Journal of Mathematics Research; Vol. 6, No. 1; 2014
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

New Approach of $\left(\acute{G}/G\right)$ Expansion Method. Applications to KdV Equation

Mohammad Ali Bashir[1] & Alaaeddin Amin Moussa[2,3]

[1] Mathematics Department, Faculty of Science, Alnillin University, Alkortom, Alsodan

[2] Mathematics Department, Faculty of post Graduate Studies, Omdurman Islamic University, Alkortom, Alsodan

[3] Department of Management Information System and Production Management, College of Business & Economics, Qassim University, Buraidah, K.S.A.

Correspondence: Alaaeddin Amin Moussa, Mathematics Department, Faculty of post Graduate Studies, Omdurman Islamic University, Alkortom, Alsodan. E-mail: Alaamath81@gmail.com

Received: November 13, 2013 Accepted: December 10, 2013 Online Published: December 24, 2013

doi:10.5539/jmr.v6n1p24 URL: http://dx.doi.org/10.5539/jmr.v6n1p24

Abstract

This paper proposes a new approach of $\left(\acute{G}/G\right)$ -expansion method for constructing more general exact solutions of nonlinear evolution equation. By using this method numerous new and more general exact solutions have been obtained. We will apply this new method to solve the nonlinear KdV equation. Various exact traveling wave solutions of these equations are obtained that include the exponential function solutions, the hyperbolic function solutions and the trigonometric function solutions. Further, the method is efficient to solve nonlinear evolution equations in many areas such as physics and engineering.

Keywords: $\left(\acute{G}/G\right)$-expansion method, KdV equation, exact solutions, travelling wave solution

1. Introduction

Several phenomena in various fields are frequently described by nonlinear partial differential equation (NLPDEs), for instance nonlinear optics, elastic media, plasma physics, fluid mechanics, chemistry, biology and many others (Rogers & Shadwick, 1982; Akbar, Ali, & Zayed, 2012). For better understanding of nonlinear phenomena and its real life applications, it is more significant to establish exact traveling wave solutions. Recently, many methods have been proposed to generate analytical solutions. For instance, the Backlund transformation method (Rogers & Shadwick, 1982), the inverse scattering method (Ablowitz & Clarkson, 1991) the truncated Painleve expansion method (Weiss, Tabor, & Carnevale, 1982), the Weirstrass elliptic function method (Kudryashov, 1990), the Hirotas bilinear transformation method (Hirota, 1971), the Jacobi elliptic function expansion method (S. K. Liu, Fu, S. D. Liu, & Zhao, 2001), the generalized Riccati equation method (Yan & Zhang, 2001; Naher & Abdullah, 2012), the tanh-coth method (Malfliet, 1992; Wazwaz, 2007), the F-expansion method (Wang & Li, 2005; Abdou, 2007), the variational iteration method (Mohyud-Din, M. Noor, K. Noor, & Hosseini, 2010), the direct algebraic method (Soliman & Abdo, 2009), the homotopy perturbation method (Mohyud-Din, Yildirim, & Sariaydin, 2011), the Exp-function method (He & Wu, 2006; Ma & Zhu, 2012), and others (Wazwaz, 2011; Bagarti, Roy, Kundu, & Dev, 2012). This work intends to generate many novel and more general exact traveling wave solutions, by proposing new approach of $\left(\acute{G}/G\right)$-expansion method for investigating NLEEs. The proposed method is used to solve the KdV equation to show its efficiency

2. Description of New Approach of $\left(\acute{G}/G\right)$ Expansion Method

Let us consider the following equation:

$$F(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, u_{xxt}, ...) = 0 \tag{1}$$

where F is a polynomial in its arguments that includes nonlinear terms and the highest order derivatives.

Step 1 Use the transformation $u(x,t) = u(\xi)$; $\xi = k_1 x + k_2 t$, where k_1, k_2 are constant, to be determined latter. permits use reducing (1) to an ODE for $u = u(\xi)$ in the form

$$P(u, k_1 u', k_2 u', k_1 k_2 u'', \ldots) = 0, \qquad u' = \frac{du}{d\xi}, \tag{2}$$

Note that the left hand side of (1) is a polynomial in u and its various derivatives.

Step 2 Suppose that the NLODE (2) has the following solution:

$$u(\xi) = \sum_{i=0}^{m} \alpha_i \left(\frac{A_1 \cdot \left(\frac{G'(\xi)}{G(\xi)}\right) + A_2}{B_1 + B_2 \cdot \left(\frac{\lambda}{2} + \frac{G'(\xi)}{G(\xi)}\right)} \right)^i \tag{3}$$

where $G = G(\xi)$ satisfies the following auxiliary ordinary differential equation:

$$G'' + \lambda G' + \mu G = 0 \tag{4}$$

where $A_1, A_2, B_1, B_2, \alpha_i\,(i = 0, 1, 2, ...m)$, λ and μ are constants to be determined later, m is a positive integer.

