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Oblique Derivative Problem for Generalized Lavrent′ev-Bitsadze Equations

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Abstract

In this article, we first give the representation of solutions for the oblique derivative boundary value problem of generalized Lavrent′ev-Bitsadze equations including the Lavrent′ev-Bitsadze equation. Next we verify the uniqueness of solutions of the above problem. Finally we prove the solvability of oblique derivative problems for quasilinear mixed (generalized Lavrent′ev-Bitsadze) equations of second order, at the same time the estimates of solutions of the above problem is also obtained. The above problem is an open problem proposed by J. M. Rassias.

Keywords: existence and uniqueness of solutions, oblique derivative problem, generalized Lavrent′ev-Bitsadze equations

AMS Mathematics Subject Classification: 35M05, 35J25, 35L70

1. Formulation of Oblique Derivative Problem for Generalized Lavrent′ev-Bitsadze Equations

In Bers (1958), Bitsadze (1988), Rassias (1990), Smirnov (1978), Zarubin (2012), and Wen (2008), the authors posed and discussed the Tricomi problem of second order equations of mixed type by using the methods of integral equations, functional analysis, energy integrals and so on, the obtained results possess the important applications to gas dynamics. Now by using the complex analytic method we handle the oblique derivative boundary value problem of generalized Lavrent′ev-Bitsadze equations, which includes the Lavrent′ev-Bitsadze equation as a special case, where the intersectional part of the elliptic closed domain and hyperbolic closed domain is a hyperbolic curse.

Let D be a simply connected bounded domain in the complex plane $\mathbb{C}$ with the boundary $\partial D = \Gamma \cup L$, where $\Gamma\,(\subset \{\hat{y} = y - 3R + \sqrt{R^2 + x^2} > 0\}) \in C_\alpha(0 < \alpha < 1)$ is a curse with the end points $z_* = R_* = -R^* = -2\sqrt{2R}$, $z^* = R^* = 2\sqrt{2R}$, R is a positive number, and $L = L_1 \cup L_2, L_1 = \{x + y = -R^* \le x \le 0\}$, $L_2 = \{x - y = R^*, 0 \le x \le R^*\}$ are line segments, $L_0 = \{-R^* \le x \le R^*,\ y + \sqrt{R^2 + x^2} = 3R\}$ is a hyperbolic curse, and denote by $D^+ = D \cap \{\hat{y} > 0\}$, $D^- = D \cap \{\hat{y} < 0\}$ the elliptic domain and hyperbolic domain respectively, $z_0 = 2iR$, and $z_1 = -2j\sqrt{2R}$ is the intersection point of L_1, L_2, where i is the imaginary unit and j is the hyperbolic unit with the condition $j^2 = 1$. Here the common boundary of elliptic domain and hyperbolic domain is a hyperbolic curse, which has not been discussed in our previous papers, and in Wen (2013), the common boundary of elliptic domain and hyperbolic domain is a circle. We consider the second order quasilinear equation of mixed type

$$u_{xx} + \operatorname{sgn}\hat{y}\ u_{yy} = au_x + bu_y + cu + d \ \text{ in } \ D, \tag{1.1}$$

where $\hat{y} = y - 3R + \sqrt{R^2 + x^2}$, a, b, c, d are functions of $z(\in D), u, u_x, u_y(\in \mathbb{R})$, its complex form is the following complex equation of second order

$$Lu_z=\begin{Bmatrix}u_{z\bar{z}}\\u_{z\bar{z}}\end{Bmatrix}=F(z,u,u_z),\ F=\mathrm{Re}[A_1u_z]+A_2u+A_3\ \text{in}\begin{Bmatrix}D^+\\D^-\end{Bmatrix},\tag{1.2}$$

where $A_j=A_j(z,u,u_z),\ j=1,2,3$, and

$$u_{z\bar{z}}=\frac{1}{2}[(u_z)_x+i(u_z)_y]=\frac{1}{4}[u_{xx}+u_{yy}]\ \text{ in }\ D^+,$$
$$u_{z\bar{z}}=\frac{1}{2}[(u_z)_x+j(u_z)_y]=\frac{1}{4}[u_{xx}-u_{yy}]\ \text{ in }\ D^-,$$
$$A_1=\begin{cases}(a+ib)/2\ \text{ in }\ D^+,\\(a-jb)/2\ \text{ in }\ D^-,\end{cases}\ A_2=\frac{c}{4},\ A_3=\frac{d}{4}\ \text{ in }\ D.$$

In which we use the complex number $z=x+iy$ in $\overline{D^+}$ and the hyperbolic number $z=x+jy$ in $\overline{D^-}$ with the hyperbolic unit j. In this article, the notations are as the same in References (Wen, 1986, 1992, 2002, 2008, 2010, 2013; Wen, Chen, & Xu, 2008; Huang, Qiao, & Wen, 2005).

