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Some Iterative Numerical Methods for a Kind of System of Mixed Nonlinear Variational Inequalities

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Abstract

In this paper, we study the numerical solution of a type of system of mixed nonlinear variational inequalities in a Banach space. Using the properties of $\eta-$proximal mapping, we construct some iterative algorithms for solving systems of mixed nonlinear variational inequalities. Moreover, we establish the convergence theorems for the proposed numerical methods.

Keywords: system of mixed nonlinear variational inequalities, $\eta-$proximal mapping, iterative algorithm, convergence

1. Introduction

Variational inequality is a kind of very important nonlinear problems and a powerful tool for studying a wide class of problems arising in engineering, physics, economics, optimal control and so on, see for example Duvaut and Lions (1976), Facchinei and Pang (2003), and references therein. During the last decades, researchers have made great progress in obtaining numerical solutions of variational inequality, such as projection method and its variations, linear approximation method, smoothing Newton method, domain decomposition method, etc. Since standard projection methods depend on the inner product on Hilbert spaces, this kind of technique cannot be extended to variational inequality in Banach spaces. Even more, for mixed variational inequalities, projection method is neither appropriate, since it is difficult to find the projection.

Recently, thanks to emergence of new kinds of numerical solutions, variational inequalities have been extended in many directions. One of the most important extensions is the system of variational inequalities, see Ferris and Pang (1997), Kazmi and Khan (2007), Verma (2007), and references therein. Huang and Noor (2007) discussed the convergence of projection method for a kind of system of variational inequalities in Hilbert spaces. Verma (2001) studied the numerical solution for a kind of a system of nonlinear variational inequalities in Hilbert space, and proposed a series of projection methods. In this paper, we will go on doing the work in this area. We study the numerical solution of system of mixed nonlinear variational inequalities in Banach space. We introduce the definition of $\eta-$proximal mapping for a proper subdifferentiable functional. Using the properties of $\eta-$proximal mapping, we construct some iterative algorithms for solving systems of mixed nonlinear variational inequalities. Morever, we establish the convergence theorems for the proposed numerical methods.

2. Preliminaries

First we give a hypothesis which will be used throughout the paper.

Hypothesis A Let B be a reflexive Banach space, B^* be the dual space B and $\langle\cdot,\cdot\rangle$ denote the pairing between B^* and B. Let ϕ: $B \to (-\infty, +\infty]$ be a proper lower semicontinuous and subdifferentiable functional.

Let 2^{B^*} denote all subsets of B^*. Let T: $B \to B^*$ be single-valued mappings. We consider the following system of mixed nonlinear variational inequalities (denoted by SMNVI): Find $(x^*, y^*) \in B \times B$, such that

$$\begin{aligned}
&\langle \gamma_1 T(y^*) + x^* - y^*, x - x^* \rangle \geq \phi(x^*) - \phi(x), \quad \gamma_1 > 0, \quad \forall x \in B,\\
&\langle \gamma_2 T(x^*) + y^* - x^*, y - y^* \rangle \geq \phi(y^*) - \phi(y), \quad \gamma_2 > 0, \quad \forall y \in B.
\end{aligned} \tag{1}$$

We first recall the following definitions and some known results.

Definition 1 The following mapping J: $B \to 2^{B^*}$ is said to be a normal dual mapping:

$$J(x) = \{f \in B^* : \langle f, x \rangle = \|f\| \cdot \|x\|, \|f\| = \|x\|\}, \quad \forall x \in B.$$

Definition 2 Let Hypothesis A hold and η: $B \to B^*$ be a mapping. Mapping $x^* \mapsto x$, denoted by $x = J_\rho^\phi(x^*)$, is said to be a $\eta-$proximal mapping for ϕ, if for any $x^* \in B^*$ and constant $\rho > 0$, there exists $x \in B$ satisfying

$$\langle \eta x - x^*, y - x \rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \quad \forall y \in B. \tag{2}$$

Definition 3 Let A: $B \to B^*$ be a single mapping. A is said to be $\alpha-$strongly monotone, if for any $x, y \in B$, there exists constant $\alpha > 0$, such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2.$$

Hypothesis B Let $\eta : B \to B^*$ be an $\alpha-$strongly continuous mapping.

Definition 4 Let B be a Banach space, T: $B \to B$ is said to be $s-$Lipschitz continuous, if there exists $s > 0$ such that

$$\|T(x) - T(y)\| \leq s\|x - y\|, \quad \forall x, y \in B.$$

By the definition of subdifferentiable and (2), we have $x^* - \eta x \in \rho\partial\phi(x)$, hence

$$x = J_\rho^\phi(x^*) = (\eta + \rho\partial\phi)^{-1}(x^*).$$

Remark 1 If $B = H$ is a Hilbert space, η: $H \to H$ is an identity mapping on H, and ϕ is a proper convex subdifferentiable functional, then $\eta-$proximal mapping for ϕ degenerates to a resolvent operator on H.