Step 3 Find the value of m by balancing the highest order derivative and nonlinear terms in (3).

Step 4 Substituting (3) into (2) and using (4), and then setting all the coefficients of $\left(\frac{\acute{G}}{G}\right)$ of the obtained systems numerator to zero, produces a set of over determined nonlinear algebraic equations for $k_1, k_2, A_1, A_2, B_1, B_2$ and $\alpha_i\,(i = 0, 1, 2, ...m)$.

Step 5 Suppose that the value of $k_1, k_2, A_1, A_2, B_1, B_2$ and $\alpha_i (i = 0, 1, 2, ...m)$ can be found by solving the algebraic system which are found in step 4. Because the solutions of (4) are known , substituting $k_1, k_2, A_1, A_2, B_1, B_2, c, \alpha_i (i = 0, 1, 2, ...m)$ and the general solutions of (4) into (3), we have the exact solutions of the nonlinear PDEs (1).

3. Exact Solutions of the Nonlinear KdV Equation

In this section, the proposed methods is applied for studying the KdV equation:

$$u_t + \delta \cdot u \cdot u_x + u_{xxx} = 0 \tag{5}$$

using the wave transformation (2) into the (5), produces:

$$k_2 \cdot \acute{u} + \delta \cdot k_1 \cdot u \cdot \acute{u} + k_1^3 \cdot u^{(3)} = 0 \tag{6}$$

the homogeneous balance between $u \cdot \acute{u}$ and $u^{(3)}$ in (6), we obtain $m = 2$. Therefore, the solution of (6) is of the form:

$$u(\xi) = \alpha_0 + \alpha_1 \left(\frac{A_1 \cdot \left(\frac{G'(\xi)}{G(\xi)}\right) + A_2}{B_1 + B_2 \cdot \left(\frac{\lambda}{2} + \frac{G'(\xi)}{G(\xi)}\right)} \right) + \alpha_2 \left(\frac{A_1 \cdot \left(\frac{G'(\xi)}{G(\xi)}\right) + A_2}{B_1 + B_2 \cdot \left(\frac{\lambda}{2} + \frac{G'(\xi)}{G(\xi)}\right)} \right)^2 \tag{7}$$

where $\alpha_0, \alpha_1, \alpha_2, A_1, A_2, B_1$ and B_2 are arbitrary constants to be determined. Substituting (7) and (4) into (6), and then setting all the coefficients of $\left(\frac{\acute{G}}{G}\right)^j$, $(j = 0, 1, 2, ...)$ of the obtaining systems numerator to zero, yields a set of over determined nonlinear algebraic equations for $\alpha_0, \alpha_1, \alpha_2, A_1, A_2, B_1, B_2, k_1$ and k_2. Solving the over determined algebraic equations by Maple, we find the following results:

Case (1)

$$\begin{aligned}
\alpha_0 &= \alpha_0, \alpha_1 = \alpha_1, \alpha_2 = \alpha_2, A_1 = A_1, B_1 = B_1, B_2 = B_2 \\
A_2 &= \frac{\left(\lambda^2 - 4\mu\right)\left(B_2^2\alpha_1 + 2B_2A_1\alpha_2\right) + 4A_1\lambda B_1\alpha_2 - 4\alpha_1 B_1^2}{8\alpha_2 B_1} \\
k_1 &= \frac{\left(\sqrt{\frac{-\delta}{48\cdot\alpha_2}}\right)(2A_1\alpha_2 + \alpha_1 B_2)}{B_1} \\
k_2 &= -\left(\frac{\delta \cdot k_1}{24 \cdot B_1^2\alpha_2}\right)\left[24B_1^2\alpha_2\alpha_0 - 6\alpha_1^2B_1^2 + \left(B_2^2\alpha_1^2 + 4\alpha_1\alpha_2B_2A_1 + 4A_1^2a_2^2\right)\left(\lambda^2 - 4\mu\right)\right]
\end{aligned} \tag{8}$$

Substituting (8) into (7), we have:

$$
\begin{aligned}
u(\xi) &= \alpha_0 + \alpha_1 \left(\frac{A_1 . \left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{\left(\lambda^2-4\mu\right)\left(B_2^2\alpha_1+2B_2A_1\alpha_2\right)+4A_1\lambda B_1\alpha_2-4\alpha_1B_1^2}{8\alpha_2B_1}}{B_1 + B_2 \cdot \left(\frac{\lambda}{2} + \frac{G'(\xi)}{G(\xi)}\right)} \right) \\
&\quad + \alpha_2 \left(\frac{A_1 \cdot \left(\frac{G'(\xi)}{G(\xi)}\right) + \frac{\left(\lambda^2-4\mu\right)\left(B_2^2\alpha_1+2B_2A_1\alpha_2\right)+4A_1\lambda B_1\alpha_2-4\alpha_1B_1^2}{8\alpha_2B_1}}{B_1 + B_2 \cdot \left(\frac{\lambda}{2} + \frac{G'(\xi)}{G(\xi)}\right)} \right)^2
\end{aligned}
\tag{9}
$$

