Journal of Mathematics Research; Vol. 6, No. 1; 2014
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

The Existence of Positive Solutions to Kirchhoff Type Equations in $\mathbb{R}^N$ With Asymptotic Nonlinearity

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Received: October 25, 2013 Accepted: November 14, 2013 Online Published: December 24, 2013

doi:10.5539/jmr.v6n1p14 URL: http://dx.doi.org/10.5539/jmr.v6n1p14

Abstract

In this paper, we are concerned with the following Kirchhoff problem

$$\begin{cases} \left(a+\lambda\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)|u|^2)\right)[-\Delta u+V(x)u]=f(x,u), & x\in\mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N), \quad u>0, & x\in\mathbb{R}^N,\end{cases}$$

where $N\geq 3$, $a>0$ is a constant, $\lambda>0$ is a parameter, the potential $V(x)$ may not be radially symmetric and $f(x,s)$ is asymptotically linear with respect to s at infinity. Under some assumptions on V and f, we prove the existence of a positive solution for λ small and the nonexistence result for λ large.

Keywords: Kirchhoff equation, asymptotically linear, positive solutions, mountain-pass theorem

1. Introduction

In this paper, we consider the existence of positive solutions to the following Kirchhoff type problem:

$$\begin{cases} \left(a+\lambda\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)|u|^2)\right)[-\Delta u+V(x)u]=f(x,u), & x\in\mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N), \quad u>0, & x\in\mathbb{R}^N,\end{cases} \tag{1}$$

where $N\geq 3$, $a>0$ is a constant and $\lambda>0$ is a parameter. We assume that $V(x)$ and $f(x,s)$ verify the following hypotheses:

(V_1) $V(x)\in C(\mathbb{R}^N)$ and there exists $\Gamma_0>0$ such that

$$\Gamma_0 \triangleq \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)}{\int_{\mathbb{R}^N}|u|^2}>0;$$

(V_2) $\lim_{|x|\to+\infty}V(x)=V(\infty)\in(0,+\infty)$;

(f_1) $f\in C(\mathbb{R}^N\times\mathbb{R}_+,\mathbb{R}_+)$ and $f(x,s)\equiv 0$ for all $s\leq 0$;

(f_2) there exist $0\leq p(x)$, $q(x)\in L^\infty(\mathbb{R}^N)$ with $|p(x)|_\infty<\Gamma_0$, $q(\infty)=\lim_{|x|\to+\infty}q(x)<V(\infty)$ such that

$$\lim_{s\to 0}\frac{f(x,s)}{as}=p(x), \quad \lim_{s\to+\infty}\frac{f(x,s)}{as}=q(x)\not\equiv 0 \ \text{ uniformly in } x\in\mathbb{R}^N$$

and $0\leq\frac{f(x,s)}{as}\leq q(x)$ for all $x\in\mathbb{R}^N$ and $s\neq 0$.

By a positive solution of problem (1), we mean $u\in H^1(\mathbb{R}^N)$ such that $u>0$ a.e. in $\mathbb{R}^N$ and

$$\left(a+\lambda\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)\right)\int_{\mathbb{R}^N}(\nabla u\nabla v+V(x)uv)=\int_{\mathbb{R}^N}f(x,u)v,$$

for any $v \in H^1(\mathbb{R}^N)$.

In recent years, the following elliptic problem

$$\begin{cases} -\left(a + b\int_{\mathbb{R}^N} |\nabla u|^2\right)\Delta u + V(x)u = f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{2}$$

has been studied extensively by many researchers, where $N = 1, 2, 3$ and $a, b > 0$ are constants. (2) is a nonlocal problem as the appearance of the term $\int_{\mathbb{R}^N} |\nabla u|^2$ implies that (2) is not a pointwise identity. This causes some mathematical difficulties which make the study of (2) particularly interesting. Problem (2) arises in an interesting physical context. Indeed, if we set $V(x) = 0$ and replace $\mathbb{R}^N$ by a bounded domain $\Omega \subset \mathbb{R}^N$ in (2), then we get the following Kirchhoff Dirichlet problem

$$\begin{cases} -\left(a + b\int_{\Omega} |\nabla u|^2\right)\Delta u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

which is related to the stationary analogue of the equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = 0$$

presented by Kirchhoff in 1883. The readers can learn some early research of Kirchhoff equations from Bernstein (1940) and Pohožaev (1975). In 1978, J. L. Lions introduced an abstract functional analysis framework to the following equation

$$u_{tt} - \left(a + b\int_{\Omega} |\nabla u|^2\right)\Delta u = f(x,u). \tag{3}$$

After that, (3) received much attention, see (Alves & Crrea, 2001; Alves & Figueiredo, 2009; Arosio & Panizzi, 1996; Cavalcanti & Cavalcanti, 2001; D'Ancona & Spagnolo, 1992) and the references therein.

Before we review some results about (2), we give several definitions.

