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Application of Generalized Fractional Integral Operators to Certain Class of Multivalent Prestarlike Functions Defined by the Generalized Operator $L_p^\lambda(a, c)$

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Abstract

In this paper we introduce certain classes of multivalent prestarlike functions with negative coefficients defined by using the Cho-Kwon Srivastava operator and investigate some distortion theorems in terms of the fractional operator involving H-functions. Classes preserving integral operator and Radius of convexity for this classes and are also included.

Keywords: multivalent function, prestarlike function, fractional calculus

1. Introduction

Let A_p denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \qquad (p \in N\{1, 2, 3, ...\}), \tag{1.1}$$

which are analytic and p-valent in the unit disk $U = \{z \colon |z| < 1\}$. And let T_p denote the subclass of A_p consisting of analytic and p-valent functions which can be expressed in the form

$$f(z) = z^p - \sum_{k=1}^{\infty} a_{k+p} z^{k+p}, \qquad (a_{k+p} \geq 0). \tag{1.2}$$

A function $f(z) \in A_p$ is said to be p-valent starlike of order α, if and only if

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \qquad (z \in U), \tag{1.3}$$

for some $\alpha(0 \leq \alpha < p)$. We denote the class of all p- valent starlike functions of order α by $S_p(\alpha)$. Further a function $f(z)$ from A_p is said to be convex of order α if and only if

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \qquad (z \in U), \tag{1.4}$$

for some $\alpha(0 \leq \alpha < p)$. We denoted the class of all p-valent starlike functions of order α by $C_p(\alpha)$. The classes $S_p(\alpha)$ and $C_p(\alpha)$ were first introduced by D. A. Patil.

The function

$$S_\gamma(z) = z^p(1 - z)^{-2(p-\gamma)} \qquad (0 \leq \gamma < p; p \in N) \tag{1.5}$$

is well-known as the extreme function for the class $S_p(\gamma)$. Sitting

$$C(\gamma, k) = \frac{\prod_{i=2}^{k+1}\Big(2(p-\gamma) + i - 2\Big)}{k!} \qquad (k \geq 1; 0 \leq \gamma < p), \tag{1.6}$$

then $S_\gamma(z)$ can be written in the form

$$S_\gamma(z) = z^p + \sum_{k=1}^{\infty} C(\gamma, k) z^{k+p}. \tag{1.7}$$

We note that $C(\gamma, k)$ is a decreasing function in γ and that

$$\lim_{k\to\infty} C(\gamma, k) = \begin{cases} \infty & (\gamma < \frac{2p-1}{2}) \\ 1 & (\gamma \doteq \frac{2p-1}{2}) \\ 0 & (\gamma > \frac{2p-1}{2}). \end{cases} \tag{1.8}$$

Let$(f * g)(z)$ denote the Hadamard product (convolution) of two analytic functions $f(z)$ and $g(z)$, that is, if $f(z)$ is given by (1.1) and $g(z)$ is given by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p}, \tag{1.9}$$

then

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} a_{k+p} z^{k+p}. \tag{1.10}$$

A function $f(z) \in A_p$ is said to be p-valent γ -prestarlike function of order $\alpha(0 \le \alpha < p; 0 \le \gamma < p)$ if

$$(f * g)(z) \in S^*_P(\alpha), \tag{1.11}$$

where $S_\gamma(z)$ is defined by (1.7). We denote by $R_p(\gamma, \alpha)$ the class of all p-valent γ-prestarlike functions of order $\alpha(0 \le \alpha < p; 0 \le \gamma < p)$.

The class $R_p(\gamma, \alpha)$ and was studied by M. K Aouf and G. M. Shenen, while the class $R_p(\gamma, \alpha) =^p (\alpha)$ is the class of p-valent prestarlike functions of order α and was studied by G. A. Kumar and others. For $\gamma \doteq \frac{2p-1}{2}; 0 \le \alpha < p$, $R_p\left(\frac{2p-1}{2}, \alpha\right) = S_p(\alpha)$.

Let

$$R^*_p(\gamma, \alpha) = R_p(\gamma, \alpha) \cap T_p, S^*_p(\alpha) = S_p(\alpha) \cap T_p \quad \text{and} \quad C^*_p(\alpha) = C_p(\alpha) \cap T_p.$$

H. Saitoh introduce a line operator:

$$L_p(a, c) : A_p \to A_p$$

defined by

$$L_p(a, c) = \phi_p(a, c; z) * f(z) \qquad (z \in U), \tag{1.12}$$

where

$$\phi_p(a, c; z) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} z^{p+k}, \tag{1.13}$$

and $(a)_k$ is the Pochammer symbol defined by

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1; & (k = 0) \\ a(a+1)(a+2)...(a+k-1); & (k \in N). \end{cases}$$

In 2004, N. E. Cho introduced the following linear operator $L^\lambda_p(a, c)$ analogous to $L_p(a, c)$

$$L^\lambda_p(a, c) : S_p \to S_p$$

defined by

$$L^\lambda_p(a, c) f(z) = \phi^*_p(a, c; z) * f(z) \qquad (z \in U; a, c \in R \setminus Z^-_0; \lambda > -p), \tag{1.14}$$

where $\phi^*_p(a, c; z)$ is the function defined in terms of the Hadamard product (or convolution) by the following condition

$$\phi_p(a, c; z) * \phi^*_p(a, c; z) = \frac{z^p}{(1-z)^{\lambda+p}}, \tag{1.15}$$