Suppose that the Equation (1.2) satisfies the following conditions, namely

Condition C 1) $A_j(z,u,u_z)\,(j=1,2,3)$ are continuous in $u\in\mathbb{R},u_z\in\mathbb{C}$ for almost every point $z\in D^+$, and measurable in $z\in D^+$ and continuous in $\overline{D^-}$ for all continuously differentiable functions $u(z)$ in $D^*=\overline{D}\backslash\{z_*,z^*\}$ and satisfy

$$L_p[A_1,\overline{D^+}]\le k_0,\ ,C[A_2,\overline{D^-}]\le\varepsilon k_0,\ L_p[A_3,\overline{D^+}]\le k_1,$$
$$C[A_j,\overline{D^-}]\le k_0,\ j=1,2,\ C[A_3,\overline{D^-}]\le k_1.\tag{1.3}$$

2) For any continuously differentiable functions $u_1(z),u_2(z)$ in D^*, there is

$$F(z,u_1,u_{1z})-F(z,u_2,u_{2z})=\mathrm{Re}[\tilde{A}_1(u_1-u_2)_z]+\tilde{A}_2(u_1-u_2)\ \text{ in }\ D,\tag{1.4}$$

where $\tilde{A}_j=\tilde{A}_j(z,u_1,u_2)\,(j=1,2)$ satisfy the conditions

$$L_p[\tilde{A}_1,\overline{D^+}]\le k_0,\ L_p[\tilde{A}_2,\overline{D^+}]\le\varepsilon k_0,\ C[\tilde{A}_j,\overline{D^-}]\le k_0,\ j=1,2\tag{1.5}$$

in (1.3),(1.5), $p\,(>2)$, k_0, k_1 are positive constants, and ε is a sufficiently small positive constant. In particular, the condition (1.4) obviously holds, when (1.2) is a linear equation.

Problem P The oblique derivative boundary value problem for (1.2) is to find a continuously differentiable solution $u(z)$ of (1.2) in $D^*=\overline{D}\backslash\{z_1,z_2\}$, which is continuous in $\overline{D}$ and satisfies the boundary conditions

$$\frac{1}{2}\frac{\partial u}{\partial\nu}=\mathrm{Re}[\overline{\lambda(z)}u_z]=r(z),\ z\in\Gamma,\ \frac{1}{2}\frac{\partial u}{\partial\nu}=\mathrm{Re}[\overline{\lambda(z)}u_{\bar{z}}]=r(z),\ z\in L_1,$$
$$u(z_0)=b_0,\ \mathrm{Im}[\overline{\lambda(z)}u_z]_{z=z_0}=b_1,\ \mathrm{Im}[\overline{\lambda(z)}u_{\bar{z}}]|_{z=z_1}=b_2,\tag{1.6}$$

where ν is a given vector at every point on $\Gamma\cup L_1$, $\lambda(z)=a(x)+ib(x)=\cos(\nu,x)\mp i\cos(\nu,y)$, and $\mp$ are determined by $z\in\Gamma$ and $z\in L_1$ respectively, b_0,b_1,b_2 are real constants, and $\lambda(z),\ r(z),\ b_0,\ b_1,b_2$ satisfy the conditions

$$C_\alpha[\lambda(z),\Gamma]\le k_0,\ C_\alpha[r(z),\Gamma]\le k_2,\ C_\alpha[\lambda(z),L_1]\le k_0,\ C_\alpha[r(z),L_1]\le k_2,$$
$$|b_0|,|b_1|,|b_2|\le k_2,\ \max_{z\in L_1}|a(z)-b(z)|=0\ \text{ or }\ \max_{z\in L_1}|[a(z)]^2-[b(z)]^2|^{-1}\le k_0,\tag{1.7}$$

in which $\alpha\,(1/2<\alpha<1),k_0,k_2$ are positive constants. The above boundary value problem is a general boundary value problem, which includes the irregular oblique derivative boundary condition. The boundary value problem for (1.2) with $A_3(z,u,u_z)=0,\ z\in D,u\in\mathbb{R},\ u_z\in\mathbb{C},\ r(z)=0,\ z\in\Gamma$ and $b_0=b_1=b_2=0$ will be called Problem P_0. The number

$$K=\frac{1}{2}(K_1+K_2)$$

is called the index of Problem P and Problem P_0, where

$$K_j=\left[\frac{\phi_j}{\pi}\right]+J_j,\ J_j=0\text{ or }1,\ e^{i\phi_j}=\frac{\lambda(t_j-0)}{\lambda(t_j+0)},\ \gamma_j=\frac{\phi_j}{\pi}-K_j,\ j=1,2,\tag{1.8}$$

in which $[a]$ is the largest integer not exceeding the real number a, $t_1 = z_*$, $t_2 = z^*$ on L_0, here we only discuss the case of $K = 0$ on ∂D^+, and the solution of Problem P is unique.

Besides, if the index $K = 1/2$, we can add a point condition

$$\mathrm{Im}[\overline{\lambda(z)}u_{\bar{z}}]|_{z=z_2} = b_3, \tag{1.9}$$

where z_2 is an inner point of Γ, b_3 is a real constant with the condition $|b_3| \le k_2$, and the boundary value problem for (1.2) will be called Problem Q.