Let $(X, \|\cdot\|)$ be a Banach space with its dual space $(X^*, \|\cdot\|_*)$, $I \in C^1(X, \mathbb{R})$ and $c \in \mathbb{R}$. We say a sequence $\{x_n\}$ in X a Palais-Smale sequence at level c ($(PS)_c$ sequence in short) if $I(x_n) \to c$ and $\| I'(x_n) \|_* \to 0$ as $n \to \infty$. We say that I satisfies $(PS)_c$ condition if for any $(PS)_c$ sequence $\{x_n\}$ in X, there exists a subsequence $\{x_{n_k}\}$ such that $x_{n_k} \to x_0$ in X for some $x_0 \in X$.

Throughout the paper, we use the standard notations. We write $\int_\Omega h$ to mean the Lebesgue integral of $h(x)$ over a domain $\Omega \subset \mathbb{R}^N$. The norm of $u \in L^p(\mathbb{R}^N)$ $(1 \leq p \leq +\infty)$ will be denoted by $|u|_p$. We use "$\to$" and "$\rightharpoonup$" to denote the strong and weak convergence in the related function space respectively. $B_r(x) \triangleq \{y \in \mathbb{R}^N |\ |x - y| < r\}$. We denote $|A|$ the Lebesgue measure of a subset $A \subset \mathbb{R}^N$. C will denote a positive constant unless specified.

There have been some works about the existence, multiplicity results to (2) by using variational methods, see e.g. (He & Zou, 2012; Jin & Wu, 2010; Liu & He, 2012; Wang et al., 2012; Wu, 2011). Clearly weak solutions of (2) correspond to critical points of the energy functional

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (a|\nabla u|^2 + V(x)|u|^2) + \frac{b}{4}\left(\int_{\mathbb{R}^N} |\nabla u|^2\right)^2 - \int_{\mathbb{R}^N} F(x,u)$$

defined on $E \triangleq \{u \in H^1(\mathbb{R}^N)|\ \int_{\mathbb{R}^N} V(x)|u|^2 < \infty\}$, where $F(x,u) = \int_0^u f(x,s)ds$. A typical way to deal with (2) is to use the Mountain-Pass Theorem. For this purpose, one usually assumes that $f(x,u)$ is subcritical, superlinear at the origin and either is 4-superlinear at infinity in the sense that

$$\lim_{|u| \to +\infty} \frac{F(x,u)}{u^4} = +\infty \text{ uniformly in } x \in \mathbb{R}^N$$

or satisfies the Ambrosetti-Rabinowitz type condition ((AR) in short):

$$(AR)\ \exists\, \mu > 4 \text{ such that } \quad 0 < \mu\, F(x,u) \leq f(x,u)u \quad \text{for all } u \neq 0.$$

Under the above conditions, one easily sees that Ψ possesses a mountain-pass geometry around $0 \in H^1(\mathbb{R}^N)$ and by the Mountain-Pass Theorem, one can get a (PS) sequence of Ψ. Moreover, the (PS) sequence is bounded if

$$(F)\ \ 4F(x,u) \le f(x,u)u \ \text{ for all } u \in \mathbb{R}$$

holds. However, it is not easy to see that Ψ' is weakly continuous by direct calculations due to the existence of the nonlocal term $\int_{\mathbb{R}^N} |\nabla u|^2$. In fact, in general, we do not know $\int_{\mathbb{R}^N} |\nabla u_n|^2 \to \int_{\mathbb{R}^N} |\nabla u|^2$ from $u_n \rightharpoonup u$ in E. This difficulty was dealt with in (Jin & Wu, 2010; Li et al., 2012), when $V(x) \equiv$ const and $f(x,u)$ is radially symmetric to x, by using the radially symmetric Sobolev space $H_r^1(\mathbb{R}^N) = \{u \in H^1(\mathbb{R}^N) | \ u(|x|) = u(x)\}$, where the embeddings $H_r^1(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ $(2 < q < 2^*)$ are compact. In (Liu & He, 2012; Wu, 2011), the potential $V(x)$ satisfies:

$$(V_3)\quad \inf_{x\in\mathbb{R}^N} V(x) \ge a_1 > 0 \ \text{ and for each } M > 0, |\{x \in \mathbb{R}^N | \ V(x) \le M\}| < +\infty.$$

By using a weighted Sobolev space $E = \{u \in H^1(\mathbb{R}^N) | \int_{\mathbb{R}^N} V(x)|u|^2 < \infty\}$, where $E \hookrightarrow L^q(\mathbb{R}^N)$ $(2 \le q < 2^*)$ are compact to guarantee that (PS) condition holds, this difficulty was overcome. In (He & Zou, 2012; Wang et al., 2012), the potential $V(x)$ satisfies:

$$(V_4)\quad 0 < V_0 = \inf_{x\in\mathbb{R}^N} V(x) < \liminf_{|x|\to+\infty} V(x),$$

then the method used above can not work. However, for the mountain-pass level c, it can be proved in (He & Zou, 2012; Wang et al., 2012) that each $(PS)_c$ sequence weakly converges to a critical point of Ψ in $H^1(\mathbb{R}^N)$ and their argument strongly depends on the fact that $c = \inf \Psi(N)$, where $N = \{u \in H^1(\mathbb{R}^N)\backslash\{0\} | \ \langle \Psi'(u), u\rangle = 0\}$ and $\frac{f(x,u)}{u^3}$ is strictly increasing for $u > 0$.