where

$$
\begin{aligned}
\xi &= \left(\frac{\left(\sqrt{\frac{-\delta}{48\cdot\alpha_2}}\right)(2A_1\alpha_2 + \alpha_1 B_2)}{B_1} \right) x \\
&\quad + \left(-\left(\frac{\delta \cdot k_1}{24 \cdot B_1^2 \alpha_2}\right) \left(24B_1^2\alpha_2\alpha_0 - 6\alpha_1^2B_1^2\left(B_2^2\alpha_1^2 + 4\alpha_1\alpha_2B_2A_1 + 4A_1^2a_2^2\right)\left(\lambda^2 - 4\mu\right)\right)\right) t
\end{aligned}
\tag{10}
$$

Therefor, the following three types of exact solution of (5) are obtained:

Case (1-1) When $\lambda^2 - 4\mu > 0$, we obtain:

$$
u(\xi) = \left[
\begin{aligned}
&\alpha_0 + \alpha_1 \left(\frac{A_1 \cdot \left(\frac{\lambda}{2} + \frac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} \right) + \frac{\left(\lambda^2-4\mu\right)\left(B_2^2\alpha_1+2B_2A_1\alpha_2\right)+4A_1\lambda B_1\alpha_2-4\alpha_1B_1^2}{8\alpha_2B_1}}{B_1 + B_2 \cdot \left(\frac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} \right)} \right) \\
&+\alpha_2 \left(\frac{A_1 \cdot \left(\frac{\lambda}{2} + \frac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} \right) + \frac{\left(\lambda^2-4\mu\right)\left(B_2^2\alpha_1+2B_2A_1\alpha_2\right)+4A_1\lambda B_1\alpha_2-4\alpha_1B_1^2}{8\alpha_2B_1}}{B_1 + B_2 \cdot \left(\frac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)} \right)} \right)^2
\end{aligned}
\right]
\tag{11}
$$

where $\alpha_0, \alpha_1, \alpha_2, A_1, B_1, B_2, c_1$ and c_2 are constants.

Particularly, setting $\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 1, A_1 = -2, B_1 = -2, B_2 = -3, \mu = 0, \lambda = 2, c_2 = 0, c_1 \neq 0, \delta = -6$. We find:

$$
\begin{aligned}
u(\xi) &= \left(\frac{\frac{17}{4} + 2\coth\xi}{2 + 3 \cdot \coth\xi} \right) + \left(\frac{\frac{17}{4} + 2\coth\xi}{2 + 3 \cdot \coth\xi} \right)^2 \\
\xi &= \sqrt{8}\left[\left(\frac{7}{16}\right) x + \left(\frac{301}{64}\right) t \right]
\end{aligned}
\tag{12}
$$

See Figure 1.

Case (1-2) When $\lambda^2 - 4\mu < 0$ we obtain:

$$
u(\xi) = \left[
\begin{aligned}
&\alpha_0 + \alpha_1 \left(\frac{A_1 \cdot \left(\frac{\lambda}{2} + \frac{-c_1\cdot\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)} \right) + \frac{\left(\lambda^2-4\mu\right)\left(B_2^2\alpha_1+2B_2A_1\alpha_2\right)+4A_1\lambda B_1\alpha_2-4\alpha_1B_1^2}{8\alpha_2B_1}}{B_1 + B_2 \cdot \left(\frac{-c_1\cdot\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)} \right)} \right) \\
&+\alpha_2 \left(\frac{A_1 \cdot \left(\frac{\lambda}{2} + \frac{-c_1\cdot\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)} \right) + \frac{\left(\lambda^2-4\mu\right)\left(B_2^2\alpha_1+2B_2A_1\alpha_2\right)+4A_1\lambda B_1\alpha_2-4\alpha_1B_1^2}{8\alpha_2B_1}}{B_1 + B_2 \cdot \left(\frac{-c_1\cdot\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)} \right)} \right)^2
\end{aligned}
\right]
\tag{13}
$$

where $\alpha_0, \alpha_1, \alpha_2, A_1, B_1, B_2, c_1$ and c_2 are constants.

Particularly, setting $\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 1, A_1 = 1, B_1 = 4, B_2 = 1, \mu = 3, \lambda = \sqrt{3}, c_2 = 0, c_1 \neq 0, \delta = -6$. We find:

$$\begin{aligned} u(\xi) &= \left(\frac{\frac{3}{2}\cdot\tan\left(\frac{3}{2}\xi\right)+\frac{91}{32}}{\frac{3}{2}\cdot\tan\left(\frac{3}{2}\xi\right)-4}\right)+\left(\frac{\frac{3}{2}\cdot\tan\left(\frac{3}{2}\xi\right)+\frac{91}{32}}{\frac{3}{2}\cdot\tan\left(\frac{3}{2}\xi\right)-4}\right)^2 \\ \xi &= \sqrt{8}\cdot\left(\left(\frac{3}{32}\right)x-\left(\frac{531}{2048}\right)t\right) \end{aligned} \tag{14}$$

See Figure 2.