Lemma 1 (Xia & Huang, 2008) *Let Hypothesises A and B hold. Then for any $x^* \in B^*$ and any $\rho > 0$, there exists unique $x \in B$ such that*

$$\langle \eta x - x^*, y - x \rangle + \rho\phi(y) - \rho\phi(x) \geq 0, \quad \forall y \in B.$$

That is, $x = J_\rho^\phi(x^*)$, $\eta-$proximal mapping for ϕ is well defined.

Lemma 2 (Xia & Huang, 2008) *Let Hypothesises A and B hold. Then $\eta-$proximal mapping $J_\rho^\phi = (\eta + \rho\partial\phi)^{-1}$ is $\frac{1}{\alpha}-$Lipschitz continuous. If the subdifferentiable $\partial\phi$: $B \to 2^{B^*}$ of ϕ is $\xi-$strongly monotone, then $\eta-$proximal mapping $J_\rho^\phi = (\eta + \rho\partial\phi)^{-1}$ is $\frac{1}{\alpha + \rho\xi}-$Lipschitz continuous.*

Lemma 3 *Let Hypothesises A and B hold. Then, $(x^*, y^*) \in B \times B$ is the solution of (1) if and only if*

$$\begin{aligned} x^* &= J_\rho^\phi[\eta(x^*) - \rho(\gamma_1 T(y^*) + x^* - y^*)], \\ y^* &= J_\rho^\phi[\eta(y^*) - \rho(\gamma_2 T(x^*) + y^* - x^*)], \end{aligned} \tag{3}$$

where $J_\rho^\phi = (\eta + \rho\partial\phi)^{-1}$, $\rho > 0$ is a constant.

Proof. Let (x^*, y^*) satisfy (3). Since $J_\rho^\phi = (\eta + \rho\partial\phi)^{-1}$, we have (3) holds if and only if (x^*, y^*) satisfies

$$\begin{aligned} \eta(x^*) - \rho(\gamma_1 T(y^*) + x^* - y^*) &\in \eta(x^*) + \rho\partial\phi(x^*), \\ \eta(y^*) - \rho(\gamma_2 T(x^*) + y^* - x^*) &\in \eta(y^*) + \rho\partial\phi(y^*). \end{aligned} \tag{4}$$

By the definition of functional subdifferentiable, (4) is equivalent to

$$\begin{aligned} \langle \gamma_1 T(y^*) + x^* - y^*, x - x^* \rangle &\geq \phi(x^*) - \phi(x), \quad \forall x \in B, \\ \langle \gamma_2 T(x^*) + y^* - x^*, y - y^* \rangle &\geq \phi(y^*) - \phi(y), \quad \forall y \in B. \end{aligned}$$

In summary, (x^*, y^*) is the solution of (1) if and only if (x^*, y^*) satisfies (3), which completes the proof. □

Lemma 4 (Verma, 2001) *Let $\{\delta_n\}_{n=0}^\infty$ be a non negative sequence, and satisfy the following inequality*

$$\delta_{n+1} \leq (1 - \lambda_n)\delta_n + \sigma_n, \quad \forall n \geq 0,$$

where, $\lambda_n \in [0, 1]$, $\sum_{n=0}^\infty \lambda_n = \infty$, and $\sigma_n = o(\lambda_n)$, then $\lim_{n\to\infty} \delta_n = 0$.

3. Numerical Methods and Convergence

Based on Lemma 3, we propose the following iterative numerical methods for (1).

Method 1 Let Hypothesises A and B hold. For any $(x_0, y_0) \in B \times B$, calculate $(x_{n+1}, y_{n+1}) \in B \times B$:

$$x_{n+1} = (1 - a_n)x_n + a_n(J_\rho^\phi[\eta(x_n) - \rho(\gamma_1 T(y_n) + x_n - y_n)]),$$
$$y_{n+1} = (1 - b_n)y_n + b_n(J_\rho^\phi[\eta(y_n) - \rho(\gamma_2 T(x_n) + y_n - x_n)]),$$

where $0 \le a_n \le 1$, $0 \le b_n \le 1$.