Recently, in 2012, Li et al. studied (1) in the case when $V(x) \equiv b > 0$ is a constant and $f(x,u) \equiv f(u)$ is of subcritical growth and superliner at the origin and at infinity, i.e.

$$\lim_{u\to 0} \frac{f(u)}{u} = 0, \qquad \lim_{u\to+\infty} \frac{f(u)}{u} = +\infty.$$

By using a truncation argument combined with the monotonicity trick, they showed that there exists $\lambda_0 > 0$ such that for any $\lambda \in [0, \lambda_0)$, (1) has at least one positive radially symmetric solution.

In this paper, we try to prove the existence of positive solutions to problem (1) under the assumptions that $V(x)$ and $f(x,s)$ may not be necessarily radially symmetric with respect to x and $f(x,s)$ is asymptotically linear in s at infinity (i.e. (f_1) (f_2) hold). As far as we know, it seems there are few results to (1) in this situation.

2. Method

We try to use the Mountain-Pass Theorem to get a positive critical point for I_λ. There are some difficulties. First, as mentioned before, (AR) condition does not hold, which makes it difficult to get a $(PS)_c$ sequence. Secondly, condition (f_2) implies that (F) is no longer true, then the boundedness of the $(PS)_c$ sequence is difficult to prove. Thirdly, due to the effect of the nonlocal term $\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)$, we still face the difficulty to verify that the weak limit of the $(PS)_c$ sequence is a critical point of I_λ if we have obtained a bounded $(PS)_c$ sequence since in general, we do not know that

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2) \to \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)$$

just from $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^N)$. Our assumptions imposed on $V(x)$ and $f(x,s)$ make the method used in (He & Zou, 2012; Jin & Wu, 2010; Liu & He, 2012; Wang et al., 2012; Wu, 2011) can not be applied in our case to deal with this difficulty. We overcome these difficulties by adapting an argument used in (del Pino & Felmer, 1996; Wang & Zhou, 2007). However, since the conditions are different and we deal with a Kirchhoff equation, the arguments should be improved and more careful analysis is needed.

By (V_1) (V_2), $V(x) \in L^\infty(\mathbb{R}^N)$ and

$$V(\infty) \le |V(x)|_\infty. \tag{4}$$

Then by (f_2) and (4), we see that

$$-q(\infty) > -V(\infty) \ge -|V(x)|_\infty. \tag{5}$$

Moreover,

$$|V(x)|_\infty = \begin{cases} \max_{x\in\mathbb{R}^N} V(x), & \text{if } |V(x)|_\infty \text{ is attained at some point in } \mathbb{R}^N, \\ V(\infty), & \text{otherwise.} \end{cases}$$

By (f_1) (f_2), we have that

$$0 \leq F(x,s) \leq \frac{aq(x)}{2}s^2 \leq \frac{a|q(x)|_\infty}{2}s^2, \ \ \forall\, x \in \mathbb{R}^N, s \in \mathbb{R}. \tag{6}$$

On the other hand, condition (AR) implies that for some $C > 0$,

$$F(x,s) \geq Cs^\mu, \ \ \forall\, x \in \mathbb{R}^N, s \in \mathbb{R}.$$

So the classical condition (AR) can not be satisfied in our case.

Since $V(x)$ satisfies (V_1) (V_2), we introduce an equivalent norm on $H^1(\mathbb{R}^N)$: the norm of $u \in H^1(\mathbb{R}^N)$ is defined as

$$\|u\| = \left(\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2\right)^{\frac{1}{2}},$$

which is induced by the corresponding inner product on $H^1(\mathbb{R}^N)$. Weak solutions of (1) correspond to critical points of the following C^1 functional

$$I_\lambda(u) = \frac{a}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2) + \frac{\lambda}{4}\left(\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)|u|^2)\right)^2 - \int_{\mathbb{R}^N} F(x,u),$$

where $F(x,u) = \int_0^u f(x,s)ds$. Note that if $\lambda = 0$ in (5), we still denote the functional by $I_0(u)$.

Define

$$L = \inf\left\{\int_{\mathbb{R}^N}(|\nabla u|^2 - q(x)u^2)|\ u \in H^1(\mathbb{R}^N),\ |u|_2 = 1\right\}. \tag{7}$$

Clearly, $L \geq -|q(x)|_\infty > -\infty$ since $\inf\limits_{u\in H^1(\mathbb{R}^N),\ |u|_2=1} \int_{\mathbb{R}^N} |\nabla u|^2 = 0$. As is well-known (Berzin & M. Shubin, 1991), $L = \inf \sigma(S)$, where $\sigma(S)$ is the spectrum of the self-adjoint operator $S\colon H^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ defined by

$$Su = -\Delta u - q(x)u, \ \ \forall\, u \in H^2(\mathbb{R}^N).$$

Furthermore, the essential spectrum of S is the interval $[-q(\infty), +\infty)$ and so we have that

$$-|q(x)|_\infty \leq L \leq -q(\infty). \tag{8}$$

The operator S plays a crucial role since it defines the asymptotic linearization of (1).