We can easily find from (1.13), (1.14) and (1.15) and for the function $f(z) \in T_p$ that

$$L_p^\lambda(a,c)f(z) = z^p - \sum_{k=1}^{\infty} \frac{(\lambda+p)_k (c)_k}{k!(a)_k} a_{k+p} z^{p+k}, \tag{1.16}$$

It is easily verified from (1.16) that

$$z(L_p^\lambda(a+1,c)f)'(z) = aL_p^\lambda(a,c)f(z) - (a-p)L_p^\lambda(a+1,c)f(z), \tag{1.17}$$

and

$$z(L_p^\lambda(a,c)f)'(z) = (\lambda+p)L_p^{\lambda+1}(a,c)f(z) - \lambda L_p^\lambda(a,c)f(z). \tag{1.18}$$

Also by specializing the parameters λ, a, c we obtain from (1.16)

$$L_p^1(p+1,1)f(z) = f(z), \qquad L_p^1(p,1)f(z) = \frac{zf'(z)}{p}, \tag{1.19}$$

and

$$L_p^n(a,a)f(z) = D^{n+p-1}f(z) \qquad (n > -p),$$

where D^{n+p-1} is the well-known Ruschewehy derivative of order $n+p-1$.

The function $f(z)$ is said to be subordinate to $g(z)$ U written $f(z) \prec g(z)$ if there exist a function $w(z)$ analytic U such that $w(0) = 0$, and $|w(z)| < 1$, such that $f(z) = g(w(z))$.

Now making use of N. E. Cho operator $L_p^\lambda(a,c)$ defined by (1.16) we introduce the following subclass $S_p^\lambda(a,c,A,B,\gamma,\alpha)$ of p-valent γ prestarlike function of order $\alpha(0 \le \alpha < p; 0 \le \gamma < p)$.

Definition 1.1 For A, B arbitrary fixed real numbers, $-1 \le B < A \le 1$, a function $f(z) \in A_p$ defined by (1.1) is said to be in the class $S_p^\lambda(a,c,A,B,\gamma,\alpha)$ if it satisfies

$$(\lambda+p)\frac{L_p^{\lambda+1}(a,c)(f*S_\gamma)(z)}{L_p^\lambda(a,c)(f*S_\gamma)(z)} \prec \frac{(\lambda+p)+[(\lambda+p)B+(A-B)(p-\alpha)]z}{1+Bz}$$

$$(z \in U), (0 \le \alpha < p; 0 \le \gamma < p; a, c \in R \setminus Z_0^-; \lambda > -p) \tag{1.20}$$

also let $T_p^\lambda(a,c,A,B,\gamma,\alpha) = S_p^\lambda(a,c,A,B,\gamma,\alpha) \cap T_p$.

Using (1.18) it is seen that (1.20) is equivalent to

$$\left| \frac{\dfrac{z(L_p^\lambda(a,c)(f*S_\gamma))'(z)}{L_p^\lambda(a,c)(f*S_\gamma)(z)} - p}{pB+(A-B)(p-\alpha) - B\dfrac{z(L_p^\lambda(a,c)(f*S_\gamma))'(z)}{L_p^\lambda(a,c)(f*S_\gamma)(z)}} \right| < 1 \qquad (z \in U). \tag{1.21}$$

We note that:

(i) $T_p^1(p+1,1,1,-1,\gamma,\alpha) = R_p^*(\gamma,\alpha)$;

(ii) $T_p^1(p+1,1,1,-1,\frac{2p-1}{2},\alpha) = S_p^*(\alpha)$;

(iii) $T_p^1(p,1,1,-1,\frac{2p-1}{2},\alpha) = C_p^*(\alpha)$.

The class $T_p^\lambda(a,c,A,B,\gamma,\alpha)$ generalizes and extends other classes studied and introduced by several researches as M. K. Aouf, G. A. Shenan, G. A. Kumar, T. Sheil-Small, and S. Owa.

2. Fractional Integral Operators

We shall be concerned with fractional integral operators involving Fox's H-function which has been recently introduced by S. L. Kalla and V. S. Kiryakova.

Definition 2.1 Let $s \in N_0$ (the set of non-negative integers), $\beta_j \in R_+$ (the set of positive real numbers) and $\delta_j, \gamma_{j \in C}$ (the set of complex numbers) for $j = 1, 2, ..., s$, while $\sum_{k=1}^{\infty} Re(\delta_j) > 0$, then the generalized fractional integral operator for the function $f(z)$ is given by

$$\begin{aligned} I_{(\beta_s);s}^{(\gamma_s;(\delta_s)} f(z) &= I_{(\beta_1,...,\beta_s);s}^{(\gamma_1,...,\gamma_s;(\delta_1,...,\delta_s)} f(z) \\ &= \tfrac{1}{z}\int_o^z H_{s,s}^{s,0}\left[\tfrac{t}{z}\middle|\begin{smallmatrix}(1+\gamma_j+\delta_j-1/\beta_j,1/\beta_j)_{1,s}\\(1+\gamma_j-1/\beta_j,1/\beta_j)_{1,s}\end{smallmatrix}\right] f(t)dt, for \sum_{k=1}^{\infty} Re(\delta_j) > 0 \\ &= f(z), \qquad for\ \delta_1 = \delta_1 = ... = \delta_s = 0 \end{aligned} \tag{2.1}$$

where $f(z)$ is an analytic function in a simply connected region of the z-plane containing the origin and

$$H_{s,s}^{s,0}[z] = \frac{1}{2\pi i}\int_L \prod_{j=1}^{s} \frac{\Gamma\Big(b_j - t/\beta_j\Big)}{\Gamma\Big(a_j - t/\beta_j\Big)} z^t dt, \tag{2.2}$$

where $a_j = \gamma_j + \delta_j + 1 - t/\beta_j$ and $b_j = \gamma_j + 1 - t/\beta_j$ $(j = 1, 2, ..., s)$.