Case (1-3) When $\lambda^2 - 4\mu = 0$, we obtain:

$$u(\xi) = \alpha_0 + \alpha_1\left(\frac{A_1\left(\frac{c_2}{c_1+c_2\xi}-\frac{\lambda}{2}\right)+\frac{4A_1\lambda B_1\alpha_2-4\alpha_1 B_1^2}{8\alpha_2 B_1}}{B_1+B_2.\left(\frac{c_2}{c_1+c_2\xi}\right)}\right)+\alpha_2\left(\frac{A_1\left(\frac{c_2}{c_1+c_2\xi}-\frac{\lambda}{2}\right)+\frac{4A_1\lambda B_1\alpha_2-4\alpha_1 B_1^2}{8\alpha_2 B_1}}{B_1+B_2\left(\frac{c_2}{c_1+c_2\xi}\right)}\right)^2 \tag{15}$$

where $\alpha_0, \alpha_1, \alpha_2, A_1, B_1, B_2, c_1$ and c_2 are constants.

In particular, setting $\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 1, A_1 = 2, B_1 = 3, B_2 = 2, \mu = 4, \lambda = 4, c_2 = -2, c_1 = 2, \delta = -6$. We find:

$$\begin{aligned} u(\xi) &= -\frac{(3\xi+5)(3\xi-7)}{4(3\xi-1)^2} \\ \xi &= \sqrt{8}\cdot\left(\left(\frac{1}{4}\right)x-\left(\frac{3}{8}\right)t\right) \end{aligned} \tag{16}$$

See Figure 3.

Case (2)

$$\begin{aligned} A_1 &= A_1, A_2 = A_2, B_2 = B_2, B_1 = 0, k_1 = k_1, \alpha_0 = \alpha_0 \\ k_2 &= \left[\frac{-k_1\left(\left(\lambda^2-4\mu\right)\left(8A_2k_1^2\right)(\lambda A_1 - A_2)+\left(k_1^2A_1^2\right)\left(\left(\lambda^2-8\mu\right)^2-16\mu^2\right)+\delta\alpha_0(\lambda A_1-2A_2)^2\right)}{(\lambda A_1-2A_2)^2}\right] \\ \alpha_1 &= \frac{\left(6A_1B_2k_1^2\right)\left(\lambda^2-4\mu\right)^2}{\delta\cdot(\lambda A_1-2A_2)^2} \\ \alpha_2 &= -\frac{\left(3B_2^2k_1^2\right)\left(\lambda^2-4\mu\right)^2}{\delta\cdot(\lambda A_1-2A_2)^2} \end{aligned} \tag{17}$$

Substituting (17) into (7), we have:

$$u(\xi) = \alpha_0 + \left(\frac{\left(3B_2k_1^2\right)\left(\lambda^2-4\mu\right)^2}{\delta(\lambda A_1-2A_2)^2}\right)\left[2A_1\cdot\left(\frac{A_1\cdot\left(\frac{G'(\xi)}{G(\xi)}\right)+A_2}{B_2\cdot\left(\frac{\lambda}{2}+\frac{G'(\xi)}{G(\xi)}\right)}\right)-B_2\cdot\left(\frac{A_1\cdot\left(\frac{G'(\xi)}{G(\xi)}\right)+A_2}{B_2\cdot\left(\frac{\lambda}{2}+\frac{G'(\xi)}{G(\xi)}\right)}\right)^2\right] \tag{18}$$

where

$$\xi = \left[k_1x+\left(\frac{-k_1\left(\left(\lambda^2-4\mu\right)\left(8A_2k_1^2\right)(\lambda A_1-A_2)+\left(k_1^2A_1^2\right)\left[\left(\lambda^2-8\mu\right)^2-16\mu^2\right]+\delta\alpha_0(\lambda A_1-2A_2)^2\right)}{(\lambda A_1-2A_2)^2}\right)t\right] \tag{19}$$

Thus, the following three types of exact solution of (6) are obtained:

Case (2-1) When $\lambda^2 - 4\mu > 0$ we obtain:

$$u(\xi) = \alpha_0 + \left(\frac{\left(6A_1B_2k_1^2\right)\left(\lambda^2-4\mu\right)^2}{\delta\left(\lambda A_1 - 2A_2\right)^2}\right)\left[\begin{array}{l} 2A_1\left(\dfrac{A_1\cdot\left(\frac{\lambda}{2}+\dfrac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right)+A_2}{B_2\cdot\left(\dfrac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right)}\right) \\ -B_2\left(\dfrac{A_1\cdot\left(\frac{\lambda}{2}+\dfrac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right)+A_2}{B_2\cdot\left(\dfrac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right)}\right)^2 \end{array}\right] \tag{20}$$

where $A_1, A_2, B_2, k_1, \alpha_0, \delta, c_1$ and c_2 are constants.

