_2, d_1 with d_2 in the above identity, we have

$$\frac{1}{2\pi i}\int_C \frac{(z^2, z^{-2}; q)_\infty}{(az, a/z, bz, b/z, cz, c/z, dz, d/z; q)_\infty}\frac{F_{n_1}(z; d_1)F_{n_2}(z; d_2)}{z}dz$$
$$= \frac{2(d_1b, d_1/b; q)_{n_1}(abcd; q)_\infty}{(q, ab, bc, bd, ac, ad, cd; q)_\infty}\sum_{k=0}^{n_1}\frac{(q^{-n_1}, ab, bc, bd; q)_k q^k (q^k d_2b, d_2/q^k b; q)_{n_2}}{(q, d_1b, q^{1-n_1}b/d_1, abcd; q)_k}$$
$$\times \sum_{k_1=0}^{n_2}\frac{(q^{-n_2}, q^k ab, q^k bc, q^k bd; q)_{k_1} q^{k_1}}{(q, q^k d_2b, q^{1-n_2+k}b/d_2, q^k abcd; q)_{k_1}}. \tag{18}$$

From (17) and (18), we obtain the following extension formula of Sears' transformations of terminating ${}_4\Phi_3$:

Proposition 2 (An extension Sears' transformations of terminating ${}_4\Phi_3$) *We have*

$$\begin{aligned}
&\sum_{k=0}^{n_2}\frac{(q^{-n},ab,ac,ad;q)_k q^k (q^k d_1 a, d_1/q^k a;q)_{n_1}}{(q,d_2 a,q^{1-n_2}a/d_2,abcd;q)_k}\\
&\times {}_4\Phi_3\left(\begin{matrix} q^{-n_1}, & q^k ab, & q^k ac, & q^k ad \\ & q^k d_1 a, & q^{1-n_1+k}a/d_1, & q^k abcd \end{matrix};\ q,\ q\right)\\
=\ &\frac{(d_1 b,d_1/b;q)_{n_1}}{(d_2 a,d_2/a;q)_{n_2}}\sum_{k=0}^{n_1}\frac{(q^{-n_1},ab,bc,bd;q)_k q^k (q^k d_2 b,d_2/q^k b;q)_{n_2}}{(q,d_1 b,q^{1-n_1}b/d_1,abcd;q)_k}\\
&\times {}_4\Phi_3\left(\begin{matrix} q^{-n_2}, & q^k ab, & q^k bc, & q^k bd \\ & q^k d_2 b, & q^{1-n_2+k}b/d_2, & q^k abcd \end{matrix};\ q,\ q\right).
\end{aligned}\tag{19}$$

Remark 3 Setting $n_2 = 0$, we get (15) after some rearrangement and simplification.

Putting $m = 2$ in (7), then applying (17), we find that

Corollary 3 *We have*

$$\begin{aligned}
&\frac{1}{2\pi i}\int_C \frac{(z^2,z^{-2};q)_\infty}{(az,a/z,bz,b/z,cz,c/z,dz,d/z;q)_\infty}\frac{F_{n_3}(z;d_3)F_{n_2}(z;d_2)F_{n_1}(z;d_1)}{z}dz\\
=\ &\frac{2(d_3 a,d_3/a;q)_{n_3}(abcd;q)_\infty}{(q,ab,ac,ad,bc,bd,cd;q)_\infty}\sum_{k=0}^{n_3}\frac{(q^{-n_3},ab,ac,ad;q)_k q^k (q^k d_2 a,d_2/q^k a;q)_{n_2}}{(q,d_3 a,q^{1-n_3}a/d_3,abcd;q)_k}\\
&\times\sum_{k_1=0}^{n_2}\frac{(q^{-n_2},q^k ab,q^k ac,q^k ad;q)_{k_1} q^{k_1}(q^{k+k_1}d_1 a,d_1/q^{k+k_1}a;q)_{n_1}}{(q,q^k d_2 a,q^{1-n_2+k}a/d_2,q^k abcd;q)_{k_1}}\\
&\times\sum_{k_2=0}^{n_1}\frac{(q^{-n_1},q^{k+k_1}ab,q^{k+k_1}ac,q^{k+k_1}ad;q)_{k_2}q^{k_2}}{(q,q^{k+k_1}d_1 a,q^{1-n_1+k_1+k}a/d_1,q^{k+k_1}abcd;q)_{k_2}}.
\end{aligned}\tag{20}$$

Similar as the process of getting Proposition 2.4, interchanging a with b, n_1 with n_2, d_1 with d_2 in the above identity, we find that:

Proposition 3 (Another extension Sears' transformations of terminating ${}_4\Phi_3$) *We have*

$$\begin{aligned}
&\sum_{k=0}^{n_3}\frac{(q^{-n_3},ab,ac,ad;q)_k q^k (q^k d_2 a,d_2/q^k a;q)_{n_2}}{(q,d_3 a,q^{1-n_3}a/d_3,abcd;q)_k}\\
&\times\sum_{k_1=0}^{n_2}\frac{(q^{-n_2},q^k ab,q^k ac,q^k ad;q)_{k_1} q^{k_1}(q^{k+k_1}d_1 a,d_1/q^{k+k_1}a;q)_{n_1}}{(q,q^k d_2 a,q^{1-n_2+k}a/d_2,q^k abcd;q)_{k_1}}\\
&\times\sum_{k_2=0}^{n_1}\frac{(q^{-n_1},q^{k+k_1}ab,q^{k+k_1}ac,q^{k+k_1}ad;q)_{k_2}q^{k_2}}{(q,q^{k+k_1}d_1 a,q^{1-n_1+k+k_1}a/d_1,q^{k+k_1}abcd;q)_{k_2}}\\
=\ &\frac{(d_3 b,d_3/b)_{n_3}}{(d_3 a,d_3/a)_{n_3}}\sum_{k=0}^{n_3}\frac{(q^{-n_3},ab,bc,bd;q)_k q^k (q^k d_1 b,d_1/q^k b;q)_{n_1}}{(q,d_3 b,q^{1-n_3}b/d_3,abcd;q)_k}\\
&\times\sum_{k_1=0}^{n_1}\frac{(q^{-n_1},q^k ab,q^k bc,q^k bd;q)_{k_1} q^{k_1}(q^{k+k_1}d_2 b,d_2/q^{k+k_1}b;q)_{n_2}}{(q,q^k d_1 b,q^{1-n_2+k}b/d_1,q^k abcd;q)_{k_1}}\\
&\times\sum_{k_2=0}^{n_2}\frac{(q^{-n_2},q^{k+k_1}ab,q^{k+k_1}bc,q^{k+k_1}bd;q)_{k_2}q^{k_2}}{(q,q^{k+k_1}d_2 b,q^{1-n_1+k+k_1}b/d_2,q^{k+k_1}abcd;q)_{k_2}}.
\end{aligned}\tag{21}$$

Remark 4 Letting $n_3 = n_2 = 0$ in the above identity, we can get (15) again after some rearrangement and simplification. It is easy to see that interchanging a with c, exchanging n_1 and n_3, d_1 and d_3, one can get other transformation formulas.

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