The fractional integral operator $I_{(\beta_s);s}^{(\gamma_s;(\delta_s)} f(z)$ contains as special case, many other fractional integral operators (M. Saigo), here we need some of them.

(i) For $t > 0$ and $f(z)$ being analytic function in a simply connected region of the z-plane containing the origin,

$$z^{\beta} I_{1,1}^{0,\beta} f(z) = z^{\beta-1}\int_0^z H_{1,1}^{1,0}\left[\frac{t}{z}\middle|\begin{smallmatrix}(\beta,1)\\(0,1)\end{smallmatrix}\right] f(t)dt = \frac{1}{\Gamma(\beta)}\int_0^z \frac{f(t)}{(z-t)^{1-\beta}} dt = D_z^{-\beta} f(z) \tag{2.3}$$

Where $Re(\beta) > 0$ and the multiplicity of $(z-t)^{\beta-1}$ is remove by requiring $\log|z-t|$ to be real when $(z-t) > 0$, $D_z^{-\beta} f(z)$ is called the Riemann-Liouville fractional integral operator of order β (S. G. Samko).

(ii) With $Re(\delta) > 0$, $\beta, \eta \in C$ and $f(z)$ being analytic function in a simply connected region of the $z-$ plane containing the origion,

$$\begin{aligned} z^{-\beta} I_{(1,1):2}^{(0,\eta-\beta);(-\beta,\delta+\beta)} f(z) &= z^{-\beta-1}\int_0^z H_{2,2}^{2,0}\left[\tfrac{t}{z}\middle|\begin{smallmatrix}(-\beta,1)(\delta+\eta,1)\\(0,1);(\eta-\beta,1)\end{smallmatrix}\right] f(t)dt \\ &= \tfrac{z^{-\delta-\beta}}{\Gamma(\delta)}\int_0^z (z-t)^{\delta-1}\ {}_2F_1\Big(\delta+\beta, -\eta; \delta; 1-\tfrac{t}{z}\Big) f(t)dt \\ &= I_{0,z}^{\delta,\beta,\eta} f(z), \end{aligned} \tag{2.4}$$

where $I_{0,z}^{\delta,\beta,\eta} f(z)$is the known Saigo fractional integral operator δ, (M. Saigo) and with the order

$$f(z) = O|z|^{\varepsilon}, z \to 0$$

$$\varepsilon > max(0, \beta - \eta) - 1$$

and the multiplicity of $(Z-1)^{\delta-1}$ is removed as in M. Saigo.

(iii) The following form is due to M. Saigo, for $Re(\eta) > 0$, $\nu, \mu, \zeta, \delta \in C, \sigma \in R$ and $f(z)$ is an analytic function in a simply connected region of z-plane containing the origin

$$\begin{aligned} z^{\mu+\sigma(1+\eta)+\upsilon+1} I_f^{(\frac{\upsilon+1}{\sigma}-1,\eta-\xi-\delta+\frac{\upsilon-1}{\sigma}-1);(\eta-\xi,\xi)}(z) &= z^{\mu+\sigma(1-\eta)+\upsilon}\int_o^z H_{2,2}^{2,0}\left[\tfrac{t}{z}\middle|\begin{smallmatrix}(\eta-\xi\frac{\nu}{\sigma},\frac{1}{\sigma})(\eta-\delta\frac{\nu}{\sigma},\frac{1}{\sigma})\\ \frac{\nu}{\sigma},\frac{1}{\sigma}(\eta-\xi-\delta+\frac{\nu}{\sigma},\frac{1}{\sigma})\end{smallmatrix}\right] f(t)dt \\ &= \tfrac{\sigma z^{\mu}}{\Gamma(\eta)}\int_0^z (z^{\sigma}-t^{\sigma})^{\eta-1}\ {}_2F_1\Big(\zeta, \delta, \eta; 1-(t/z)^{\sigma}\Big) t^{\nu} f(t)dt \\ &= I_{+,\upsilon}^{\eta,\sigma,\mu,\xi,\delta} f(z) \end{aligned} \tag{2.5}$$

with the order

$$f(z) = O|z|^{\varepsilon}, z \to 0$$

$$\sigma + \varepsilon > max(0, \delta + \zeta - \eta) - 1$$

and the multiplicity of $(Z^\sigma - t^\sigma)^{1-\eta}$ is removed by requiring $\log(Z^\sigma - t^\sigma)$ to be real as $(Z^\sigma - t^\sigma) > 0$. For the function $f(z)$ defined by (1.1) the operator $J^{(\gamma_s);(\delta_s)}_{(\beta_s^{-1});s}$ is defined by

$$J^{(\gamma_s);(\delta_s)}_{(\beta_s^{-1});s} f(z) = \prod_{j=1}^{s}\left[\frac{\Gamma(1+\gamma_j+\delta_j+p\beta_j)}{\Gamma(1+\gamma_j+p\beta_j)}\right] I^{(\gamma_s);(\delta_s)}_{(\beta_s^{-1});s} f(z) \tag{2.6}$$

Where $Re(\gamma_j) > p\beta_j - 1, \beta_j \in R_+$ and $Re(\delta_j) > 0$, $j = 1, 2, ..., s$.