Particularly, setting, $A_1 = -1, A_2 = -2, B_2 = 1, \mu = 0, \lambda = 2, c_2 = -1, c_1 = 2, \delta = -6, k_1 = 1, \alpha_0 = 0$. We find:

$$\begin{aligned} u(\xi) &= \frac{-6}{\left(2\cosh(\xi) - \sinh(\xi)\right)^2} \\ \xi &= x - 4t \end{aligned} \tag{21}$$

See Figure 4.

Case (2-2) When $\lambda^2 - 4\mu < 0$ we obtain:

$$u(\xi) = \alpha_0 + \left(\frac{\left(6A_1B_2k_1^2\right)\left(\lambda^2-4\mu\right)^2}{\delta\left(\lambda A_1 - 2A_2\right)^2}\right)\left[\begin{array}{l} 2A_1\cdot\left(\dfrac{A_1\cdot\left(\frac{\lambda}{2}+\dfrac{-c_1\cdot\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right)+A_2}{B_2\cdot\left(\dfrac{-c_1\cdot\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right)}\right) \\ -B_2\cdot\left(\dfrac{A_1\cdot\left(\frac{\lambda}{2}+\dfrac{-c_1\cdot\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right)+A_2}{B_2\cdot\left(\dfrac{-c_1\cdot\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right)}\right)^2 \end{array}\right] \tag{22}$$

where $A_1, A_2, B_2, k_1, \alpha_0, \delta, c_1$ and c_2 are constants.

Particularly, setting: $A_1 = \sqrt{3}, A_2 = 1, B_2 = 1, \mu = 3, \lambda = \sqrt{3}, c_2 = 1, c_1 = -1, \delta = -6, k_1 = 1, \alpha_0 = 0$. We find:

$$\begin{aligned} u(\xi) &= -9\cdot\left(\frac{14\sin(3\xi) + 13}{\sin(3\xi) + 1}\right) \\ \xi &= x - (747)\,t \end{aligned} \tag{23}$$

See Figure 5.

Case (2-3) When $\lambda^2 - 4\mu = 0$, we get the not important solution in the form:

$$u(\xi) = \alpha_0 \tag{24}$$

where α_0 is arbitrary constant.

Case (3)

$$\begin{aligned}\alpha_0 &= \alpha_0, \alpha_1 = \alpha_1, B_1 = B_1, B_2 = 0, A_1 = A_1, k_1 = k_1 \\ \alpha_2 &= -\frac{12B_1^2k_1^2}{\delta A_1^2} \\ A_2 &= \frac{A_1\left(12\lambda B_1k_1^2 + \alpha_1\delta A_1\right)}{24B_1k_1^2} \\ k_2 &= \frac{96k_1^4B_1^2\left(\lambda^2 - 4\mu\right) - \left(\alpha_1^2\delta^2A_1^2 + 48\alpha_0\delta k_1^2B_1^2\right)}{48k_1B_1^2}\end{aligned} \tag{25}$$

Substituting (25) into (7), we have:

$$u(\xi) = \begin{bmatrix} \alpha_0 + \frac{\alpha_1}{B_1}\left[A_1 \cdot \left(\frac{G'(\xi)}{G(\xi)}\right) + \left(\frac{A_1\left(12\lambda B_1k_1^2+\alpha_1\delta A_1\right)}{24B_1k_1^2}\right)\right] + \\ \left(-\frac{12k_1^2}{\delta A_1^2}\right)\left[A_1 \cdot \left(\frac{G'(\xi)}{G(\xi)}\right) + \left(\frac{A_1\left(12\lambda B_1k_1^2+\alpha_1\delta A_1\right)}{24B_1k_1^2}\right)\right]^2 \end{bmatrix} \tag{26}$$

where

$$\xi = k_1x + \left[\frac{96k_1^4B_1^2\left(\lambda^2 - 4\mu\right) - \left(\alpha_1^2\delta^2A_1^2 + 48\alpha_0\delta k_1^2B_1^2\right)}{48k_1B_1^2}\right]t \tag{27}$$

Therefore, the three we following types of exact solution of (6) are obtained:

Case (3-1) When $\lambda^2 - 4\mu > 0$, we obtain:

$$u(\xi) = \begin{bmatrix} \alpha_0 + \frac{\alpha_1}{B_1}\left[A_1 \cdot \left(\frac{\lambda}{2} + \frac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right) + \left(\frac{A_1\left(12\lambda B_1k_1^2+\alpha_1\delta A_1\right)}{24B_1k_1^2}\right)\right] + \\ \left(-\frac{12k_1^2}{\delta A_1^2}\right)\left[A_1 \cdot \left(\frac{\lambda}{2} + \frac{c_1\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}{c_1\sinh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)+c_2\cosh\left(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi\right)}\right) + \left(\frac{A_1\left(12\lambda B_1k_1^2+\alpha_1\delta A_1\right)}{24B_1k_1^2}\right)\right]^2 \end{bmatrix} \tag{28}$$

where $\alpha_0, \alpha_1, B_1, k_1, A_1, \delta, c_1$ and c_2 are constants.