Setting that

$$w(z) = u_z = \begin{cases} [u_x - iu_y]/2 = U(z) + iV(z) \ \text{ in } \ D^+, \\ [u_x - ju_y]/2 = U(z) + jV(z) \ \text{ in } \ D^-, \end{cases} \tag{1.10}$$

it is clear that Problem P for (1.2) is equivalent to the Riemann-Hilbert boundary value problem (Problem A) for the first order complex equation of mixed type

$$w_{\bar{z}} = F, \ F = \mathrm{Re}[A_1(z)w] + A_2(z)u + A_3(z) \ \text{ in } \ D \tag{1.11}$$

with the boundary conditions

$$\begin{gathered} \mathrm{Re}[\overline{\lambda(z)}w(z)] = r(z), \ z \in \Gamma, \ u(z_0) = b_0, \ \mathrm{Im}[\overline{\lambda(z_0)}w(z_0)] = b_1, \\ \mathrm{Re}[\overline{\lambda(z)}w(z)] = r(z), \ z \in L_1, \ \mathrm{Im}[\overline{\lambda(z_1)}w(z_1)] = b_2, \end{gathered} \tag{1.12}$$

and the relation

$$u(z) = \begin{Bmatrix} 2\mathrm{Re}\int_{z_0}^{z} w(z)dz \\ 2\mathrm{Re}\int_{z_0}^{z} \overline{w(z)}dz \end{Bmatrix} + b_1 = \begin{Bmatrix} 2\int_{z_0}^{z}[U(z)dx - V(z)dy] \\ 2\int_{z_0}^{z}[U(z)dx - V(z)dy] \end{Bmatrix} + b_1 \ \text{ in } \begin{Bmatrix} \overline{D^+} \\ \overline{D^-} \end{Bmatrix}. \tag{1.13}$$

From the formula (2.10), Chapter II, Wen (2002), we see that the above integral is independent of integral path in D and $u(z)$ is continuously differentiable in $D^* = \overline{D}\backslash\{z_*, z^*\}$.

Obviously the Lavrent′ev-Bitsadze equation

$$u_{xx} + \mathrm{sgn}\hat{y}u_{yy} = 0, \ \text{ i.e, } \ w_{\bar{z}} = 0 \ \text{ in } \ D, \tag{1.14}$$

is a special case of generalized Lavrent′ev-Bitsadze Equation (1.1), the relation of $u(z)$ and $w(z)$ is as stated in (1.10) and (1.13).

2. Solvability of Oblique Derivative Problem for Lavrent′ev-Bitsadze Equation

We first prove the existence and representation of solutions for Problem A of the equation

$$\begin{gathered} w_{\bar{z}} = 0 \ \text{ in } \ D, \ \text{ i.e. } \\ w_{\bar{z}} = (U + iV)_{\bar{z}} = 0 \ \text{ in } \ D^+, \\ w_{\bar{z}} = e_1(U + V)_\mu + e_2(U - V)_\nu = 0, \text{ i.e. } (U + V)_\mu = 0, \ (U - V)_\nu = 0 \ \text{ in } \overline{D^-}, \end{gathered} \tag{2.1}$$

with the boundary conditions

$$\begin{gathered} \mathrm{Re}[\overline{\lambda(z)}w(z)] = r(z), \ z \in \Gamma, \ \mathrm{Re}[\overline{\lambda(z)}w(z)] = r(z), z \in L_1, \\ \mathrm{Im}[\overline{\lambda(z_1)}w(z_1)] = b_2, \ \mathrm{Re}[\overline{\lambda(z)}w(z)] = U(z) - V(z) = r_0(z), \ z \in L_0, \end{gathered} \tag{2.2}$$

where $w = U + iV$ in D^+, $w = U + jV$ in D^-, $\mu = x + y$, $\nu = x - y$, $\lambda(z) = a(z) + jb(z) \neq 0$ on L_1, $\lambda(z) = 1 + j$ on L_0, $r(z)$ on L_1 is a known real function, $r_0(z)$ on L_0 is an undetermined real constant, b_2 is a real constant, and $\lambda(z)$, $r(z)$, b_2 satisfy the conditions

$$\begin{gathered} C_\alpha[\lambda(z), \Gamma] \le k_0, C_\alpha[\lambda(z), L_1] \le k_0, C_\alpha[r(z), \Gamma] \le k_2, \ C_\alpha[r(z), L_1] \le k_2, \\ |b_j| \le k_2, j = 0, 2, \ \max_{z \in L_1} |a(z) - b(z)| = 0 \ \text{ or } \ \max_{z \in L_1} |[a(z)]^2 - [b(z)]^2|^{-1} \le k_0, \end{gathered} \tag{2.3}$$

in which $\alpha\,(0<\alpha<1), k_0, k_2\,(\geq k_0)$ are positive constants; and

$$\begin{aligned} w(z) &= U+jV=(U+V)e_1+(U-V)e_2=f(x-y)e_1+g(x+y)e_2 \\ &= f(\nu)e_1+g(\mu)e_2=\tfrac{1}{2}\{f(\nu)+g(\mu)+j[f(\nu)-g(\mu)]\}, \end{aligned} \tag{2.4}$$

in which $e_1=(1+j)/2,\ e_2=(1-j)/2$.