Lemma 1 *Assume that* (V_1) (V_2), (f_1) (f_2) *hold and* $L + |V(x)|_\infty < 0$, *then*

(i) there exist $\rho, \alpha > 0$ *satisfying* $I_\lambda(u) \geq \alpha > 0$ *for all* $\|u\| = \rho$;

(ii) there exist $e \in H^1(\mathbb{R}^N)$ *with* $\|e\| > \rho$ *and* $\lambda_0 > 0$ *such that* $I_\lambda(e) < 0$ *for* $\lambda \in (0, \lambda_0)$.

Proof. (i) By (f_1) (f_2), for any $\varepsilon > 0$ and $r \in (2, 2^*)$, there exists $C_\varepsilon > 0$ such that for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}_+$,

$$F(x,s) \leq \frac{a}{2}(|p(x)|_\infty + \varepsilon)s^2 + C_\varepsilon s^r. \tag{9}$$

Taking $\varepsilon > 0$ such that $|p(x)|_\infty + \varepsilon < \Gamma_0$ since $|p(x)|_\infty < \Gamma_0$, then by (V_1), (9) and the Sobolev inequality, we see that

$$I_\lambda(u) \geq \frac{a}{2}\left(1 - \frac{|p(x)|_\infty + \varepsilon}{\Gamma_0}\right)\|u\|^2 - C_\varepsilon\|u\|^r.$$

So (i) is proved if we choose $\|u\| = \rho > 0$ small enough.

(ii) Since $L + |V(x)|_\infty < 0$, there exists $\delta > 0$ such that $L + |V(x)|_\infty + \delta < 0$. For such a $\delta > 0$, we conclude from the definition of L that there exists a nonnegative $w \in H^1(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N}(|\nabla w|^2 - q(x)w^2) \leq (L+\delta)\int_{\mathbb{R}^N} w^2. \tag{10}$$

Hence by Fatou's Lemma and (10), for any $t > 0$, we see that

$$\begin{aligned}\lim_{t\to+\infty}\frac{I_0(tw)}{t^2} &= \tfrac{a}{2}\int_{\mathbb{R}^N}(|\nabla w|^2+V(x)w^2)-\lim_{t\to+\infty}\int_{\mathbb{R}^N}\tfrac{F(x,tw)}{t^2w^2}w^2\\ &\le \tfrac{a}{2}\left[\int_{\mathbb{R}^N}(|\nabla w|^2-q(x)w^2)+\int_{\mathbb{R}^N}V(x)w^2\right]\\ &\le \tfrac{a}{2}\int_{\mathbb{R}^N}(L+\delta+|V(x)|_\infty)w^2<0.\end{aligned}$$

So $\lim\limits_{t\to+\infty} I_0(tw) = -\infty$, which implies that there exists $e \in H^1(\mathbb{R}^N)$ with $\|e\| > \rho$ such that $I_0(e) < 0$. Since $\lim\limits_{\lambda\to 0^+} I_\lambda(e) = I_0(e)$, there exists $0 < \lambda_0 < 1$ small such that $I_\lambda(e) < 0$ for $\lambda \in (0, \lambda_0)$. □

By Lemma 1 and the Mountain-Pass Theorem (Ambrosetti & Rabinowitz, 1973), for $\lambda \in (0, \lambda_0)$, there is a $(PS)_{c_\lambda}$ sequence $\{u_n\} \subset H^1(\mathbb{R}^N)$ such that

$$I_\lambda(u_n) \to c_\lambda \quad\text{and}\quad I'_\lambda(u_n) \to 0 \ \text{ in } H^{-1}(\mathbb{R}^N), \tag{11}$$

where

$$c_\lambda = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) \ge \alpha > 0$$

and $\Gamma = \{\gamma \in C([0,1], H^1(\mathbb{R}^N))|\ \gamma(0) = 0,\ \gamma(1) = e\}$.

Remark 1 If we assume $(f_2)'$ instead of (f_2), then Lemma 3 is also true since $b > \Gamma_0$ implies that there exists a nonnegative $w \in H^1(\mathbb{R}^N)$ satisfying

$$\int_{\mathbb{R}^N}(|\nabla w|^2 + V(x)w^2) < b\int_{\mathbb{R}^N} w^2.$$

Lemma 2 *Assume that* (V_1) (V_2), (f_1) (f_2) *hold, then there exists* $\lambda_1 > 0$ *such that the* $(PS)_{c_\lambda}$ *sequence* $\{u_n\}$ *given by* (11) *is bounded in* $H^1(\mathbb{R}^N)$ *for each* $\lambda \in (0, \lambda_1)$.