Particularly, setting $\alpha_0 = 0, \alpha_1 = 1, \alpha_2 = 2, k_1 = 1, \mu = 0, \lambda = 2, c_2 = 0, \delta = -6, B_1 = 1, A_1 = 1$. We find:

$$u(\xi) = \left(\coth(\xi) - \frac{1}{4}\right) + 2.\left(\coth(\xi) - \frac{1}{4}\right)^2 \tag{29}$$

where

$$\xi = x + \frac{29}{4}t \tag{30}$$

See Figure 6.

Case (3-2) When $\lambda^2 - 4\mu < 0$, we obtain:

$$u(\xi) = \begin{bmatrix} \alpha_0 + \frac{\alpha_1}{B_1}\left[A_1 \cdot \left(\frac{\lambda}{2} + \frac{-c_1\cdot\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right) + \left(\frac{A_1\left(12\lambda B_1k_1^2+\alpha_1\delta A_1\right)}{24B_1k_1^2}\right)\right] + \\ \left(-\frac{12k_1^2}{\delta A_1^2}\right)\left[A_1 \cdot \left(\frac{\lambda}{2} + \frac{-c_1\cdot\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}{c_1\cos\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)+c_2\sin\left(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi\right)}\right) + \left(\frac{A_1\left(12\lambda B_1k_1^2+\alpha_1\delta A_1\right)}{24B_1k_1^2}\right)\right]^2 \end{bmatrix} \tag{31}$$

where $\alpha_0, \alpha_1, k_1, A_1, B_1, \delta, c_1$ and c_2 are constants.

Particularly, setting $\alpha_0 = 0, \alpha_1 = 1, k_1 = 1, A_1 = 1, B_1 = 1, \mu = 1, \lambda = \sqrt{3}, c_2 = 0, \delta = -6$. We find:

$$u(\xi) = -\left(\frac{1}{2}\tan\left(\frac{\xi}{2}\right) + \frac{1}{4}\right) + 2\left(\frac{1}{2}\tan\left(\frac{\xi}{2}\right) + \frac{1}{4}\right)^2 \tag{32}$$

where

$$\xi = x - \frac{11}{4}t \tag{33}$$

See Figure 7.

Case (3-3) When $\lambda^2 - 4\mu = 0$, we obtain:

$$u(\xi) = \left[\begin{array}{l} \alpha_0 + \frac{\alpha_1}{B_1}\left[A_1.\left(\frac{c_2}{c_1+c_2\xi} - \frac{\lambda}{2}\right) + \left(\frac{A_1\left(12\lambda B_1 k_1^2 + \alpha_1\delta A_1\right)}{24B_1k_1^2}\right)\right] + \\ \left(-\frac{12k_1^2}{\delta A_1^2}\right)\left[A_1.\left(\frac{c_2}{c_1+c_2\xi} - \frac{\lambda}{2}\right) + \left(\frac{A_1\left(12\lambda B_1 k_1^2 + \alpha_1\delta A_1\right)}{24B_1k_1^2}\right)\right]^2 \end{array}\right] \tag{34}$$

where $\alpha_0, \alpha_1, A_1, B_1, \delta, k_1, c_1$ and c_2 are constants.

Particularly, setting $\alpha_0 = 1, \alpha_1 = 1, k_1 = 1, B_1 = 1, A_1 = 1, \mu = 1, \lambda = 2, c_1 = 0, \delta = -6$. We find:

$$u(\xi) = \frac{7}{8} + \frac{2}{\xi^2} \tag{35}$$

where

$$\xi = x + \frac{21}{4}t \tag{36}$$

See Figure 8.