From the boundary condition (1.15) of Tricomi problem in Section 1, we can find the directive derivation for (1.15) according to the arc length parameter s on $\Gamma\cup L_1$, and get the boundary conditions of oblique derivative problem (Problem P) as follows

$$\begin{aligned} &u_s/2=[u_xx_s+u_yy_s]/2=\mathrm{Re}[(x_s+ix_y)(u_x-iu_y)/2] \\ &=\mathrm{Re}[\overline{(x_s-iy_s)}w(z)]=\mathrm{Re}[\overline{(a(z)+ib(z))}w(z)]=\phi'(s)/2\ \text{ on } \Gamma, \\ &u_s/2=[u_x+u_yy_x]/2\sqrt{2}=\mathrm{Re}[(1-jy_x)(u_x-ju_y)/2\sqrt{2}] \\ &=\mathrm{Re}[\overline{(1-j)}w(z)]/\sqrt{2}=\mathrm{Re}[\overline{(a(z)+ib(z))}w(z)]=\phi'(s)/2\ \text{ on } L_1, \end{aligned} \tag{2.5}$$

in which $a(z)+ib(z)=x_s-iy_s$ on Γ, and $s=x\sqrt{2}$, $a(z)+jb(z)=(1-j)/\sqrt{2}$ on L_1, i.e. $a(z)+b(z)=0$ on L_1. Later on we shall use the condition on L_1.

For this, we shall find the solution of the last system of (2.1) in D^- with the boundary conditions

$$\begin{aligned} &\mathrm{Re}[\overline{\lambda(z)}(U+jV)]=r(z),\ z\in L_1=\{-R^*\leq x\leq 0, x+y=-R^*\}, \\ &\mathrm{Re}[\overline{\lambda(z)}(U+jV)]=r_0(z),\ z\in L_0=\{-R^*<x<R^*,\ \hat{y}=y-3R+\sqrt{R^2+x^2}=0\}. \end{aligned} \tag{2.6}$$

In fact the solution of Problem A for (2.1) in D^- can be expressed as

$$\begin{aligned} &\xi=U+V=f(\nu),\ \nu=x-y, \\ &\eta=U-V=g(\mu),\ \mu=x+y, \\ &U(z)=[f(\nu)+g(\mu)]/2,\ V(z)=[f(\nu)-g(\mu)]/2,\ \text{and} \\ &w(z)=[(1+j)f(\nu)+(1-j)g(\mu)]/2, \end{aligned} \tag{2.7}$$

in which $f(t), g(t)$ are two arbitrary real continuous functions on $L_0=[-R^*\leq x\leq R^*,\ y-3R+\sqrt{R^2+x^2}=0]$, thus the formulas in (2.6) can be rewritten as

$$\begin{aligned} &a(z)U(z)-b(z)V(z)=r(z)\ \text{ on } L_1, \\ &U(x+y)-V(x+y)=r_0(x+y)\ \text{ on } L_0,\ \text{ i.e.} \\ &[a(z)-b(z)]f(x-y)+[a(z)+b(z)]g(x+y)=2r(z)\ \text{ on } L_1, \\ &U(x+y)-V(x+y)=r_0(x+y)\ \text{ on } L_0,\ \text{ i.e.} \\ &[a((1+j)x+jR_*)-b((1+j)x+jR_*)]f(2x+R^*) \\ &+[a((1+j)x+jR_*)+b((1+j)x+jR_*)]g(-R^*) \\ &=2r((1+j)x+jR_*)\ \text{ on } L_1=\{-R^*\leq x\leq 0,\ x+y=-R^*\}, \\ &U(x-y)-V(x-y)=r_0(x-y)\ \text{ on } L_0=\{-R^*\leq x\leq R^*,\ \hat{y}=0\},\ \text{ i.e.} \\ &[a((1+j)t/2-(1-j)R^*/2)-b((1+j)t/2-(1-j)R^*/2)]f(t) \\ &+[a((1+j)t/2-(1-j)R^*/2)+b((1+j)t/2-(1-j)R^*/2)]g(-R^*) \\ &=2r((1+j)t/2-(1-j)R^*/2),\ t=2x+R^*\in[-R^*,R^*], \\ &U(t)-V(t)=r_0(t),\ t=\mu\in[-R^*,R^*],\ \text{ i.e.} \\ &f(t)=f(\nu)=U(\nu)+V(\nu) \\ &=\frac{2r((1+j)t/2-R^*(1-j)/2)-H(t)}{a((1+j)t/2-R^*(1-j)/2)-b((1+j)t/2-R^*(1-j)/2)},\ t=\nu\in[-R^*,R^*], \\ &U(\mu)-V(\mu)=g(\mu)=r_0(\mu),\ t=\mu\in[-R^*,R^*], \end{aligned} \tag{2.8}$$

in which $H(t) = [a((1+j)t/2 - R^*(1-j)/2) + b((1+j)t/2 - R^*(1-j)/2)]g(-R^*)$. When $z = z_0$, we have

$$\overline{\lambda(z_1)}w(z_1) = a(z_1)U(z_1) - b(z_1)V(z_1) + j[a(z_1)V(z_1) - b(z_1)U(z_1)] = r(z_1) + jb_1,$$

and can find

$$\begin{aligned} U(z_1) &= (a(z_1)r(z_1) + b_0 b(z_1))/([a(z)]^2 - [b(z)]^2), \\ V(z_1) &= (b(z_1)r(z_1) + b_0 a(z_1))/([a(z_1)]^2 - [b(z_1)]^2), \\ g(-R^*) = U(z_1) - V(z_1) &= \frac{(a(z_1)r(z_1) + b_0 b(z_1) - b(z_1)r(z_1) - b_0 a(z_1))}{[a(z_0)]^2 - [b(z_0)]^2}. \end{aligned} \tag{2.9}$$

Thus we can derive

$$\begin{aligned} U &= \frac{1}{2}\{\frac{2r((1+j)\nu/2 - (1-j)R^*/2) - H(\nu)g(-R^*)}{a((1+j)\nu/2 - (1-j)R^*/2) - b((1+j)\nu/2 - (1-j)R^*/2)} + r_0(\mu)\}, \\ V &= \frac{1}{2}\{\frac{2r((1+j)\nu/2 - (1-j)R^*/2) - H(\nu)g(-R^*)}{a((1+j)\nu/2 - (1-j)R^*/2) - b((1+j)\nu/2 - (1-j)R^*/2} - r_0(\mu))\}, \end{aligned} \tag{2.10}$$

if $[a(z)]^2 - [b(z)]^2 \neq 0$ on L_1.