For the function $f(z)$ defined by (1.1) the operator $J^{(\gamma_s);(\delta_s)}_{(\beta_s^{-1});s}$ is defined by

$$J^{-\gamma} f(z) = \frac{\Gamma(1+p+\gamma)}{\Gamma(1+p)} z^{-\gamma} D_z^{-\gamma} f(z) \tag{2.7}$$

Similarly for $s = 2, \beta_1 = \beta_2 = 1, \gamma_1 = 0, \gamma_2 = \eta - \phi, \delta_2 = \beta + \phi$, we have the following operator

$$J^{\beta,\phi,\eta}_{0,z} f(z) = \frac{\Gamma(1-\phi+p)\Gamma(1+\beta+\eta)}{\Gamma(1+p)\Gamma(1-\phi+\eta+p)} z^{\phi} J^{\beta,\phi,\eta}_{0,z} f(z) min(P-\phi+\eta, P-\phi, P+\beta+\eta) > -1 \tag{2.8}$$

The operator $J^{-\gamma} f(z)$ and $J^{\beta,\phi,\eta}_{0,z} f(z)$ have been studied by J. H. Choi.

Lemma 2.1 *Due to V. S. Kiryakova, if* $Re(\gamma_j) > \frac{-p}{\beta_j} - 1, \beta_j \in R_+$ *and* $Re(\delta_j) > 0$, $j = 1, 2, ..., s$. then

$$I^{(\gamma_s);(\delta_s)}_{(\beta_s);s}[z^p] = \prod_{j=1}^{s}\left[\frac{\Gamma(1+\gamma_j+\delta_j+p/\beta_j)}{\Gamma(1+\gamma_j+\delta_j+p/\beta_j)}\right] z^p. \tag{2.9}$$

3. Coefficient Estimates

Theorem 3.1 *For A, B arbitrary fixed real numbers, $-1 \le B < A \le 1$, a function $f(z) \in T_p$ defined by* (1.2) *belongs to the class $T_p^\lambda(a, c, A, B, \gamma, \alpha)$ if and only if*

$$\sum_{k=1}^{\infty}[(1-\beta)k + (A-B)(p-\alpha)]\phi_k^\lambda(a, c, p, \gamma)a_{p+k} \le (A-B)(p-\alpha), \tag{3.1}$$

where

$$\phi_k^\lambda(a, c, p, \gamma) = \frac{(\lambda+p)_k(c)_k}{(1)_k(a)_k} C(\gamma, k). \qquad (0 \le \alpha < p; 0 \le \gamma < p; a, c, \in R \setminus Z_0^\lambda; \lambda > -p) \tag{3.2}$$

The result is sharp.

Proof. Let the function $f(z) \in T_P$ defined by (1.2), then from (1.7) and (1.16) we have

$$L_p^\lambda(a, c)(f * S_\gamma)(z) = z^p - \sum_{k=1}^{\infty}\phi_k^\lambda(a, c, p, \gamma)a_{k+p}z^{k+p}, \tag{3.3}$$

assume that the inequality (3.1) holds true and let $|z| = 1$, then from (1.21) we have

$$\begin{aligned} & \left|\frac{z(L_p^\lambda(a,c)(f*S_\gamma))'(z)}{L_p^\lambda(a,c)(f*S_\gamma)(z)} - p\right| - \left|pB + (A-B)(p-\alpha) - B\frac{z(L_p^\lambda(a,c)(f*S_\gamma))'(z)}{L_p^\lambda(a,c)(f*S_\gamma)(z)}\right| \\ = \; & \left|-\sum_{k=1}^{\infty}\phi_k^\lambda(a, c, p, \gamma)a_{k+p}z^{k+p}\right| - \left|(A-B)(p-\alpha)z^p + \sum_{k=1}^{\infty}(Bk - (A-B)(p-\alpha))\phi_k^\lambda(a, c, p, \gamma)a_{k+p}z^{k+p}\right| \\ \le \; & \sum_{k=1}^{\infty}\left[(1-B)k + (A-B)(p-\alpha)\right]\phi_k^\lambda(a, c, p, \gamma)a_{k+p} - (A-B)(p-\alpha)) \le 0. \end{aligned}$$

Hence by the principle of maximum modulus $f(z) \in T_p^\lambda(a, c, A, B, \gamma, \alpha)$.

Conversely, assume that $f(z)$ defined by (3.1) is in the class $T_p^\lambda(a, c, A, B, \gamma, \alpha)$, then from (3.3) we have

$$\begin{aligned} & \left|\frac{\frac{z(L_p^\lambda(a,c)(f*S_\gamma))'(z)}{L_p^\lambda(a,c)(f*S_\gamma)(z)} - p}{pB + (A-B)(p-\alpha) - B\frac{z(L_p^\lambda(a,c)(f*S_\gamma))'(z)}{L_p^\lambda(a,c)(f*S_\gamma)(z)}}\right| \\ = \; & \left|\frac{-\sum_{k=1}^{\infty}k\phi_k^\lambda(a, c, p, \gamma)a_{k+p}z^{k+p}}{(A-B)(p-\alpha)z^p + \sum_{k=1}^{\infty}(Bk - (A-B)(p-\alpha))\phi_k^\lambda(a, c, p, \gamma)a_{k+p}z^{k+p}}\right| < 1. \end{aligned}$$

