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# Fuzzy Anti-n-Continuous and n-Bounded Linear Operators

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**Abstract**

In this paper we study the concept of Fuzzy-anti-n-normed linear operator as a generalization of Fuzzy-anti-2-normed linear operator. Fuzzy-anti-n-continuous linear operator and three types (strongly, weakly, and sequentially) of Fuzzy-anti-n-continuous linear operators are defined and relation between strongly, weakly and sequentially Fuzzy-anti-n-continuous linear operator is developed. Also strongly and weakly fuzzy-anti n-bounded linear operators are defined and relation between Fuzzy-anti-n-continuous linear operator and Fuzzy-anti-n-bounded linear operators is established.

**Keywords:** fuzzy-anti-n-linear operator, fuzzy-anti-n-continuous-linear operator, strongly, weakly, sequentially fuzzy-anti-n-continuous-linear operators, fuzzy-anti-n-bounded-linear operators

## 1. Introduction

The idea of Fuzzy norm was initiated by Katsaras (1984). In 1993, Felbin introduced an idea of Fuzzy norm on a linear space by assigning a Fuzzy Real number to each element of the linear space, so that the corresponding metric associated this Fuzzy norm is a Kaleva type fuzzy metric. Narayanan and Vijayabalaji (2005) extended the notion of n-normed linear space to fuzzy-n-normed-linear space. In 2010, Jebril and Samanta introduced fuzzy-anti-norm on a linear space depending on the idea of fuzzy-anti-norm was introduced by Bag and Samanta (2003) and investigated their important properties. In 2011, Reddy studied fuzzy-anti-2-norm and some results are established in fuzzy-anti-2-normed linear space and Reddy (2011) introduced fuzzy-anti-n-norm on linear space and studied the notion of convergent sequence, Cauchy sequence in fuzzy-anti-n-normed linear space. Sinha, Mishra, Lal (2011, 2012) introduced the concept of fuzzy-anti-2-continous linear operator and fuzzy-anti-2-bounded linear operator on fuzzy-anti-2-normed linear space. In this paper we introduced the concept of fuzzy-anti-n-continuous linear operator on a fuzzy-anti-n-normed linear space to another fuzzy-anti-n-normed linear space and defined three types (strongly, weakly and sequentially) of fuzzy-anti-n-continuous linear operators and relation between strongly, weakly and sequentially fuzzy-anti-n-continuous linear operator is developed. Also introduced the concept of fuzzy-anti-n-bounded linear operator on a fuzzy-anti-n-normed linear space to another fuzzy-anti n-normed linear space and defined two types (strongly ane weakly) of fuzzy-anti-n-bounded linear operators and relation between strongly, weakly fuzzy-anti-n-bounded linear operator is established.

## 2. Preliminaries

This section contains a few basic definitions and preliminary results which will be needed in the sequel.

**Definition 2.1** Let $n \in N$ and let $X$ be a real linear space of dimension $d \geq n$. A real valued function $\|\bullet, \bullet, ..., \bullet\|$: $X \times X \times ... \times X \to R$ satisfying the following four properties

$nN_1$: $\|x_1, x_2, ..., x_n\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent vectors.

$nN_2$: $\|x_1, x_2, ..., x_n\| = \left\|x_{j_1}, x_{j_2}, ..., x_{j_n}\right\|$ for every permutation $(j_1, j_2, ..., j_n)$ of $(1, 2, ..., n)$, *i.e.*, $\|x_1, x_2, ..., x_n\|$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

$nN_3$: $\|x_1, x_2, ..., x_{n-1}, \alpha x_n\| = |\alpha|\|x_1, x_2, ..., x_n\|$ for all $\alpha \in R$.

$nN_4$: $\|x_1, x_2, ..., x_{n-1}, y + z\| \leq \|x_1, x_2, ..., x_{n-1}, y\| + \|x_1, x_2, ..., x_{n-1}, z\|$ for all $y, z, x_1, x_2, ..., x_{n-1} \in X$, is called an n-norm on $X$ and the pair $(X, \|\bullet, \bullet, ..., \bullet\|)$ is called $n$ -normed linear space.