Due to the above formula (2.5), we can obtain the oblique derivative condition of Tricomi problem as follows

$$\begin{gathered} [a((1+j)\nu/2 - (1-j)R^*/2) + b((1+j)\nu/2 - (1-j)R^*/2)g(-R^*) = 0, \\ U(\nu) + V(\nu) = f(\nu) = \frac{2r((1+j)\nu/2 - (1-j)R^*/2)}{a((1+j)\nu/2 - (1-j)R^*/2) - b((1+j)\nu/2 - (1-j)R^*/2)}, \\ t = \nu \in [-R^*, R^*]. \end{gathered} \tag{2.11}$$

Substituting $y = 3R - \sqrt{R^2 + x^2}$ into the formula (2.10), we obtain the boundary condition

$$\begin{aligned} &\mathrm{Re}[(1+j)w(z)] = U(x - 3R + \sqrt{R^2 + x^2}) + V(x - 3R + \sqrt{R^2 + x^2}) = r_1(z) \\ &= \frac{2r((1+j)(x - 3R + \sqrt{R^2 + x^2})/2 - (1-j)R^*/2) - H(x - 3R + \sqrt{R^2 + x^2})}{a(x - 3R + \sqrt{R^2 + x^2}) - b(x - 3R + \sqrt{R^2 + x^2})}, \text{ i.e.} \\ &\mathrm{Re}[(1-i)w(z)] = U(x - 3R + \sqrt{R^2 + x^2}) + V(x - 3R + \sqrt{R^2 + x^2}) = r_1(z) \text{ on } L_0, \end{aligned} \tag{2.12}$$

where $H(x-3R+\sqrt{R^2+x^2}) = [a((1+j)(x-3R+\sqrt{R^2+x^2})/2 - (1-j)R^*/2) + b((1+j)(x-3R+\sqrt{R^2+x^2})/2 - (1-j)R^*/2)]g(-R^*)$.

In addition, from the above condition and the first boundary condition in (2.2), noting that the index $K = 0$, there exists a unique solution $w(z) = U + iV$ of discontinuous Riemann-Hilbert problem with the boundary conditions (2.12) and the first condition in (2.2) for the Equation (1.14) in D^+ (see Theorem 6.6, Chapter V, Wen, 1992), and then the solution of Problem A for (1.14) is obtained as follows

$$\begin{aligned} w(z) &= U(z) + iV(z) \text{ in } D^+, \\ w(z) &= \frac{1}{2}[(1+j)\frac{2[r(h(\nu)) - (a(h(\nu)) + b(h(\nu)))g(-R^*)}{a(h(\nu)) - b(h(\nu))} + (1-j)r_0(\mu) \\ &= \frac{1}{2}[\frac{2[r(h(\nu)) - (a(h(\nu)) + b(h(\nu)))g(-R^*)}{a(h(\nu)) - b(h(\nu))} + r_0(\mu)] \\ &+ \frac{j}{2}[\frac{2[r(h(\nu)) + (a(h(\nu)) + b(h(\nu)))g(-R^*)}{a(h(\nu)) - b(h(\nu))} - r_0(\mu)] \text{ in } D^-, \end{aligned} \tag{2.13}$$

where $h(\nu) = (1+j)\nu/2 - (1-j)R^*/2 = h(x-y)$. Hence we have the following theorem.

Theorem 2.1 *Problem* A *for* (2.1) *in D has a unique solution in the form* (2.13)*, which satisfies the estimates*

$$\begin{gathered} C_\eta[w(z), D^\pm_\varepsilon] = C_\eta[U(z) + iV(z), D^\pm_\varepsilon] \leq M_1, \\ C_\eta[f(\nu), D^\pm_\varepsilon] \leq M_1, \ C_\alpha[g(\mu), D^\pm_\varepsilon] \leq M_1, \end{gathered} \tag{2.14}$$

where $\nu = x - y, \mu = x + y$, $D^{\pm}_{\varepsilon} = \overline{D^{\pm}} \cap \{|z - R_*| \geq \varepsilon\} \cap \{|z - R^*| \geq \varepsilon\}$, *$\varepsilon$ is a sufficiently small positive number,* $\eta = \min(1 - 2/p_0, \alpha)$, $p_0\ (2 < p_0 \leq p)$ *and* $M_1 = M_1(\eta, k_0, k_2, D^{\pm}_{\varepsilon})$ *are positive constants.*

Let the solution $w(z)$ of Problem A for (1.14) be substituted in (1.13). Then the solution $u(z)$ of Problem P for Lavrent′ev-Bitsadze Equation (1.14) is obtained, which can represented by the formula (1.13). We mention that the method in this section is completely solved the solvability of oblique derivative problem for Lavrent'ev-Bitsadze equation, which is differed with our previous researches including (Wen, 2013), and the reasoning is stringent and very intersenting.