4. Table Graphics

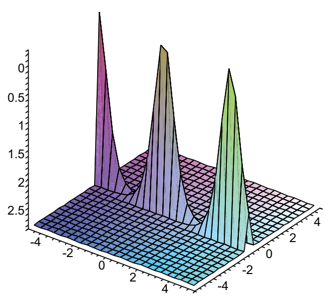

Figure 1. $u(\xi) = \left(\frac{\frac{17}{4}+2\coth\xi}{2+3\cdot\coth\xi}\right) + \left(\frac{\frac{17}{4}+2\coth\xi}{2+3\cdot\coth\xi}\right)^2$

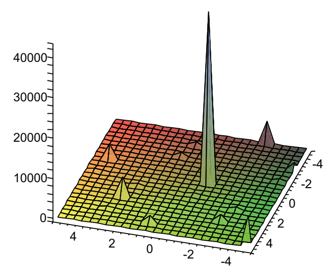

Figure 2. $u(\xi) = \left(\frac{\frac{3}{2}\cdot\tan\left(\frac{3}{2}\xi\right)+\frac{91}{32}}{\frac{3}{2}\cdot\tan\left(\frac{3}{2}\xi\right)-4}\right) + \left(\frac{\frac{3}{2}\cdot\tan\left(\frac{3}{2}\xi\right)+\frac{91}{32}}{\frac{3}{2}\cdot\tan\left(\frac{3}{2}\xi\right)-4}\right)^2$

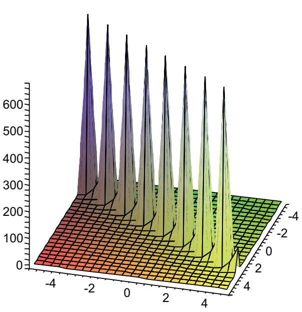

Figure 3. $u(\xi) = -\frac{(3\xi+5)(3\xi-7)}{4(3\xi-1)^2}$

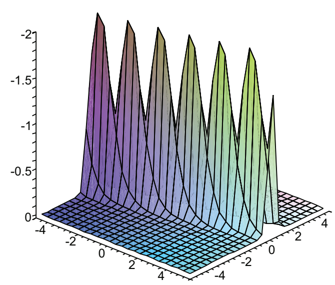

Figure 4. $u(\xi) = \frac{-6}{(2\cosh(\xi)-\sinh(\xi))^2}$

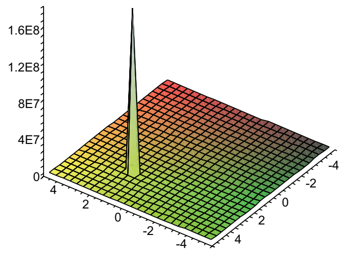

Figure 5. $u(\xi) = -9 \cdot \left(\frac{14\sin(3\xi)+13}{\sin(3\xi)+1}\right)$

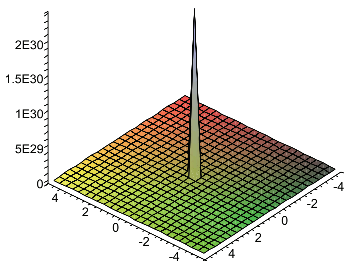

Figure 6. $u(\xi) = \left(\coth(\xi) - \frac{1}{4}\right) + 2 \cdot \left(\coth(\xi) - \frac{1}{4}\right)^2$

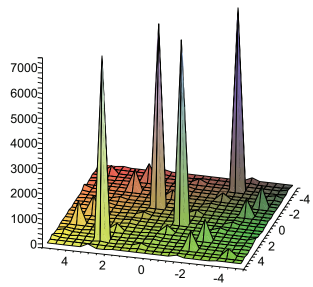

Figure 7. $u(\xi) = -\left(\frac{1}{2}\tan\left(\frac{\xi}{2}\right) + \frac{1}{4}\right) + 2\left(\frac{1}{2}\tan\left(\frac{\xi}{2}\right) + \frac{1}{4}\right)^2$

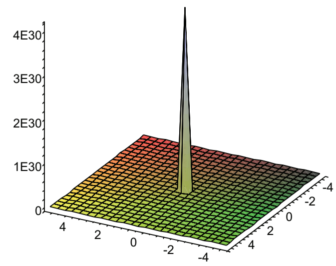

Figure 8. $u(\xi) = \frac{7}{8} + \frac{2}{\xi^2}$

5. Remarks and Conclusion

We note that the solution in the Case (3), a perfect fit with solutions that give them the classic $\left(\frac{\acute{G}}{G}\right)$-expansion method. However, the solution in both cases (1) and (2) are new, and cannot be obtained from the classic $\left(\frac{\acute{G}}{G}\right)$-expansion method.

For example, in the case (1), if we substitute $A_1 = B_1; B_2 = 0; \alpha_1 = \lambda\alpha_2$. We have $\left[u = \alpha_0 + \alpha_1\left(\frac{\acute{G}}{G}\right) + \alpha_2\left(\frac{\acute{G}}{G}\right)^2\right]$.

The proposed solutions have been tested with Maple by substituting them in Equation (5).

In this article, a novel approach of $\left(\frac{\acute{G}}{G}\right)$-expansion method is proposed, and applied to find the exact solutions for the nonlinear KdV equation. These solutions comprise the trigonometric function solutions, the hyperbolic function solutions, and the rational function solutions. This work shows that, the new approach of $\left(\frac{\acute{G}}{G}\right)$-expansion method is efficient and can be applied to several other NLPDEs in mathematical physics.