**Definition 2.2** Let $X$ be a linear space over a real field $F$. A fuzzy subset $N$ of $X \times X \times ... \times X \times R \to R$ is called a fuzzy n-norm on $X$ if the following conditions are satisfied for all $x_1, x_2, ..., x_n, x'_n \in X$ and

$(n-N1)$: For all $t \in R$ with $t \leq 0$, $N(x_1, x_2, ..., x_n, t) = 0$.

$(n-N2)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) = 1$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

$(n-N3)$: $N(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

$(n-N4)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., x_{n-1}, cx_n, t) = N(x_1, x_2, ..., x_n, \frac{t}{|c|})$ if $c \neq 0$, $c \in F$.

$(n-N5)$: $\forall s, t \in R$,

$$N(x_1, x_2, ..., x_{n-1}, x_n + x'_n, s + t) \geq \min\{N(x_1, x_2, ..., x_{n-1}, x_n, s), N(x_1, x_2, ..., x_{n-1}, x'_n, t)\}$$

$(n-N6)$: $N(x_1, x_2, ..., x_n, t)$ is a non-decreasing function of $t \in R$ and $\lim_{t\to\infty} N(x_1, x_2, ..., x_n, t) = 1$.

Then $N$ is said to be a fuzzy n-norm on a linear space $X$ and the pair $(X, N)$ is said to be a fuzzy n-normed linear space (briefly F-n-NLS).

The following condition of fuzzy n-norm $N$ will be required later on

$(n-N7)$: For all $t \in R$ with $t > 0$, $N(x_1, x_2, ..., x_n, t) > 0$, implies that $x_1, x_2, ..., x_n$ are linearly dependent.

**Definition 2.3** Let $X$ be a linear space over a real field $F$. A fuzzy subset $N^*$ of $X \times X \times ... \times X \times R \to R$ such that for all $x_1, x_2, ..., x_n, x'_n \in X$ and $c \in F$

$(n-N^*1)$: For all $t \in R$ with $t \leq 0$, $N^*(x_1, x_2, ..., x_n, t) = 1$.

$(n-N^*2)$: For all $t \in R$ with $t > 0$, $N^*(x_1, x_2, ..., x_n, t) = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

$(n-N^*3)$: $N^*(x_1, x_2, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, ..., x_n$.

$(n-N^*4)$: For all $t \in R$ with $t > 0$, $N^*(x_1, x_2, ..., cx_n, t) = N^*(x_1, x_2, ..., x_n, \frac{t}{|c|})$ if $c \neq 0$.

$(n-N^*5)$: For all $s, t \in R$,

$$N^*(x_1, x_2, ..., x_{n-1}, x_n + x'_n,\ s + t) \leq \max\{N^*(x_1, x_2, ..., x_{n-1}, x_n,\ s), N^*(x_1, x_2, ..., x_{n-1}, x'_n,\ t)\}.$$

$(n-N^*6)$: $N^*(x_1, x_2, ..., x_n, t)$ is a non-increasing function of $t \in R$ and

$$\lim_{t\to\infty} N^*(x_1, x_2, ..., x_n, t) = 0.$$

Then $N^*$ is said to be a fuzzy anti-n-norm on a linear space $X$ and the pair $(X, N^*)$ is called a fuzzy anti-n-normed linear space (briefly Fa-n-NLS).

The following condition of fuzzy anti-n-norm $N^*$ will be required later on.

$(n-N^*7)$: For all $t \in R$ with $t > 0$, $N^*(x_1, x_2, ..., x_n, t) < 1$, implies that $x_1, x_2, ..., x_n$ are linearly dependent.

## 3. Fuzzy Anti n-Continuous Linear Operators

Let $(X, N_1^*)$ and $(Y, N_2^*)$ are fuzzy-anti-n-normed-linear spaces defined on the same field.