Method 2 Let Hypothesises A and B hold. For any$(x_0, y_0) \in B \times B$, calculate $(x_{n+1}, y_{n+1}) \in B \times B$:

$$x_{n+1} = (1 - a_n)x_n + a_n(J_\rho^\phi[\eta(x_n) - \rho(\gamma_1 T(y_n) + x_n - y_n)]),$$
$$y_{n+1} = (1 - b_n)y_n + b_n(J_\rho^\phi[\eta(y_n) - \rho(\gamma_2 T(x_{n+1}) + y_n - x_{n+1})]),$$

where $0 \le a_n \le 1$, $0 \le b_n \le 1$.

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$$x_{n+1} = (1 - a_n)x_n + a_n(J_\rho^\phi[\eta(x_n) - \rho(\gamma_1 T(y_n) + x_n - y_n)]),$$
$$y_{n+1} = J_\rho^\phi[\eta(y_n) - \rho(\gamma_2 T(x_n) + y_n - x_n)],$$

where $0 \le a_n \le 1$.

Method 4 Let Hypothesises A and B hold. For any$(x_0, y_0) \in B \times B$, calculate $(x_{n+1}, y_{n+1}) \in B \times B$:

$$x_{n+1} = (1 - a_n)x_n + a_n(J_\rho^\phi[\eta(x_n) - \rho(\gamma_1 T(y_n) + x_n - y_n)]),$$
$$y_{n+1} = J_\rho^\phi[\eta(y_n) - \rho(\gamma_2 T(x_{n+1}) + y_n - x_{n+1})],$$

where $0 \le a_n \le 1$.

Remark 2 By Lemma 1, methods 1-4 are well pose.

Theorem 1 *Let Hypothesises A and B hold. Let $\eta - I$ be $s-$Lipschitz continuous. Suppose that (x^*, y^*) is a solution of SMNVI (1). Additionally, we assume that operators $\gamma_1 T - I$ and $\gamma_2 T - I$ are s_1-Lipschitz and s_2-Lipschitz continuous, respectively, such that $s_2 b_n - a_n > 0$, $s_1 a_n - b_n > 0$, and $1 > \rho > 0$ satisfies*

$$\rho < \frac{a_n(\alpha - s - 1)}{b_n s_2 - a_n}, \qquad \rho < \frac{b_n(\alpha - s - 1)}{a_n s_1 - b_n}. \tag{5}$$

Then the sequence $\{(x_n, y_n)\}$ generated by Method 1 converges to (x^*, y^*).

Proof. Since (x^*, y^*) is a solution of SMNVI (1), we have

$$x^* = J_\rho^\phi[\eta(x^*) - \rho(\gamma_1 T(y^*) + x^* - y^*)],$$

then

$$x^* = (1 - a_n)x^* + a_n(J_\rho^\phi[\eta(x^*) - \rho(\gamma_1 T(y^*) + x^* - y^*)]).$$

Since $\gamma_1 T - I$ is s_1-Lipschitz continuous, η: $B \to B^*$is $\alpha-$strongly monotone and $\eta - I$ is $s-$Lipschitz continuous, we have

$$\begin{aligned}
& \|x_{n+1} - x^*\| \\
= \ & \|(1 - a_n)x_n + a_n(J_\rho^\phi[\eta(x_n) - \rho(\gamma_1 T(y_n) + x_n - y_n)]) - (1 - a_n)x^* - a_n(J_\rho^\phi[\eta(x^*) - \rho(\gamma_1 T(y^*) + x^* - y^*)])\| \\
\le \ & (1 - a_n)\|x_n - x^*\| + a_n\|J_\rho^\phi[\eta(x_n) - \rho(\gamma_1 T(y_n) + x_n - y_n)] - J_\rho^\phi[\eta(x^*) - \rho(\gamma_1 T(y^*) + x^* - y^*)]\| \\
= \ & (1 - a_n)\|x_n - x^*\| + a_n\|J_\rho^\phi[\eta(x_n) - \rho(\gamma_1 T(y_n) + x_n - y_n)] - J_\rho^\phi[\eta(x_n) - \rho(\gamma_1 T(y^*) + x_n - y^*)] + \\
& J_\rho^\phi[\eta(x_n) - \rho(\gamma_1 T(y^*) + x_n - y^*)] - J_\rho^\phi[\eta(x^*) - \rho(\gamma_1 T(y^*) + x^* - y^*)]\| \\
\le \ & (1 - a_n)\|x_n - x^*\| + \frac{a_n}{\alpha}\|\rho(\gamma_1 T(y_n) - y_n - \gamma_1 T(y^*) + y^*)\| + \frac{a_n}{\alpha}\|\eta(x_n) - x_n + (1 - \rho)x_n - \eta(x^*) + x^* + (\rho - 1)x^*\| \\
\le \ & (1 - a_n)\|x_n - x^*\| + \frac{\rho a_n s_1}{\alpha}\|y_n - y^*\| + \frac{a_n(s + 1 - \rho)}{\alpha}\|x_n - x^*\|,
\end{aligned}$$

where the second inequality obtained by Lemma 2.