Proof. For any fixed $R > 0$, let $\eta_R \in C^\infty(\mathbb{R}^N, \mathbb{R})$ such that $\eta_R(x) \equiv 0$ for $|x| \le \frac{R}{2}$, $\eta_R(x) \equiv 1$ for $|x| \ge R$ and $|\nabla\eta_R(x)| \le \frac{4}{R}$ for all $x \in \mathbb{R}^N$. Then for any $u \in H^1(\mathbb{R}^N)$ and $R > 1$, we easily see that there exists a constant $C > 0$ such that $\eta_R u \in H^1(\mathbb{R}^N)$ and $\|\eta_R u\| \le C\|u\|$.

Since $I'_\lambda(u_n) \to 0$ in $H^{-1}(\mathbb{R}^N)$, for n large, we see that

$$\langle I'_\lambda(u_n), \eta_R u_n\rangle \le \|I'_\lambda(u_n)\|_{H^{-1}(\mathbb{R}^N)}\|\eta_R u_n\| \le \|u_n\|,$$

i.e. for n large,

$$(a+\lambda\|u_n\|^2)\left(\int_{\mathbb{R}^N}(|\nabla u_n|^2+V(x)u_n^2)\eta_R+\int_{\mathbb{R}^N}\nabla u_n u_n\nabla\eta_R\right)\le\int_{\mathbb{R}^N}f(x,u_n)u_n\eta_R+\|u_n\|. \tag{12}$$

Since $\lim\limits_{|x|\to+\infty} q(x) < V(\infty) = \lim\limits_{|x|\to+\infty} V(x)$, there are $\delta > 0$, $R_1 > 1$ such that $V(x) > \lim\limits_{|x|\to+\infty} q(x) + \delta$ for $|x| \ge R_1$. On the other hand, there exists $R_2 > R_1$ such that

$$q(x) \le V(x) - \delta \ \text{ for } |x| \ge R_2.$$

By (f_2), $f(x, u_n)u_n \le aq(x)u_n^2$ for all $x \in \mathbb{R}^N$. Then choosing $R > 2R_2$ in the definition of η_R and by (12), we see that

$$a\int_{\mathbb{R}^N}(|\nabla u_n|^2+\delta u_n^2)\eta_R \le \frac{4}{R}(a+\lambda\|u_n\|^2)\left(\int_{\mathbb{R}^N}|\nabla u_n|^2+\int_{\mathbb{R}^N}u_n^2\right)+\|u_n\|.$$

Therefore, there exists a constant $C > 0$ independent of R such that

$$a\int_{|x|\ge R}u_n^2 \le \frac{C}{R}(a\|u_n\|^2+\lambda\|u_n\|^4)+C\|u_n\|. \tag{13}$$

Similarly, we see that $\langle I'_\lambda(u_n), u_n\rangle \le \|u_n\|$, i.e.

$$a\int_{\mathbb{R}^N}(|\nabla u_n|^2+V(x)u_n^2)+\lambda\|u_n\|^4-\int_{\mathbb{R}^N}f(x,u_n)u_n \le \|u_n\|. \tag{14}$$

For each $n \in \mathbb{N}$, $\lambda > 0$ and $\beta > 1$, we consider the following function $h_\lambda(t)$: $\mathbb{R}_+ \to \mathbb{R}$:

$$h_\lambda(t) = t^4\left(\frac{\lambda}{2}\|u_n\|^4 - \lambda^\beta \int_{\mathbb{R}^N} |u_n|^4\right) + \frac{a\|u_n\|^2}{2}t^2.$$

Denote

$$C_4 = \inf_{H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 + V(x)|u|^2}{|u|_4^2} > 0. \tag{15}$$

Then for all $t \geq 0$,

$$\begin{aligned} h_\lambda(t) &= t^2\left[t^2\left(\tfrac{\lambda}{2}\|u_n\|^4 - \lambda^\beta \int_{\mathbb{R}^N} |u_n|^4\right) + \tfrac{a}{2}\|u_n\|^2\right] \\ &\geq 0 \quad \text{if} \quad \int_{\mathbb{R}^3} |u_n|^4 \leq \tfrac{1}{2\lambda^{\beta-1}}\|u_n\|^4, \end{aligned}$$

hence there exists $\lambda_1 = \left(\sqrt{2}C_4\right)^{-\frac{2}{\beta-1}}$ such that $h_\lambda(t) \geq 0$ for all $t \geq 0$ and $\lambda \in (0, \lambda_1)$. In particular, $h_\lambda(1) \geq 0$, i.e.

$$\frac{\lambda}{2}\|u_n\|^4 \geq \lambda^\beta \int_{\mathbb{R}^N} |u_n|^4 - \frac{a}{2}\|u_n\|^2 \quad \text{for } \lambda \in (0, \lambda_1). \tag{16}$$

So it follows from (14) (16) and (f_2) that for $\lambda \in (0, \lambda_1)$,

$$\frac{a}{2}\|u_n\|^2 + \frac{\lambda}{2}\|u_n\|^4 + \int_{\mathbb{R}^N} (\lambda^\beta |u_n|^4 - a|q(x)|_\infty |u_n|^2) \leq \|u_n\|.$$