**Definition 3.1** $T$ is a mapping from $X_1 \times X_2 \times ... \times X_n$ to $Y_1 \times Y_2 \times ... \times Y_n$ where $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are subspaces of $(X, N_1^*)$, $(Y, N_2^*)$ respectively. Then $T$ is said to be fuzzy-anti-n-linear operator, if

$$T(\sum_{i_n=1}^{n} x_1^{(i_n)}, \sum_{i_{n-1}=1}^{n} x_2^{(i_{n-1})}, \sum_{i_{n-2}=1}^{n} x_3^{(i_{n-2})}, ...., \sum_{i_2=1}^{n} x_{n-1}^{(i_2)}, \sum_{i_1=1}^{n} x_n^{(i_1)}) = \sum_{i_1=1}^{n}\sum_{i_2=1}^{n}\sum_{i_3=1}^{n}...\sum_{i_n=1}^{n} T(x_1^{(i_n)}, x_2^{(i_{n-1})}, x_3^{(i_{n-2})}, ...., x_{n-1}^{(i_2)}, x_n^{(i_1)})$$

and

$$T(\alpha_1 x_1, \alpha_2 x_2, ..., \alpha_n x_n) = \alpha_1\alpha_2...\alpha_n T(x_1, x_2, ..., x_n), \forall (x_1, x_2, ..., x_n) \in X_1 \times X_2 \times ... \times X_n.$$

**Definition 3.2** Let $T$ be a fuzzy-anti-n-linear map from $X_1 \times X_2 \times ... \times X_n$ to $Y_1 \times Y_2 \times ... \times Y_n$, $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are subspaces of $(X, N_1^*)$,$(Y, N_2^*)$ respectively. Then $T$ is called fuzzy-anti-n-continuous at $(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \in X_1 \times X_2 \times ... \times X_n$ if given $\varepsilon > 0$, $\alpha \in (0, 1)$ $\exists \delta = \delta(\alpha, \varepsilon) > 0$, $\beta = \beta(\alpha, \varepsilon) \in (0, 1)$, such that for all $(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) \in X_1 \times X_2 \times ... \times X_n$

$$N_1^*[(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) - (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \delta] < \beta$$

$$\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \varepsilon] < \alpha.$$

If $T$ is fuzzy-anti-n-continuous at every point of $T$: $X_1 \times X_2 \times ... \times X_n \rightarrow Y_1 \times Y_2 \times ... \times Y_n$ then $T$ is fuzzy-anti-n-continuous on $X_1 \times X_2 \times ... \times X_n$.

From now we will denote fuzzy-anti-n-continuous map by fa-n-continuous map.

**Definition 3.3** Let $T$: $X_1 \times X_2 \times ... \times X_n \rightarrow Y_1 \times Y_2 \times ... \times Y_n$ be a fuzzy-anti-n-linear mapping, $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are subspaces of $(X, N_1^*)$, $(Y, N_2^*)$ respectively. Then $T$ is called sequentially-fuzzy-anti-n-continuous at $(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \in X_1 \times X_2 \times ... \times X_n$ if

$$\forall k, (x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \rightarrow (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})$$

$$\Rightarrow T(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \rightarrow T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}).$$

i.e.

$$\lim_{k\to\infty} N_1^*[(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), t] = 0, \forall t > 0$$

$$\Rightarrow \lim_{k\to\infty} N_2^*[T(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), t] = 0, \forall t > 0.$$

From now we will denote sequentially-fuzzy-anti-n-continuous map by Sq-fa-n-continuous map.

If $T$ is Sq-fa-n-continuous at every point of $X_1 \times X_2 \times ... \times X_n$ then $T$ is called Sq-fa-n-continuous on $X_1 \times X_2 \times ... \times X_n$.

**Definition 3.4** Let $T$: $X_1 \times X_2 \times ... \times X_n \rightarrow Y_1 \times Y_2 \times ... \times Y_n$ be a fuzzy-anti-n-linear mapping, $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are subspaces of $(X, N_1^*)$,$(Y, N_2^*)$ respectively. Then $T$ is called strongly-fuzzy-anti-n-continuous at $(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \in X_1 \times X_2 \times ... \times X_n$, if for each $\varepsilon > 0$, $\exists \delta > 0$ such that $\forall (x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) \in X_1 \times X_2 \times ... \times X_n$,

$$N_2^*[T(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \varepsilon]$$

$$\leq N_1^*[(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) - (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \delta].$$

From now we will denote strongly-fuzzy-anti-n-continuous map by St-fa-n-continuous map.