since $|Re(z)| \leq |z|$ for all z, we have

$$= Re\left[\frac{\sum_{k=1}^{\infty} k\phi_k^{\lambda}(a,c,p,\gamma)a_{k+p}z^{k+p}}{(A-B)(p-\alpha)z^p + \sum_{k=1}^{\infty}(Bk-(A-B)(p-\alpha))\phi_k^{\lambda}(a,c,p,\gamma)a_{k+p}z^{k+p}}\right] < 1. \tag{3.4}$$

choose the values z on the real axis so that $\frac{z(L_p^{\lambda}(a,c)(f*S_\gamma))'(z)}{L_p^{\lambda}(a,c)(f*S_\gamma)(z)}$ is real. Upon clearing the denominator of (3.4) and letting $z \to 1$ through real values we get

$$\sum_{k=1}^{\infty} k\phi_k^{\lambda}(a,c,p,\gamma)a_{k+p} \leq (A-B)(p-\alpha) + \sum_{k=1}^{\infty}(Bk-(A-B)(p-\alpha))\phi_k^{\lambda}(a,c,p,\gamma)a_{k+p},$$

which implies the inequality (3.1). Sharpness of the result follows by setting

$$f(z) = z^p - \frac{(A-B)(p-\alpha)}{[(1-B)k+(A-B)(p-\alpha)]\phi_k^{\lambda}(a,c,p,\gamma)a_{k+p}}z^{k+p} \quad (k \geq 1) \tag{3.5}$$

4. Distortion Theorem for the Class $T_p^{\lambda}(a,c,A,B,\gamma,\alpha)$

Theorem 4.1 *Let $\beta_j \in N, \delta_j \in R, (j = 1,2,3,...,s)$ be such that $Re(\gamma_j) > -Re(p\beta_j - 1)$, and*

$$\prod_{j=1}^{s}\left[\frac{(1+\gamma_j+(p+1)\beta_j)_{\beta_j}}{(1+\gamma_j+\delta_j+(p+1)\beta_j)_{\beta_j}}\right] \leq 1, \tag{4.1}$$

if the function $f(z)$ defined by (1.2) is in the class $T_p^{\lambda}(a,c,A,B,\gamma,\alpha)$, then

$$\begin{aligned}\left|I^{(\gamma_s);(\delta_s)}_{(\beta_s^{-1});s}f(z)\right| \quad \geq \quad & \prod_{j=1}^{s}\frac{\Gamma(1+\gamma_j+p\beta_j)}{\Gamma(1+\gamma_j+\delta_j+p\beta_j)}|z|^p \\ & \times\left\{1 - \frac{a(A-B)(p-\alpha)}{2c(\lambda+p)(p-\gamma)[1-B+(A-B)(p-\alpha)]}\prod_{j=1}^{s}\frac{\Gamma(1+\gamma_j+p\beta_j)}{\Gamma(1+\gamma_j+\delta_j+p\beta_j)}|z|\right\}\end{aligned} \tag{4.2}$$

and

$$\begin{aligned}\left|I^{(\gamma_s);(\delta_s)}_{(\beta_s^{-1});s}f(z)\right| \quad \leq \quad & \prod_{j=1}^{s}\frac{\Gamma(1+\gamma_j+p\beta_j)}{\Gamma(1+\gamma_j+\delta_j+p\beta_j)}|z|^p \\ & \times\left\{1 + \frac{a(A-B)(p-\alpha)}{2c(\lambda+p)(p-\gamma)[1-B+(A-B)(p-\alpha)]}\prod_{j=1}^{s}\frac{\Gamma(1+\gamma_j+p\beta_j)}{\Gamma(1+\gamma_j+\delta_j+p\beta_j)}|z|\right\}. \\ & (0 \leq \alpha < p; 0 \leq \gamma < p; a, c, \in R \setminus Z_0^-; \lambda > -p; -1 \leq B < A \leq 1).\end{aligned} \tag{4.3}$$

For $z \in U$, the equalities in (4.2) and (4.3) are attained by the function

$$f(z) = z^p - \frac{a(A-B)(p-\alpha)}{2c(\lambda+p)(p-\gamma)[1-B+(A-B)(p-\alpha)]}z^{p+1} \tag{4.4}$$

Proof. Making use of (1.2) and Lemma 2.1, we obtain

$$I^{(\gamma_s);(\delta_s)}_{(\beta_s^{-1});s}f(z) = \prod_{j=1}^{s}\left[\frac{\Gamma(1+\gamma_j+p\beta_j)}{\Gamma(1+\gamma_j+\delta_j+p\beta_j)}\right]z^p - \sum_{k=1}^{\infty}a_{p+k}\prod_{j=1}^{s}\left[\frac{\Gamma(1+\gamma_j+(p+k)\beta_j)}{\Gamma(1+\gamma_j+\delta_j+(p+k)\beta_j)}\right]z^{p+k}. \tag{4.5}$$

From (2.6) we have

$$J^{(\gamma_s);(\delta_s)}_{(\beta_s^{-1});s}f(z) = \prod_{j=1}^{s}\left[\frac{(1+\gamma_j+\delta_j+p\beta_j)}{\Gamma(1+\gamma_j+p\beta_j)}\right]I^{(\gamma_s);(\delta_s)}_{(\beta_s^{-1});s}f(z) = z^p - \sum_{k=1}^{\infty}\psi(k)a_{k+p}z^{k+p}, \tag{4.6}$$

where

$$\psi(k) = \prod_{j=1}^{s}\left[\frac{(1+\gamma_j+p\beta_j)_{\beta_j k}}{(1+\gamma_j+\delta_j+p\beta_j)_{\beta_j k}}\right] \qquad (k = 1,2,...). \tag{4.7}$$