Set $g(t) = \lambda^\beta t^4 - a|q(x)|_\infty t^2$, then

$$\frac{a}{2}\|u_n\|^2 + \frac{\lambda}{2}\|u_n\|^4 + \int_{\mathbb{R}^N} g(u_n) \leq \|u_n\|. \tag{17}$$

Let $b = \inf_{t\in\mathbb{R}} g(t)$, then $b \in (-\infty, 0)$. By (13), we have that

$$\begin{aligned} \int_{\mathbb{R}^N} g(u_n) &\geq \int_{|x|\leq R} b - a|q(x)|_\infty \int_{|x|\geq R} |u_n|^2 \\ &\geq b|B_R(0)| - \tfrac{C|q(x)|_\infty}{R}(a\|u_n\|^2 + \lambda\|u_n\|^4) - C|q(x)|_\infty\|u_n\|. \end{aligned} \tag{18}$$

By (17) (18), we see that

$$\frac{a}{2}\|u_n\|^2 + \frac{\lambda}{2}\|u_n\|^4 \leq \frac{C|q(x)|_\infty}{R}(a\|u_n\|^2 + \lambda\|u_n\|^4) + (C|q(x)|_\infty + 1)\|u_n\| + |b||B_R(0)|. \tag{19}$$

Choosing $R > 0$ large in (19) satisfying $\frac{C|q(x)|_\infty}{R} < \frac{1}{2}$, then $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$. □

Lemma 3 *Assume that* (V_1) (V_2), (f_1) (f_2) *hold, then the* $(PS)_{c_\lambda}$ *condition holds for* I_λ *if* $\lambda \in (0, \lambda_1)$, *where* λ_1 *is given in Lemma 2.*

Proof. By Lemma 2, for $\lambda \in (0, \lambda_1)$, the $(PS)_{c_\lambda}$ sequence $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and then

$$\langle I'_\lambda(u_n), \eta_R u_n\rangle = o(1),$$

where η_R is given in Lemma 2, $o(1) \to 0$ as $n \to +\infty$. Then similar to (13), we have that

$$\int_{|x|\geq R} u_n^2 \leq \frac{C}{R}(\|u_n\|^2 + \|u_n\|^4) + o(1). \tag{20}$$

Hence for any $\varepsilon > 0$, there exists $R > 0$ such that for n large

$$\int_{|x|\geq R} u_n^2 \leq \varepsilon. \tag{21}$$

Since $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$, up to a subsequence, we may assume that for some $u \in H^1(\mathbb{R}^N)$,

$$\begin{cases} u_n \rightharpoonup u \ \text{ in } H^1(\mathbb{R}^N), \\ u_n \to u \ \text{ in } L^s_{loc}(\mathbb{R}^N), \\ u_n(x) \to u(x) \ \ a.e. \text{ in } \mathbb{R}^N. \end{cases} \tag{22}$$

Then by (f_2), (22) and Hölder inequality, we have that

$$\begin{aligned}\left|\int_{\mathbb{R}^N} f(x,u_n)(u_n-u)\right| &\le \int_{|x|\le R} |f(x,u_n)(u_n-u)| + \int_{|x|\ge R} |f(x,u_n)(u_n-u)| \\ &\le C|u_n-u|_{L^2(B_R(0))} + C|q(x)|_\infty \left(\int_{|x|\ge R} |u_n|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Thus

$$\int_{\mathbb{R}^N} f(x,u_n)(u_n-u) = o(1). \tag{23}$$

By (22), we see that

$$\|u_n\|^2 = \|u\|^2 + \|u_n-u\|^2 + o(1). \tag{24}$$

So by (22)-(24), we have that

$$\begin{aligned} o(1) &= \langle I'_\lambda(u_n), u_n-u\rangle \\ &= (a+\lambda\|u_n\|^2)\int_{\mathbb{R}^N}[\nabla u_n(\nabla u_n-\nabla u) + V(x)u_n(u_n-u) + \int_{\mathbb{R}^N} f(x,u_n)(u_n-u) \\ &= (a+\lambda\|u_n\|^2)(\|u_n\|^2-\|u\|^2) + o(1) \\ &\ge a\int_{\mathbb{R}^N}\|u_n-u\|^2 + o(1).\end{aligned}$$

By (22), we easily see that $\|u_n\| \to \|u\|$ as $n\to+\infty$, hence $u_n \to u$ in $H^1(\mathbb{R}^N)$. □

Remark 2 If we assume $(f_2)'$ instead of (f_2), then Lemma 4 and Lemma 5 are also true since $b < V(\infty)$ and $0 \le f(x,s) \le bs^2$ for all $(x,s)\in\mathbb{R}^N\times\mathbb{R}$.

In order to get the non-existence result, we need the following lemma which gives the decay estimate of the solution at infinity.