**Definition 3.5** Let $T$: $X_1 \times X_2 \times ... \times X_n \rightarrow Y_1 \times Y_2 \times ... \times Y_n$ be fuzzy-anti-n-linear mapping, $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are subspaces of $(X, N_1^*)$, $(Y, N_2^*)$ respectively. Then $T$ is called weakly-fuzzy-anti-n-continuous at $(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \in X_1 \times X_2 \times ... \times X_n$, if for a given $\varepsilon > 0$, $\alpha \in (0, 1)$, $\exists \delta = \delta(\alpha, \varepsilon) > 0$, such that $\forall (x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) \in X_1 \times X_2 \times ... \times X_n$,

$$N_1^*[(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) - (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \delta] \leq 1 - \alpha$$

$$\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \varepsilon] \leq 1 - \alpha.$$

From now we will denote weakly-fuzzy-anti-n-continuous map by Wk-fa-n-continuous map.

**Theorem 3.6** *Let $T$: $X_1 \times X_2 \times ... \times X_n \rightarrow Y_1 \times Y_2 \times ... \times Y_n$ be a fuzzy-anti-n-linear mapping, $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are subspaces of $(X, N_1^*)$, $(Y, N_2^*)$ respectively. If $T$ is St-fa-n-continuous then $T$ is Sq-fa-n-continuous.*

*Proof.* Let us assume that $T$ is St-fa-n-continuous at $(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \in X_1 \times X_2 \times ... \times X_n$, then for each $\varepsilon > 0$, $\exists\ \delta = \delta(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n-1)}, x_0^{(n)}, \varepsilon) > 0$, such that for all $(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) \in X_1 \times X_2 \times ... \times X_n$,

$$N_2^*[T(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \varepsilon]$$

$$\leq N_1^*[(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) - (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \delta] \tag{1}$$

Let $(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)})$ be a sequence in $X_1 \times X_2 \times ... \times X_n$, such that

$$(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \rightarrow (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})$$

i.e.,

$$\lim_{k\to\infty} N_1^*[(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), t] = 0, \forall t > 0. \tag{2}$$

Now from Equation (1), by (2) we have

$$N_2^*[T(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \varepsilon] \leq N_1^*[(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \delta]$$

$$\Rightarrow \lim_{k\to\infty} N_2^*[T(x_k^{(1)}, x_k^{(2)}, ..., x_k^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), \varepsilon] \leq \lim_{k\to\infty} N_1^*[(x_k^{(1)}, x_k^{(2)}, ..., x_k^{(n)}) - (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), \delta]$$

$$\Rightarrow \lim_{k\to\infty} N_2^*[T(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \varepsilon] = 0.$$

Since $\varepsilon$ is arbitrarily small positive real, it immediately follows that $T(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \to T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})$. Therefore $T$ is Sq-fa-n-continuous.

**Theorem 3.7** *Let $T: X_1 \times X_2 \times ... \times X_n \to Y_1 \times Y_2 \times ... \times Y_n$ be a fuzzy-anti-n-linear mapping, $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are subspaces of $(X, N_1^*)$, $(Y, N_2^*)$respectively. If T is Fa-n-continuous if and only if T is Sq-fa-n-continuous.*

*Proof.* Let us assume that $T$ is Fa-n-continuous at $(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \in X_1\times X_2\times...\times X_n$. Let $(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)})$ be a sequence in $X_1 \times X_2 \times ... \times X_n$, such that $(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \to (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})$. Let $\varepsilon > 0$ be given, choose $\alpha \in (0, 1)$, since $T$ is Fa-n-continuous at $(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})$ then $\exists\ \delta = \delta(\alpha, \varepsilon) > 0, \beta = \beta(\alpha, \varepsilon) \in (0, 1)$, such that for all $(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) \in X_1 \times X_2 \times ... \times X_n$,

$$N_1^*[(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) - (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \delta] < \beta$$

$$\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \varepsilon] < \alpha.$$

Since $(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \to (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})$ in $(X, N_1^*)$ $\exists$ a positive integer $n_0$, such that

$$N_1^*[(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \delta] < \beta, \forall n \geq n_0$$

$$\Rightarrow N_2^*[T(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \varepsilon] < \alpha, \forall n \geq n_0$$

$$\Rightarrow N_2^*[T(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \varepsilon] = 0.$$

Since $\varepsilon$ is arbitrary thus $T(x_k^{(1)}, x_k^{(2)}, x_k^{(3)}, ..., x_k^{(n)}) \to T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})$ in $Y_1 \times Y_2 \times ... \times Y_n$. Therefore $T$ is Sq-fa-n-continuous.