Under the assumption of the theorem we see that $\psi(k)$ is non-increasing on k, i.e.

$$0 < \psi(k) \leq \psi(1) = \prod_{j=1}^{s}\left[\frac{(1+\gamma_j+p\beta_j)_{\beta_j}}{(1+\gamma_j+\delta_j+p\beta_j)_{\beta_j}}\right]. \tag{4.8}$$

Now employing (4.8) and theorem (3.1) in (4.6), we get

$$\begin{aligned}\left|J^{(\gamma_s);(\delta_s)}_{(\beta_s^{-1});s} f(z)\right| &\geq |z|^p - \psi(1)|z|^{p+1}\textstyle\sum_{k=1}^{\infty} a_{k+p}\\ &\geq |z|^p - \prod_{j=1}^{s}\frac{\Gamma(1+\gamma_j+p\beta_j)_{\beta_j}}{\Gamma(1+\gamma_j+\delta_j+p\beta_j)_{\beta_j}}\frac{(A-B)(p-\alpha)}{[1-B+(A-B)(p-\alpha)]\phi_1^{\lambda}(a,c,p,\gamma)}|z|^{p+1},\end{aligned}$$

where $\phi_1^{\lambda}(a,c,p,\gamma) = \frac{2c(\lambda+p)(p-\gamma)}{a}$, which implies the assertion (4.2) of Theorem (4.1).
Also, we have

$$\left|J^{(\gamma_s);(\delta_s)}_{(\beta_s^{-1});s} f(z)\right| \leq |z|^p - \prod_{j=1}^{s}\frac{\Gamma(1+\gamma_j+p\beta_j)_{\beta_j}}{\Gamma(1+\gamma_j+\delta_j+p\beta_j)_{\beta_j}}\frac{(A-B)(p-\alpha)}{[1-B+(A-B)(p-\alpha)]\phi_1^{\lambda}(a,c,p,\gamma)}|z|^{p+1}$$

which implies the assertion (4.3) of Theorem (4.1). □

Corollary 4.1 *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $T_p^{\lambda}(a,c,A,B,\gamma,\alpha)$, *then*

$$\left|D_z^{-\beta} f(z)\right| \geq \frac{\Gamma(1+p)}{\Gamma(1+\gamma+p)}|z|^{p+\beta}\left\{1 - \frac{a(A-B)(p-\alpha)(p+1)}{2c(\lambda+p)(p-\gamma)[1-B+(A-B)(p-\alpha)](p+\beta+1)}|z|\right\} \tag{4.9}$$

and

$$\left|D_z^{-\beta} f(z)\right| \leq \frac{\Gamma(1+p)}{\Gamma(1+\gamma+p)}|z|^{p+\beta}\left\{1 + \frac{a(A-B)(p-\alpha)(p+1)}{2c(\lambda+p)(p-\gamma)[1-B+(A-B)(p-\alpha)](p+\beta+1)}|z|\right\}$$

$$(\beta > 0, 0 \leq \alpha < p; 0 \leq \gamma < p; a, c, \in R \setminus Z_0^-; \lambda > -p; -1 \leq B < A \leq 1). \tag{4.10}$$

For $z \in U$, *the equalities in* (4.9) *and* (4.10) *are attained by the function given by* (4.4).

Proof. Setting $s = 1, \gamma_1 = 0, \delta_1 = \beta$ and $\beta_1 = 1$ in Theorem (4.1) and using (2.3) we obtain the result. □

Corollary 4.2 Let the function $f(z)$ defined by (1.2) be in the class $T_p^{\lambda}(a,c,A,B,\gamma,\alpha)$, then under the assumption $\frac{\beta(\eta+\delta)}{\delta} - p \leq 2$

$$\begin{aligned}\left|I_{0,z}^{\delta,\beta,\eta} f(z)\right| \geq\ & \frac{\Gamma(1+p)\Gamma(p-\beta+\eta+1)}{\Gamma(1-\gamma+p)\Gamma(p+\delta+\eta+1)}|z|^{p-\beta}\\ & \left\{1 - \frac{a(A-B)(p-\alpha)(p+1)(p-\beta+\eta+1)}{2c(\lambda+p)(p-\gamma)[1-B+(A-B)(p-\alpha)](p-\beta+1)p+\delta+\eta+1}|z|\right\}\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}\left|I_{0,z}^{\delta,\beta,\eta} f(z)\right| \geq\ & \frac{\Gamma(1+p)\Gamma(p-\beta+\eta+1)}{\Gamma(1-\gamma+p)\Gamma(p+\delta+\eta+1)}|z|^{p-\beta}\\ & \left\{1 + \frac{a(A-B)(p-\alpha)(p+1)(p-\beta+\eta+1)}{2c(\lambda+p)(p-\gamma)[1-B+(A-B)(p-\alpha)](p-\beta+1)p+\delta+\eta+1}|z|\right\}\\ & (0 \leq \alpha < p; 0 \leq \gamma < p; a, c, \in R \setminus Z_0^-; \lambda > -p; -1 \leq B < A \leq 1).\end{aligned} \tag{4.12}$$

For $(z \in U_0)$, *where*

$$U_0 = \begin{cases} U & \beta \leq p \\ U - \{0\} & \beta > p \end{cases}$$

The equality in (4.11) *and* (4.12) *are attained by the function given by* (4.4).