Lemma 4 *Assume that (V_1) (V_2) (f_1), (f_2) or $(f_2)'$ hold. If $u\in H^1(\mathbb{R}^N)$ is a nontrivial solution of* (1)*, then $u>0$ a.e. in $\mathbb{R}^N$ and $\lim_{|x|\to+\infty} u(x) = 0$.*

Proof. Let $M = \|u\|^2$. Then problem (1) can be rewritten as

$$-\Delta u + V(x)u = \frac{1}{a+\lambda M} f(x,u).$$

Hence by using standard boot-strap arguments and the strong maximum principle, we see that $u>0$ a.e. in $\mathbb{R}^N$. The proof of $\lim_{|x|\to+\infty} u(x) = 0$ is initiated in the Morse iterative method of (Moser, 1960), which is similar to that of Lemma 4.5 in (He & Zou, 2012), so we omit it. □

3. Results

Theorem 5 *Assume that $V(x)$ satisfies (V_1) (V_2) and $f(x,u)$ satisfies (f_1) (f_2).*

(i) If problem (1) *has a positive solution, then $L < -q(\infty)$.*

(ii) If $L + |V(x)|_\infty < 0$, there exists $\lambda_* > 0$ such that problem (1) *has a positive solution for any $\lambda\in(0,\lambda_*)$.*

Proof. (i) Suppose that $0 < u \in H^1(\mathbb{R}^N)$ is a positive solution of (1), then

$$a\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2) + \lambda\left(\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)\right)^2 = \int_{\mathbb{R}^N} f(x,u)u.$$

By (f_2) and the definition of L, we see that

$$L\int_{\mathbb{R}^N} u^2 \le \int_{\mathbb{R}^N}(|\nabla u|^2 - q(x)u^2) \le -\int_{\mathbb{R}^N} V(x)u^2,$$

then $V(x) \le -L$ in $\mathbb{R}^N$, hence

$$|V(x)|_\infty \le -L. \tag{25}$$

On the other hand, by (8), we have that $L \le -q(\infty)$. Just suppose that $L = -q(\infty)$. By (5) (25), we see that

$$-|V(x)|_\infty < -q(\infty) = L \le -|V(x)|_\infty,$$

which is impossible. So $L < -q(\infty)$.

(ii) Set

$$\lambda_* = \min\{\lambda_0, \lambda_1\}.$$

Then Theorem 5 is a direct consequence of Lemmas 1-3. □

Theorem 6 *Assume that $V(x)$ satisfies (V_1) (V_2) and $f(x,u)$ satisfies (f_1) (f_2).*

(i) If $L + |V(x)|_\infty > 0$, then problem (1) has no positive solution;

(ii) For the case where $L + |V(x)|_\infty = 0$, if $|V(x)|_\infty$ is attained at some point in $\mathbb{R}^N$ and there exist $\varepsilon_0 > 0$, $R_0 > 0$ such that

$$(pq) \quad p(x) + \varepsilon_0 < q(x) \ \text{for } |x| \ge R_0,$$

then problem (1) has no positive solution;

(iii) If $L + |V(x)|_\infty < 0$, then there exists $\lambda^* > 0$ such that problem (1) has no positive solution for any $\lambda > \lambda^*$.

proof Suppose that $0 < u \in H^1(\mathbb{R}^N)$ is a positive solution of (1), then

$$a \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) + \lambda \|u\|^4 = \int_{\mathbb{R}^N} f(x,u)u. \tag{26}$$

(i) Similar to the proof of Theorem 6 (i), we have that $|V(x)|_\infty \le -L$, which implies that if $L + |V(x)|_\infty > 0$, then (1) has no positive solution.

(ii) By condition (pq), there exist $\varepsilon_0 > 0$, $R_0 > 0$ such that

$$p(x) + \frac{\varepsilon_0}{2} < q(x) \ \text{ for } |x| \ge R_0. \tag{27}$$

By (f_1) (f_2), there exists $\delta_0 > 0$ such that for all $(x,s) \in \mathbb{R}^N \times (0, \delta_0)$,

$$0 \le \frac{f(x,s)}{as} \le p(x) + \frac{\varepsilon_0}{2}. \tag{28}$$

By Lemma 5, $u(x) \to 0$ as $|x| \to +\infty$, then there exists $R_1 > 0$ such that

$$0 < u(x) \le \delta_0 \ \text{ for } |x| \ge R_1. \tag{29}$$

Set $R = \max\{R_0, R_1\}$. Therefore, by (26)-(29) and (f_2), we see that

$$\begin{aligned} \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2) &\le \int_{|x|\ge R} \left(p(x) + \tfrac{\varepsilon_0}{2}\right) u^2 + \int_{|x|\le R} q(x)u^2 \\ &\le \int_{|x|\ge R} \left[p(x) + \tfrac{\varepsilon_0}{2} - q(x)\right] u^2 + \int_{\mathbb{R}^N} q(x)u^2 \\ &< \int_{\mathbb{R}^N} q(x)u^2, \end{aligned}$$

i.e.

$$L \int_{\mathbb{R}^N} u^2 \le \int_{\mathbb{R}^N} (|\nabla u|^2 - q(x)u^2) < -\int_{\mathbb{R}^N} V(x)u^2.$$

Hence $V(x) < -L$ in $\mathbb{R}^N$. Since $|V(x)|_\infty$ is attained at some point in $\mathbb{R}^N$, $|V(x)|_\infty = \max_{x\in\mathbb{R}^N} V(x) < -L$. So (1) has no positive solution if $L + |V(x)|_\infty = 0$.