In brief, the proof of the solvability for Problem P of (1.14) can be divided into four steps:

(1) From the second and third conditions in (2.2) for the Equation (1.14) in $\overline{D^-}$, the boundary condition

$$\mathrm{Re}[(1 - i)w(z)] = r_1(z) \ \text{ on } \ L_0 \tag{2.15}$$

is found, and cannot determine $\mathrm{Re}[(1 - j)w(z)] = U(x + y) - V(x + y) = r_0(x + y)$ on L_0.

(2) From the first boundary condition in (2.2) and the above condition (2.15), the continuous solution $w(z)$ of Problem A in $\overline{D^+}\backslash\{z_*, z^*\}$ is obtained, at the same time we determine the boundary condition

$$\mathrm{Re}[(1 - j)w(z)] = U(\mu) - V(\mu) = r_0(\mu) \ \text{ on } \ L_0. \tag{2.16}$$

(3) From the boundary conditions in (2.2) and (2.16), we can find the solution $w(z)$ of Problem A in D^- as stated in (2.13).

(4) To substitute the solution $w(z)$ of Problem A for the Equation (1.14) into the formula (1.13), thus the solution $u(z)$ of the oblique derivative boundary value problem (Problem P) for the Lavrent′ev-Bitsadze Equation (1.14) is gotten.

3. Unique Solvability for Problem P for Generalized Lavrent′ev-Bitsadze Equations

First of all we give the representation theorem of Problem P for the Equation (1.2).

Theorem 3.1 *Let the Equation* (1.2) *satisfy Condition* C*. Then any solution of Problem* P *for* (1.2) *can be expressed as*

$$u(z) = 2\mathrm{Re}\int_{z_0}^{z} \hat{w}(z)dz + b_0 = \left\{ \begin{array}{l} 2\mathrm{Re}\int_{z_0}^{z} w(z)dz \\ 2\mathrm{Re}\int_{z_0}^{z} \overline{w(z)}dz \end{array} \right. + b_0 \ \text{ in } \left\{ \begin{array}{c} D^+ \\ D^- \end{array} \right\}, \tag{3.1}$$

where $w(z) = u_z = [u_x - iu_y]/2 = w_0(z) + W(z)$ *in* D^+, $u_z = w(z) = [u_x - ju_y]/2 = w_0(z) + W(z)$ *in* D^-, *and* $w_0(z)$ *is a solution of Problem* A *for the equation*

$$Lw = \left\{ \begin{array}{c} w_{\bar{z}} \\ w_{\bar{z}} \end{array} \right\} = 0 \ \text{ in} \left\{ \begin{array}{c} D^+ \\ D^- \end{array} \right\}, \tag{3.2}$$

with the boundary conditions (1.6) ($w_0(z) = u_{0z}$)*, and* $w(z),\ W(z)$ *possess the form*

$$\begin{aligned} &w(z) = w_0(z) + W(z) \ \text{ in } D,\ W(z) = \tilde{\Phi}(z)e^{\tilde{\phi}(z)} + \tilde{\psi}(z) \ \text{ in } D^+, \\ &\tilde{\psi}(z) = Tf, \tilde{\phi}(z) = \tilde{\phi}_0(z) + Tg,\ Tg = -\frac{1}{\pi}\int\int_{D^+}\frac{g(\zeta)}{t - z}d\sigma_t \ \text{ in } D^+, \\ &W(z) = \Phi(z) + \Psi(z),\ \Psi(z) = \int_{R^*}^{\nu} g_1(z)d\nu e_1 + \int_{-R^*}^{\mu} g_2(z)d\mu e_2 \ \text{ in } D^-, \end{aligned} \tag{3.3}$$

in which $e_1 = (1 + i)/2$, $e_2 = (1 - i)/2$, $\mu = x + y$, $\nu = x - y$, , $\tilde{\phi}_0(z)$ *is an analytic function in* D^+ *and continuous in* $\overline{D^+}$*, such that* $\mathrm{Im}[\tilde{\phi}(x)] = 0$ *on* L_0*,*

$$\begin{aligned} &g(z) = \left\{ \begin{array}{l} A_1/2 + \overline{A_1}\bar{w}/(2w),\ w(z) \neq 0, \\ 0,\ w(z) = 0, \end{array} \right. \quad f(z) = \mathrm{Re}[A_1\tilde{\phi}_z] + A_2u + A_3 \text{ in } D^+, \\ &g_1(z) = g_2(z) = A\xi + B\eta + Cu + D,\ \xi = \mathrm{Re}w + \mathrm{Im}w,\ \eta = \mathrm{Re}w - \mathrm{Im}w, \\ &A = (\mathrm{Re}A_1 + \mathrm{Im}A_1)/2,\ B = (\mathrm{Re}A_1 - \mathrm{Im}A_1)/2,\ C = A_2,\ D = A_3 \text{ in } D^-, \end{aligned} \tag{3.4}$$

where $\tilde{\Phi}(z)$ *is an analytic function in* D^+ *and* $\Phi(z)$ *is a solution of the Equation* (3.2) *in* D^- *satisfying the boundary conditions*