Proof. Setting $s = 2, \beta_1 = \beta_2 = 1, \gamma_1 = 0, \gamma_2 - \beta, \delta_1 = beta$ and $\delta_2 = \delta + \beta$ in Theorem (4.1) and using (2.4) we obtain the result. □

Corollary 4.3 *Let the function* $f(z)$ *defined by* (1.2) *be in the class* $T_p^{\lambda}(a, c, A, B, \gamma, \alpha)$*, then*

$$\left|I_{+,\nu}^{\eta,1/\sigma,\mu,\zeta,\delta} f(z)\right| \geq \frac{\Gamma[\sigma(\nu+p+1)]\Gamma[\sigma(\nu+p+1)+\eta-\zeta-\delta]}{\Gamma[\sigma(\nu+p+1)+\eta-\zeta]\Gamma[\sigma(\nu+p+1)+\eta-\delta]}|z|^{p-\{(\frac{1-\eta}{\sigma})-p-\mu-\nu-1\}}$$
$$\times\left\{1-\frac{a(A-B)(p-\alpha)[\sigma(\nu+p+1)]_{\sigma}[\sigma(\nu+p+1)+\eta-\zeta-\delta]_{\sigma}}{2c(\lambda+p)(p-\gamma)[1-B+(A-B)(p-\alpha)][\sigma(\nu+p+1)+\eta-\zeta]_{\sigma}[\sigma(\nu+p+1)+\eta-\delta]_{\sigma}}|z|\right\} \tag{4.13}$$

and

$$\left|I_{+,\nu}^{\eta,1/\sigma,\mu,\zeta,\delta} f(z)\right| \leq \frac{\Gamma[\sigma(\nu+p+1)]\Gamma[\sigma(\nu+p+1)+\eta-\zeta-\delta]}{\Gamma[\sigma(\nu+p+1)+\eta-\zeta]\Gamma[\sigma(\nu+p+1)+\eta-\delta]}|z|^{p-\{(\frac{1-\eta}{\sigma})-p-\mu-\nu-1\}}$$
$$\times\left\{1+\frac{a(A-B)(p-\alpha)[\sigma(\nu+p+1)]_{\sigma}[\sigma(\nu+p+1)+\eta-\zeta-\delta]_{\sigma}}{2c(\lambda+p)(p-\gamma)[1-B+(A-B)(p-\alpha)][\sigma(\nu+p+1)+\eta-\zeta]_{\sigma}[\sigma(\nu+p+1)+\eta-\delta]_{\sigma}}|z|\right\} \tag{4.14}$$

where σ *is a positive integer and* $z \in U_0$*, where*

$$U_0 = \begin{cases} U; & (\frac{1-\eta}{\sigma}) - p - \mu - \nu - 1 \leq p \\ U - \{0\} \quad ; & (\frac{1-\eta}{\sigma}) - p - \mu - \nu - 1 > p \end{cases}$$

The equality in (4.13) *and* (4.14) *are attained by the function given by* (4.4)*.*

Proof. Setting $s = 2, \beta_1 = \beta_2 = \sigma, \gamma_1 = \sigma(\nu+1)-1, \gamma_2 = \eta-\zeta-\delta+(\sigma(\nu+1)-1), \delta_1 = \eta-\zeta, \delta_2 = \zeta$ in Theorem (4.1) and using (2.5) we obtain the result. □

Corollary 4.4 *Under the assumption of Theorem* (4.1)*, let the function* $f(z)$ *defined by* (1.2) *be in the class* $R_p^*(\gamma, \alpha)$

$$\left|I_{(\beta_s^{-1});s}^{(\gamma_s);(\delta_s)} f(z)\right| \geq \prod_{j=1}^{s} \frac{\Gamma(1+\gamma_j+p\beta_j)}{\Gamma(1+\gamma_j+\delta_j+p\beta_j)}|z|^p \tag{4.15}$$

$$\times\left\{1-\frac{(p-\alpha)}{2(p-\gamma)(1+p-\alpha)}\prod_{j=1}^{s}\frac{(1+\gamma_j+p\beta_j)_{\beta_j}}{(1+\gamma_j+\delta_j+p\beta_j)_{\beta_j}}|z|\right\}. \tag{4.16}$$

and

$$\left|I_{(\beta_s^{-1});s}^{(\gamma_s);(\delta_s)} f(z)\right| \leq \prod_{j=1}^{s} \frac{\Gamma(1+\gamma_j+p\beta_j)}{\Gamma(1+\gamma_j+\delta_j+p\beta_j)}|z|^p \tag{4.17}$$

$$\times\left\{1+\frac{(p-\alpha)}{2(p-\gamma)(1+p-\alpha)}\prod_{j=1}^{s}\frac{(1+\gamma_j+p\beta_j)_{\beta_j}}{(1+\gamma_j+\delta_j+p\beta_j)_{\beta_j}}|z|\right\}. \tag{4.18}$$

The equality in (4.15) *and* (4.16) *are attained by the function given by*

$$f(z) = z^p - \frac{(p-\alpha)}{2(p-\gamma)(1+p-\alpha)}z^{p+1}.$$

Proof. Setting $a = p+1$, $c = 1$, $A = 1$, $B = -1$ and $\lambda = 1$ in theorem (4.1) we get the result. □

Remark 4.1 Setting $a = p+1$, $c = 1$, $A = 1$, $B = -1$ and $\lambda = 1$, $\gamma\frac{2p-1}{2}$ and $\lambda = 1$ in Theorem (4.1) we get the result, we get the corresponding result by P. K. Banerji (Theorem (2.1), p. 113).