Next let us assume $T$ is Sq-fa-n-continuous at $(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) \in X_1\times X_2\times...\times X_n$ If it is possible let us assume $T$ is not Fa-n-continuous at $(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})$. Thus $\exists\varepsilon > 0$ and $\alpha > 0$ such that for any $\delta > 0$ and $\beta \in (0, 1)$ $\exists$ $(y^{(1)}, y^{(2)}, y^{(3)}, ..., y^{(n)})$ (depending on $\delta$, $\beta$), such that $N_1^*[(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) - (y^{(1)}, y^{(2)}, y^{(3)}, ..., y^{(n)}), \delta] < \beta$, but $N_2^*[T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) - T(y^{(1)}, y^{(2)}, y^{(3)}, ..., y^{(n)}), \varepsilon] \geq \alpha$. Thus for $\beta = \frac{1}{k+1}$, $\delta = 1 - \frac{1}{k+1}$, $k = 1, 2, 3, \dots$, $\exists(y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)})$, such that

$$N_1^*\left[(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) - (y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)}), (1 - \frac{1}{k+1})\right] < \frac{1}{k+1}$$

but $N_2^*[T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) - T(y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)}), \varepsilon] \geq \alpha$.

Taking $\delta > 0$, $\exists\ k_0$, such that $(1 - \frac{1}{k+1}) < \delta\ \forall k \geq k_0$, then

$$N_1^*[(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) - (y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)}), \delta]$$

$$< N_1^*\left[(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) - (y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)}), (1 - \frac{1}{k+1})\right] < \frac{1}{k+1}, \forall k > k_0$$

$$\lim_{k\to\infty} N_1^*[(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) - (y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)}), \delta] < 0$$

$$\Rightarrow (y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)}) \to (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}).$$

But from Equation (1) $N_2^*[T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) - T(y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)}), \varepsilon] \geq \alpha$. So, $N_2^*[T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}) - T(y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)}), \varepsilon]$ does not converges to zero as k $\to \infty$. Thus $T(y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)})$ does not converges to $T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})$, where as $(y_k^{(1)}, y_k^{(2)}, y_k^{(3)}, ..., y_k^{(n)}) \to (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})$ (with respect to $N_1^*$). This would be contradiction to above assumption. Therefore $T$ is Fa-n-continuous at $(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)})$.

## 4. Fuzzy Anti n-Bounded Linear Operators

**Definition 4.1** Let $T$: $X_1 \times X_2 \times ... \times X_n \to Y_1 \times Y_2 \times ... \times Y_n$ be a fuzzy-anti-n-linear mapping, $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are subspaces of $(X, N_1^*)$, $(Y, N_2^*)$ respectively. Then $T$ is said to be strongly-fuzzy-anti-n-bounded (St-fa-n-bounded) on $X_1 \times X_2 \times ... \times X_n$ if and only if $\exists$ a positive real number $M$, such that for all $(x_1, x_2, x_3, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$ and $\forall t \in R$,

$$N_2^*[T(x_1, x_2, x_3, ..., x_n), t] \leq N_1^*[(x_1, x_2, x_3, ..., x_n), \frac{t}{M}]$$

*Example 4.2* Let $(X, \|\bullet, \bullet, ..., \bullet\|)$ be a n-normed-linear-space over the field $K$, where $K = R$ or $C$. Let $k_1, k_2 \in R$ such that $k_1 > k_2 > 0$. Let $N_1^*, N_2^*$: $X \times X \times ... \times X \times R^+ \to [0, 1]$ be defined by

$$N_1^*[(x_1, x_2, x_3, ..., x_n, t)] = \frac{k_1 \|x_1, x_2, x_3, ..., x_n\|}{t + k_1 \|x_1, x_2, x_3, ..., x_n\|},$$

$$N_2^*[(x_1, x_2, x_3, ..., x_n, t)] = \frac{k_2 \|x_1, x_2, x_3, ..., x_n\|}{t + k_2 \|x_1, x_2, x_3, ..., x_n\|}.$$

Clearly $(X, N_1^*)$ and $(Y, N_2^*)$ are fuzzy-anti-n-normed linear spaces.