(iii) Since $|p(x)|_\infty < \Gamma_0$, there exists $\varepsilon > 0$ such that

$$|p(x)|_\infty + 2\varepsilon \le \Gamma_0. \tag{30}$$

By (f_1) (f_2), there exists $C = C(|p(x)|_\infty, \Gamma_0) > 1$ such that

$$0 \le f(x,u)u \le (|p(x)|_\infty + \varepsilon)u^2 + Cu^4. \tag{31}$$

Similarly to the argument in Lemma 2, for $\lambda > 0, \bar{\beta} \in (0, 1)$ and all $t \geq 0$, the following function

$$\begin{aligned} \bar{h}_\lambda(t) &= t^4\left(\frac{\lambda}{2}\|u\|^4 - \lambda^{\bar{\beta}}\int_{\mathbb{R}^N} u^4\right) + a\varepsilon\|u\|^2 t^2 \\ &\geq 0 \quad \text{if} \quad \int_{\mathbb{R}^3} |u_n|^4 \leq \frac{\lambda^{1-\bar{\beta}}}{2}\|u\|^4, \end{aligned}$$

hence there exists $\lambda_2 = \left(\sqrt{2}C_4\right)^{\frac{2}{1-\bar{\beta}}}$ such that

$$\frac{\lambda}{2}\|u\|^4 \geq \lambda^{\bar{\beta}}\int_{\mathbb{R}^N} u^4 - a\varepsilon\|u\|^2 \quad \text{for } \lambda > \lambda_2, \tag{32}$$

where C_4 is given in (15). So by (V_1) and (26) (30)-(32), for $\lambda > \lambda_2$, we have that

$$\begin{aligned} 0 &= a\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2) + \lambda\|u\|^4 - \int_{\mathbb{R}^N} f(x, u)u \\ &\geq a\left(1 - \frac{|p(x)|_\infty + 2\varepsilon}{\Gamma_0}\right)\|u\|^2 + (\lambda^{\bar{\beta}} - C)\int_{\mathbb{R}^N} u^4 \\ &\geq (\lambda^{\bar{\beta}} - C)\int_{\mathbb{R}^N} u^4. \end{aligned}$$

Therefore, let

$$\lambda^* = \max\{\lambda_2, C^{\frac{1}{\bar{\beta}}}\},$$

u must be zero if $\lambda > \lambda^*$. □

Remark 3 By (f_1) (f_2), $p(x), q(x) \in C(\mathbb{R}^N)$. For $V(x)$ satisfying (V_1) (V_2), it easily see that (f_1) (f_2) are satisfied by the following function:

$$f(x, s) = \begin{cases} \frac{p(x)+q(x)s}{1+s}, & x \in \mathbb{R}^N, \ s > 0, \\ 0, & x \in \mathbb{R}^N, \ s \leq 0, \end{cases}$$

where

$$q(x) = \begin{cases} |V(x)|_\infty |x|, & |x| \leq 1, \\ \left(\frac{V(\infty)}{4} - |V(x)|_\infty\right)|x| + 2|V(x)|_\infty - \frac{V(\infty)}{4}, & 1 < |x| < 2, \\ \frac{V(\infty)}{2}\frac{1}{1+|x|}, & |x| \geq 2 \end{cases}$$

and $p(x) = \frac{\Gamma_0}{2|V(x)|_\infty} q(x)$.

Remark 4 Under assumptions in Theorem 5 (ii), $q(x) \not\equiv$ const. In fact, if $q(x) \equiv b > 0$ is a constant, then $L = -b = -q(\infty)$, hence by Theorem 5 (i), problem (1) has no positive solution. So the case where $q(x) \equiv$ const can not be contained in Theorem 5. However, we still can use a similar method to obtain the existence and non-existence results for the case where $q(x) \equiv$ const under simple conditions (V_1) (V_2) (f_1) and

$$(f_2)' \quad \lim_{s\to 0}\frac{f(x, s)}{as} = 0, \ \text{ and } \ \lim_{s\to+\infty}\frac{f(x, s)}{as} = b \in (\Gamma_0, V(\infty)) \ \text{ uniformly in } x \in \mathbb{R}^N$$

and $0 \leq f(x, s) \leq bs^2$ for all $x \in \mathbb{R}^N$ and $s \neq 0$, i.e. there exist $\bar{\lambda}_*, \bar{\lambda}^* > 0$ such that (1) has a positive solution for $\lambda \in (0, \bar{\lambda}_*)$ and has no positive solution for $\lambda > \bar{\lambda}^*$.

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