$$\begin{aligned}&\mathrm{Re}[\overline{\lambda(z)}e^{\tilde{\phi}(z)}\tilde{\Phi}(z)] = -\mathrm{Re}[\overline{\lambda(z)}\tilde{\psi}(z)],\ z\in\Gamma,\\&\mathrm{Re}[\overline{\lambda(z)}(\tilde{\Phi}(z)e^{\tilde{\phi}(z)}+\tilde{\psi}(z))] = s(z),\ z\in L_0,\\&\mathrm{Re}[\overline{\lambda(z)}\Phi(z)] = -\mathrm{Re}[\overline{\lambda(z)}\Psi(z)],\ z\in L_0,\\&\mathrm{Re}[\overline{\lambda(z)}\Phi(z)] = -\mathrm{Re}[\overline{\lambda(z)}\Psi(z)],\ z\in L_1,\\&\mathrm{Im}[\overline{\lambda(z_1)}\Phi(z_1)] = -\mathrm{Im}[\overline{\lambda(z_1)}\Psi(z_1)],\end{aligned}\tag{3.5}$$

in which $\lambda(z) = 1 - i$ *or* $\lambda(z) = 1 + j$ *on* L_0. *Moreover by Theorem* 1.1, *Chapter* 5, *Wen (2002), the solution* $w_0(z)$ *of Problem* A *for* (3.2) *and* $u_0(z)$ *satisfy the estimate in the form*

$$C_\beta[u_0(z),\overline{D}] + C_\beta[w_0(z)X(z),\overline{D^+}] + C_\beta[w_0^\pm(\mu,\nu)Y^\pm(\mu,\nu),\overline{D^-}] \le M_2(k_1+k_2),\tag{3.6}$$

where

$$\begin{aligned}&X(z)=\prod_{j=1}^{2}|z-t_j|^{\eta_j},\ Y^\pm(z)=Y^\pm(\mu,\nu)=[|\nu-R^*||\mu-R_*|]^{\eta_0},\\&\eta_0=\max(\eta_1,\eta_2),\ \eta_j=\begin{cases}|\gamma_j|+\delta,\ \gamma_j<0,\\ \delta,\ \gamma_j\ge 0,\end{cases}\quad j=1,2,\end{aligned}\tag{3.7}$$

herein $t_1 = R_*, t_2 = R^*$, $w_0^\pm(\mu,\nu) = \mathrm{Re}w_0(z) \pm \mathrm{Im}w_0(z)$, $w_0(z) = w_0(\mu,\nu)$, $\mu = x+y$, $\nu = x-y$, *and* γ_1,γ_2 *are the real constants in* (1.8), $\beta\,(=\min(\alpha, 1-2/p_0,\delta))$, δ *are positive constants,*

$$u_0(z) = 2\mathrm{Re}\int_{z_0}^{z}\hat{w}_0(z)dz + b_0\ \text{ in }\ \overline{D}\tag{3.8}$$

and $p_0\,(2 < p_0 \le p)$, $M_2 = M_2(p_0,\beta,k_0,D)$ *are non-negative constants.*

Proof. Let $u(z)$ be a solution of Problem P for the Equation (1.2), and $w(z) = u_z$, $u(z)$ be substituted in the positions of w,u in (3.4), thus the functions $f(z), g(z), g_1(z), g_2(z)$, and $\tilde{\Psi}(z)$ in $\overline{D^+}$ and $\Psi(z)$ in $\overline{D^-}$ in (3.3), (3.4) can be determined. Moreover we can find the solutions $\tilde{\Phi}(z)$ in D^+ and $\Phi(z)$ in $\overline{D^-}$ of (3.2) with the boundary conditions in (3.5), thus using the method as in Section 2, we can get

$$\begin{aligned}s(z)=&\frac{2r((1+j)(x-y)/2-R^*(1-j)/2)-2R((1+j)(x-y)/2-R^*(1-j)/2)-H(x-y)}{a((1+j)(x-y)/2-R^*(1-j)/2)-b((1+j)(x-y)/2-R^*(1-j)/2)}\\&-\mathrm{Re}[\overline{\lambda(z)}\Psi(z))]\ \text{ on }\ L_0=\{-R^*\le x\le R^*,\ \hat{y}=y-3R+\sqrt{R^2+x^2}=0\},\end{aligned}\tag{3.9}$$

here and later on $H(x-y) = [a((1+j)(x-y)/2 - R^*(1-j)/2)) + b((1+j)(x-y)/2 - R^*(1-j)/2))]g(-R^*)$, $R(z) = \mathrm{Re}[\overline{\lambda(z)}\Psi(z)]$ on L_0, and then

$$w(z)=w_0(z)+W(z)=\begin{cases}w_0(z)+\tilde{\Phi}(z)e^{\tilde{\phi}(z)}+\tilde{\psi}(z)\ \text{ in }\ D^+,\\ w_0(z)+\Phi(z)+\Psi(z)\ \text{ in }\ D^-,\end{cases}$$

is the solution of Problem A for the complex equation

$$w_{\bar z} = \mathrm{Re}[A_1w] + A_2u + A_3\ \text{ in }\ D,\tag{3.10}$$

and $u(z)$ is a solution of Problem P for (1.2) as stated in the formula in (3.1). □