Analogously for y^*, we obtain

$$\|y_{n+1} - y^*\| \leq (1 - b_n)\|y_n - y^*\| + \frac{\rho b_n s_2}{\alpha}\|x_n - x^*\| + \frac{b_n(s + 1 - \rho)}{\alpha}\|y_n - y^*\|.$$

Hence, we have

$$\|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \leq \max\{\theta_1, \theta_2\}(\|x_n - x^*\| + \|y_n - y^*\|), \tag{6}$$

where, $\theta_1 = 1 - a_n + \dfrac{a_n(s + 1 - \rho) + \rho b_n s_2}{\alpha}, \theta_2 = 1 - b_n + \dfrac{\rho a_n s_1 + b_n(s + 1 - \rho)}{\alpha}$. Define the norm $\|\cdot\|_*$ on $B \times B$ as:

$$\|(u, v)\|_* = \|u\| + \|v\|, \quad \forall (u, v) \in B \times B.$$

Obviously, $(B \times B, \|\cdot\|_*)$ is a Banach space. Hence, (6) implies

$$\|(x_{n+1}, y_{n+1}) - (x^*, y^*)\|_* \leq \max\{\theta_1, \theta_2\}\|(x_n, y_n) - (x^*, y^*)\|_*.$$

By (5), we have $\theta_1, \theta_2 \in (0, 1)$. Hence, by Lemma 4, we have

$$(x_n, y_n) \to (x^*, y^*), \quad n \to \infty,$$

which completes the proof. □

Similarly to the proof of theorem 1, we have the following theorems.

Theorem 2 *Let Hypothesises A and B hold. Let $\eta - I$ be s−Lipschitz continuous. Suppose that (x^*, y^*) is a solution of SMNVI (1). Additionally, we assume that operators $\gamma_1 T - I$ and $\gamma_2 T - I$ are s_1−Lipschitz and s_2−Lipschitz continuous, respectively, such that $s_2 b_n - a_n > 0$, $s_1 a_n - b_n > 0$, and $1 > \rho > 0$ satisfies*

$$b_n s_2 a_n s_1 \rho^2 + (\alpha a_n s_1 - \alpha b_n)\rho - \alpha^2 b_n < 0,$$

$$-a_n b_n s_2 \rho^2 + \hat{b}\rho - \alpha^2 < 0,$$

where

$$\hat{b} = \alpha b_n s_2 - \alpha a_n - \alpha a_n b_n s_2 + a_n s b_n s_2 + a_n b_n s_2.$$

Then the sequence $\{(x_n, y_n)\}$ generated by Method 2 converges to (x^*, y^*).

Theorem 3 *Let Hypothesises A and B hold. Let $\eta - I$ be s−Lipschitz continuous. Suppose that (x^*, y^*) is a solution of SMNVI (1). Additionally, we assume that operators $\gamma_1 T - I$ and $\gamma_2 T - I$ are s_1−Lipschitz and s_2−Lipschitz continuous, respectively, such that $s_2 - a_n > 0$, $s_1 a_n - 1 > 0$, $1 > \rho > 0$ satisfies*

$$\rho < \frac{a_n(\alpha - s - 1)}{s_2 - a_n}, \qquad \rho < \frac{\alpha - 1 - s}{s_1 a_n - 1},$$

Then the sequence $\{(x_n, y_n)\}$ generated by Method 3 converges to (x^*, y^*).

Theorem 4 *Let Hypothesises A and B hold. Let $\eta - I$ be s−Lipschitz continuous. Suppose that (x^*, y^*) is a solution of SMNVI (1). Additionally, we assume that operators $\gamma_1 T - I$ and $\gamma_2 T - I$ are s_1−Lipschitz and s_2−Lipschitz continuous, respectively, such that $1 > \rho > 0$ satisfies*

$$s^2\rho + (\alpha - a_n s_2 s - a_n s_2 - \alpha s_2 + \alpha a_n s_2)\rho + a_n\alpha^2 - a_n\alpha s + a_n\alpha > 0,$$

$$a_n s_1 s_2 \rho^2 + (a_n s_1 \alpha - \alpha)\rho + \alpha s + \alpha - \alpha^2 < 0,$$

Then the sequence $\{(x_n, y_n)\}$ generated by Method 4 converges to (x^*, y^*).

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