Remark 4.2 Setting $a = p$, $c = 1$, $A = 1$, $B = -1$ and $\lambda = 1$, $\gamma\frac{2p-1}{2}$ and $\lambda = 1$ in Theorem (4.1) we get the result, we get the corresponding result by P. K. Banerji (Theorem (2.2), p. 115).

Remark 4.3 Several other particular studied cases studied by different authors can be obtained from Theorem (4.1) by specializing the parameters $s, \beta_s, \delta_s, \gamma, \alpha, \gamma, \lambda, A, B, a$ and c see for example H. M. Srivastava and P. K. Banerji.

5. Integral Transform of the Class $T_p^{\lambda}(a, c, A, B, \gamma, \alpha)$

The generalized Komatu integral operator $L_{c,p}^{\delta}$: $T_P \to T_P$ is defined for $\delta > 0$ and $c > -p$ as (see, e.g. S. M. Khairnar and T. O. Salim)

$$H(z) = L_{c,p}^{\delta} f(z) = \frac{(c+p)^{\delta}}{\Gamma(\delta)z^c}\int_0^z t^{c-t}\Big(log\frac{z}{t}\Big)^{\delta-1} f(t)dt. \tag{5.1}$$

Notice that for $c = 1$ we get the integral operator introduced by I. B. Jung.

Theorem 5.1 *Let $f(z) \in T_p^\lambda(a, c, A, B, \gamma, \alpha)$. Then $L_{c,p}^\delta f(z) \in T_p^\lambda(a, c, A, B, \gamma, \alpha)$.*

Proof. By using the definition of $L_{c,p}^\delta f(z)$, we have

$$L_{c,p}^\delta f(z) = \frac{(c+1)^\delta}{\Gamma(\delta)} \int_0^1 \Big(log\frac{1}{t}\Big)^{\delta-1} t^{c+p-t}\Big(z^p - \sum_{k=1}^{\infty} a_{k+p} t^k z^{k+p}\Big)dt \tag{5.2}$$

Simplifying by using the definition of gamma function we get

$$L_{c,p}^\delta f(z) = z^p - \sum_{k=1}^{\infty} \left(\frac{c+p}{c+p+k}\right)^\delta a_{k+p} z^{k+p}. \tag{5.3}$$

Now $L_{c,p}^\delta f(z) \in T_p^\lambda(a, c, A, B, \gamma, \alpha)$, if

$$\sum_{k=1}^{\infty} \frac{[(1-B)k + (A-B)(p-\alpha)]\phi_k^\lambda(a, c, p, y)}{(A-B)(p-\alpha)} \Big(\frac{c+p}{c+p+k}\Big)^\delta a_k \leq 1. \tag{5.4}$$

From Theorem (3.1), we have $f(z) \in T_p^\lambda(a, c, A, B, \gamma, \alpha)$, if and only if

$$\sum_{k=1}^{\infty} \frac{[(1-B)k + (A-B)(p-\alpha)]\phi_k^\lambda(a, c, p, y)}{(A-B)(p-\alpha)} a_{k+p} \leq 1. \tag{5.5}$$

Thus in view of (5.5) and the fact that$\Big(\frac{c+p}{c+p+k}\Big) < 1$ for $k \geq 1$, (5.4) holds true, and hence $L_{c,p}^\delta f(z) \in T_p^\lambda(a, c, A, B, \gamma, \alpha)$. □

6. Radius of Convexity for the Class $T_p^\lambda(a, c, A, B, \gamma, \alpha)$

Theorem 6.1 *Let the function $f(z)$ defined by* (1.2) *be in the class $T_p^\lambda(a, c, A, B, \gamma, \alpha)$. Then $f(z)$ is convex in the disk $|z| < r$, where*

$$r = \inf_{k \geq 1} \left\{ \frac{p^2[(1-B)k + (A-B)(p-\alpha)]\phi_k^\lambda(a, c, p, y)}{(A-B)(p-\alpha)(k+p)^2} \right\}^{\frac{1}{k}}. \tag{6.1}$$

The result is sharp for the function $f(z)$ defined by (3.5).

Proof. To establish the required result it is sufficient to show that

$$\left|1 + \frac{zf''(z)}{f'(z)} - p\right| \leq p, \qquad for \quad |z| < r,$$

or equivalently

$$\frac{\sum_{k=1}^{\infty} k(k+p)a_{k+p}|z|^k}{p - \sum_{k=1}^{\infty}(k+p)a_{k+p}|z|^k} \leq p,$$

which is equivalent to show that

$$\sum_{k=1}^{\infty} \frac{(k+p)^2}{p^2} a_{k+p}|z|^k \leq 1. \tag{6.2}$$

In view of (5.5), (6.2) is true if

$$\frac{(k+p)^2}{p^2}|z|^k \leq \frac{[(1-B)k + (A-B)(p-\alpha)]\phi_k^\lambda(a, c, p, y)}{(A-B)(p-\alpha)}. \tag{6.3}$$

sitting $|z| = r$ in (6.3) and simplifing we get the result. □

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