Theorem 3.2 *Suppose that the Equation* (1.2) *satisfies Condition* C. *Then Problem* P *for* (1.2) *has a unique solution* $u(z)$ *in D.*

Proof. Let $u_1(z), u_2(z)$ be any two solutions of Problem P for (1.2). By Condition C, we see that $u(z) = u_1(z) - u_2(z)$ and $w(z) = u_z$ satisfy the homogeneous equation and boundary condition

$$w_{\bar z} = \mathrm{Re}[A_1w] + A_2u\ \text{ in }\ D,\tag{3.11}$$

$$\begin{aligned}&\mathrm{Re}[\overline{\lambda(z)}w(z)] = 0,\ z\in\Gamma,\ u(z_0)=0,\ \mathrm{Im}[\overline{\lambda(z))}w(z_0)]=0,\\&\mathrm{Re}[\overline{\lambda(z)}w(z)]=0,\ z\in L_1,\ \mathrm{Im}[\overline{\lambda(z_1)}w(z_1)]=0.\end{aligned}\tag{3.12}$$

By Theorem 3.1, the solution $w(z)$ can be expressed in the form

$$w(z) = \begin{cases} w_0(z) + \tilde{\Phi}(z)e^{\tilde{\phi}(z)} + \tilde{\psi}(z), \tilde{\psi}(z) = Tf, \tilde{\phi}(z) = \tilde{\phi}_0(z) + \tilde{T}g \text{ in } D^+, \\ w_0(z) + \Phi(z) + \Psi(z) \text{ in } D^-, \end{cases}$$
$$\Psi(z) = \int_{-R^*}^{\mu} [A\xi + B\eta + Cu]e_1 d\mu + \int_{R^*}^{\nu} [A\xi + B\eta + Cu]e_2 d\nu \text{ in } D^-, \quad (3.13)$$

where $f(z), g(z)$ are stated as in (3.4), $\tilde{\Phi}(z)$ in D^+ is an analytic function and $\Phi(z)$ is a solution of (3.2) in $\overline{D^-}$ satisfying the boundary condition (3.5). □

According to the proof of Theorem 3.3, Chapter I, Wen (2002), Suppose $w(z) \not\equiv 0$ in the neighborhood ($\subset \overline{D}$) of two characteristic lines through the point z_1, we may choose a sufficiently small positive number $R_0 < 1$, such that $8M_3M_4R_0 < 1$, where $M_3 = \max\{C[A, Q], C[B, Q], C[C, Q], C[D, Q]\}$, $M_4 = 1 + 4k_0^2(1 + k_0^2)$ is a positive constant, and $M_5 = C[w(z), \overline{Q_0}] > 0$, herein $Q_0 = \{-R^* \le \mu \le -R^* + R_0, R^* - R_0 \le \nu \le R^*\}$. From (2.4), (2.13), (3.12), (3.13) and Condition C, we have

$$\|\Psi(z)\| \le 8M_3M_5R_0, \ \|\Phi(z)\| \le 16M_3k_0^2(1 + k_0^2)M_5R_0, \quad (3.14)$$

thus an absurd inequality $M_5 \le 8M_3M_4M_5R_0 < M_5$ is derived. It shows $w(z) = 0, (x, y) \in Q_0$. Moreover, we extend along the positive directions of $\mu = x + y$ and the negative directions of $\nu = x - y$ successively, and finally obtain $w(z) = 0$ for $(x, y) \in D^-$ and D^+. Hence we have $w_1(z) - w_2(z) = 0$, $u_1(z) - u_2(z) = 0$ in D. This proves the uniqueness of solutions of Problem P for (1.2).

Theorem 3.3 *Suppose that the mixed Equation* (1.2) *satisfies Condition* C. *Then Problem* P *for* (1.2) *has a solution in D.*

Proof. It is clear that Problem P for (1.2) is equivalent to Problem A for the complex equation of first order and boundary conditions:

$$w_{\bar{z}} = F, \ F = \text{Re}[A_1 w] + A_2 u + A_3 \text{ in } D, \quad (3.15)$$

$$\text{Re}[\overline{\lambda(z)}w(z)] = r(z), \ z \in \Gamma,$$
$$u(z_0) = b_0, \ \text{Im}[\overline{\lambda(z)}u_z]|_{z=z_0} = b_1, \quad (3.16)$$
$$\text{Re}[\overline{\lambda(z)}w(z)] = r(z), z \in L_1, \ \text{Im}[\overline{\lambda(z_1)}w(z_1)] = b_2,$$

and the relation (3.1). From (3.1), it follows that

$$C[u(z), \bar{D}] \le M_6[C(X(z)w(z), \overline{D^+}) + C(Y^{\pm}(z)w^{\pm}(z), \overline{D^-})] + k_2, \quad (3.17)$$

where $X(z), Y^{\pm}(z), w^{\pm}(z)$ are as stated in (3.7), $M_6 = M_6(D)$ is a positive constant. In the following, by using successive approximation as stated in the proof of Theorem 2.6, Chapter IV, Wen (2008), we can find a solution $w(z)$ of Problem A for the complex Equation (1.2) in D, and substitue $w(z)$ in the formula (3.1), thus the solution $u(z)$ of Problem P for (1.2) can be derived. □

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