Closed Expression for Characteristic Function of CEPE Distribution

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Abstract

The recent paper by Maturi and Elsayigh [The Correlation between Variate-Values and Ranks in Samples from Complete Fourth Power Exponential Distribution, Journal of Mathematics Research 1 (2009), No. 1, 14–18] contains an expression for the characteristic function of CFPE distribution in the §2 as double sum (one infinite) of terms involving special functions. Here, I would like to point that this formula turns out to be a corollary of closed form expression for the characteristic function of so-called Complete Eventh Power Exponential (CEPE) distribution.

Keywords: Complete Fourth Power Exponential distribution, Complete Eventh Power Exponential distribution, Fox–Wright generalized hypergeometric \( \psi_q \) function, Wright’s hypergeometric \( \phi \) function

The recent paper by Maturi and Elsayigh [The Correlation between Variate-Values and Ranks in Samples from Complete Fourth Power Exponential Distribution, Journal of Mathematics Research 1 (2009), No. 1, 14–18] derived the correlation between variate-values and ranks in a sample from the distribution referred as Complete Fourth Power Exponential (CFPE). The paper contained an expression for the characteristic function of CFPE distribution in the §2 as double sum (one infinite) of terms involving special functions. Here, I would like to reduce this formula to an explicit and closed form via Wright’s hypergeometric type \( \phi(\alpha, \beta, x) \) function. Moreover, this result turns out to be a corollary of closed form expression for the characteristic function of so-called Complete Eventh Power Exponential (CEPE) distribution such that one expresses in terms of Fox–Wright generalized hypergeometric \( \psi_q \) function.

Let us given a r.v. \( \xi_k \), on a standard probability space \( (\Omega, \mathcal{F}, P) \), with the density function

\[
f_k(\alpha, \beta; x) = \frac{k}{\beta^1(1/2k)} \exp \left\{ -\left( \frac{x - \alpha}{\beta} \right)^2 \right\} \quad k \in \mathbb{N}, \alpha \in \mathbb{R}, \beta > 0, x \in \mathbb{R}.
\]  

(1)

Such distribution we will call Complete Eventh Power Exponential (CEPE). Note that \( \xi_1 \) has standard normal \( N(\alpha, \beta^2/2) \), while \( \xi_2 \) has Complete Fourth Power Exponential (CFPE) distribution considered by Maturi & Elsayigh (2009) in this journal. They recall (by other origins) the following formula for the characteristic function:

\[
\varphi_2(i) = \mathbb{E} \exp \{ i \xi_2 \} = \sum_{m=0}^{\infty} \sum_{j=0}^{\infty} \frac{m!}{(2j)!^m} \frac{\Gamma(j/2 + 1/4)}{\Gamma(1/4)} \frac{(i\beta)^m}{m!}.
\]

(2)

Here \([x]\) stands for the integer part of some real \( x \).

Now, we will derive a more general result. To do this, we give instead of infinite double sum for expression for \( \varphi_2(i) \). In this goal let us introduce the Fox–Wright generalized hypergeometric function \( \psi_q(\cdot) \) with \( p \) numerator and \( q \) denominator parameters, defined by series (cf., e.g., Pogany et al. (2009; Eq. (9.8)):

\[
\psi_q\left[ \left( \alpha_1, A_1 \right), \ldots, \left( \alpha_p, A_p \right); (\beta_1, B_1), \ldots, (\beta_q, B_q); z \right] = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} \Gamma(\alpha_j + A_j n)}{\prod_{j=1}^{q} \Gamma(\beta_j + B_j n)} \frac{z^n}{n!}
\]

(3)

such that converges for \( A_1, B_1 > 0, 1 + \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0 \). Let us mention that the confluent case

\[
\psi_1\left[ \left( \alpha, A \right); (\beta, B); z \right] = \frac{1}{\beta} \frac{z^n}{n!} \quad \beta \in \mathbb{C}, B > 0
\]

(4)

one calls Wright’s function with lower parameters \( \beta, B \), compare Gorenflo et al. (1999).

**Theorem:** Let r.v. \( \xi_k \) has CEPE distribution, \( k \in \mathbb{N} \). Then the characteristic function

\[
\varphi_k(i) = \mathbb{E} \exp \{ i \xi_k \} = \sqrt{\pi} \frac{\exp \{ i \alpha \}}{\Gamma(1/(2k))} \psi_q\left[ \left( (1/(2k)), 1/k \right); (1/2, 1); \left( -\frac{(\beta)^2}{4} \right) \right].
\]

(5)

**Proof:** By direct calculation we have

\[
\varphi_k(i) = \mathbb{E} \exp \{ i \xi_k \} = \int_{\mathbb{R}} e^{i\alpha x} f_k(\alpha, \beta; x) \, dx = \frac{k \exp \{ i\alpha \}}{\Gamma(1/(2k))} \int_{\mathbb{R}} \exp \{ i\beta y - \beta^2/4 \} \, dy.
\]
Expanding the Fourier–kernel $\exp[iyt]$ into Maclaurin series and making use of the legitimate interchange of sum and the integral we conclude
\[
\varphi_k(t) = \frac{k \exp \left[ ita \Gamma(1/(2k)) \right]}{\Gamma(1/(2k))} \sum_{m=0}^{\infty} \frac{(i\beta t)^m}{m!} \int_R y^m \exp[-y^2] \, dy.
\]

Being the integrand odd function for $m$ odd, all integrals in summands vanish for these values. So, the last expression reduces to
\[
\varphi_k(t) = \frac{\exp \left[ ita \Gamma(1/(2k)) \right]}{\Gamma(1/(2k))} \sum_{m=0}^{\infty} \frac{\Gamma(m/k + 1/(2k))}{\Gamma(2m + 1)} \left[ - (\beta t)^2 \right]^m.
\]

Applying now the Legendre duplication formula
\[
\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma(z + 1/2)
\]
to the Gamma–function term $\Gamma(2m + 1) = 2m\Gamma(2m)$ in the denominator, we clearly deduce
\[
\varphi_k(t) = \frac{\sqrt{\pi} \exp \left[ ita \Gamma(1/(2k)) \right]}{\Gamma(1/(2k))} \sum_{m=0}^{\infty} \frac{\Gamma(m/k + 1/(2k))}{\Gamma(m + 1/2)} \left[ - (\beta t)^2/4 \right]^m.
\]

Corollary: For the r.v. $\xi_2$ we have
\[
\varphi_2(t) = \frac{\sqrt{\pi} \exp \left[ ita \Gamma(1/4) \right]}{\Gamma(1/4)} \cdot \phi(3/4; 1/2; -(\beta t)^2/8),
\]
where $\phi$ stands for the Wright function defined in (4).

Proof: Specifying $k = 2$ in (6) we get
\[
\varphi_2(t) = \frac{\sqrt{\pi} \exp \left[ ita \Gamma(1/4) \right]}{\Gamma(1/4)} \sum_{m=0}^{\infty} \frac{\Gamma(m/2 + 1/4)}{\Gamma(m + 1/2)} \left[ - (\beta t)^2/4 \right]^m.
\]

Apply once more Legendre’s duplication formula to the denominator term $\Gamma(m + 1/2)$, so the result.

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