References

Abdou, M. A. (2007). The extended F-expansion method and its application for a class of nonlinear evolution equations. *Chaos, Solitons, and Fract, 31*(1), 95-104. http://dx.doi.org/10.1016/j.chaos.2005.09.030

Ablowitz, M. J., & Clarkson, P. A. (1991). Cambridge Univ. Press, Cambridge.

Akbar, M. A., Ali, N. H. M., & Zayed, E. M. E. (2012). A Generalized and Improved-Expansion Method for Nonlinear Evolution Equations. *Math. Prob. Engr., 2012*(2012), Article ID 459879. http://dx.doi.org/10.1155/2012/459879

Bagarti, T., Roy, A., Kundu, K., & Dev, B. N. (2012). A reaction diffusion model of pattern formation in clustering of adatoms on silicon surfaces. *AIP Advances, 2*(4), 042101. http://dx.doi.org/10.1063/1.4757592

He, J. H., & Wu, X. H. (2006). Exp-function method for nonlinear wave equations. *Chaos, Solitons and Fract, 30*(3), 700-708. http://dx.doi.org/10.1016/j.chaos.2006.03.020

Hirota, R. (1971). Exact Solution of the Korteweg-de Vries Equation for Multiple Collisions of Solitons. *Phys.*

Rev. Lett., 27, 1192-1194.

Kudryashov, N. A. (1990). Exact solutions of the generalized Kuramoto-Sivashinsky equation. *Phys. Lett. A, 147*, 287-291. http://dx.doi.org/10.1016/0375-9601(90)90449-X

Liu, S. K., Fu, Z., Liu, S. D., & Zhao, Q. (2001). Jacobi elliptic function expansion method and periodic wave solutions of nonlinear wave equations. *Phys. Lett. A, 289*(1-2), 69. http://dx.doi.org/10.1016/S0375-9601(01)00580-1

Ma, W. X., & Zhu, Z. (2012). Solving the (3+1)-dimensional generalized KP and BKP equations by the multiple exp-function algorithm. *Appl. Math. Compu., 218*(24), 11871-11879. http://dx.doi.org/10.1016/j.amc.2012.05.049

Malfliet, W. (1992). Solitary wave solutions of nonlinear wave equations. *Am. J. Phys., 60*, 650. http://dx.doi.org/10.1119/1.17120

Mohyud-Din, S. T., Noor, M. A., Noor, K. I., & Hosseini, M. M. (2010). Solution of singular equations by He's variational iteration method. *Int. J. Nonlinear Sci. Numer. Simulat, 11*, 81-86.

Mohyud-Din, S. T., Yildirim, A., & Sariaydin, S. (2011). Numerical soliton solution of the Kaup-Kupershmidt equation. *Int. J. Numerical Meth. Heat Fluid Flow, 21*(3), 272-281. http://dx.doi.org/10.1108/09615531111108459

Naher, H., & Abdullah, F. A. (2012). New Traveling Wave Solutions by the Extended Generalized Riccati Equation Mapping Method of the -Dimensional Evolution Equation. *J. Appl. Math., 2012*(2012), Article ID 486458. http://dx.doi.org/10.1155/2012/486458

Rogers, C., & Shadwick, W. F. (1982). *Bäcklund Transformations and Their Applications*. New York: Academic Press.

Soliman, A. A., & Abdo, H. A. (2009). New exact Solutions of nonlinear variants of the RLW, the PHI-four and Boussinesq equations based on modified extended direct algebraic method. *Int. J. Nonlinear Sci., 7*(3), 274-282.

Wang, M. L., & Li, X. Z. (2005). Applications of F-expansion to periodic wave solutions for a new Hamiltonian amplitude equation. *Chaos, Solitons and Fract, 24*(5), 1257-1268. http://dx.doi.org/10.1016/j.chaos.2004.09.044

Wazwaz, A. M. (2007). The tanhCcoth method for solitons and kink solutions for nonlinear parabolic equations. *Appl. Math. and Comput, 188*(2), 1467. http://dx.doi.org/10.1016/j.amc.2006.11.013

Wazwaz, A. M. (2011). A new (2+1)-dimensional Korteweg-de Vries equation and its extension to a new (3+1)-dimensional Kadomtsev-Petviashvili equation. *Phys. Scr., 84*(3). http://dx.doi.org/10.1088/0031-8949/84/03/035010

Weiss, J., Tabor, M., & Carnevale, G. (1983). The Painlevé property for partial differential equations. *J. Math. Phys., 24*, 522. http://dx.doi.org/10.1063/1.525721

Yan, Z., & Zhang, H. Q. (2001). New explicit solitary wave solutions and periodic wave solutions for Whitham-Broer-Kaup equation in shallow water. *Phy. Lett. A, 258*(5-6), 355. http://dx.doi.org/10.1016/S0375-9601(01)00376-0

Copyrights