Consider the mapping $T$: $X_1 \times X_2 \times ... \times X_n \to Y_1 \times Y_2 \times ... \times Y_n$ defined by $T(x_1, x_2, x_3, ..., x_n) = r(x_1, x_2, x_3, ..., x_n)$, where $r(\neq 0) \in R$ is fixed.

Clearly $T$ is a linear operator. Let us choose an arbitrary but fixed $M > 0$ such that $M \geq |r|$ and $(x_1, x_2, x_3, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$. Now

$$M \geq |r|$$

$$\Rightarrow \quad k_1 M \|x_1, x_2, x_3, ..., x_n\| \geq k_2 |r| \|x_1, x_2, x_3, ..., x_n\|$$

$$\Rightarrow \quad t + k_1 M \|x_1, x_2, x_3, ..., x_n\| \geq t + k_2 |r| \|x_1, x_2, x_3, ..., x_n\|, \forall t > 0$$

$$\Rightarrow \quad \frac{t}{t + k_2 |r| \|x_1, x_2, x_3, ..., x_n\|} \geq \frac{t}{t + k_1 M \ \|x_1, x_2, x_3, ..., x_n\|}, \forall t > 0$$

$$\Rightarrow \quad \frac{t}{t + k_2 \|r(x_1, x_2, x_3, ..., x_n)\|} \geq \frac{\frac{t}{M}}{\frac{t}{M} + k_1 \|x_1, x_2, x_3, ..., x_n\|}, \forall t > 0$$

$$\Rightarrow \quad 1 - \frac{t}{t + k_2 \|r(x_1, x_2, x_3, ..., x_n)\|} \leq 1 - \frac{\frac{t}{M}}{\frac{t}{M} + k_1 \|x_1, x_2, x_3, ..., x_n\|}, \forall t > 0$$

$$\Rightarrow \quad \frac{k_2 \|r(x_1, x_2, x_3, ..., x_n)\|}{t + k_2 \|r(x_1, x_2, x_3, ..., x_n)\|} \leq \frac{k_1 \|x_1, x_2, x_3, ..., x_n\|}{\frac{t}{M} + k_1 \|x_1, x_2, x_3, ..., x_n\|}, \forall t > 0.$$

$$N_2^*[r(x_1, x_2, x_3, ..., x_n), t] \leq N_1^*[(x_1, x_2, x_3, ..., x_n), \frac{t}{M}], \forall t > 0$$

and

$$(x_1, x_2, x_3, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$$

(i.e.)

$$N_2^*[T(x_1, x_2, x_3, ..., x_n), t] \leq N_1^*[(x_1, x_2, x_3, ..., x_n), \frac{t}{M}], \forall t > 0$$

and

$$(x_1, x_2, x_3, ..., x_n) \in X_1 \times X_2 \times ... \times X_n.$$

Therefore $T$ is St-fa-n-bounded.

**Definition 4.3** Let $T$: $X_1 \times X_2 \times ... \times X_n \to Y_1 \times Y_2 \times ... \times Y_n$ be a fuzzy-anti-n-linear mapping, $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_n$ are subspaces of $(X, N_1^*)$, $(Y, N_2^*)$ respectively. Then $T$ is said to be weakly-fuzzy-anti-n-bounded (Wk-fa-n-bounded) on $X_1 \times X_2 \times ... \times X_n$ iff for any $\alpha \in (0, 1)$ $\exists$ $M_\alpha > 0$, such that for all $(x_1, x_2, x_3, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$ and $\forall t \in R$,

$$N_1^*[(x_1, x_2, x_3, ..., x_n), \frac{t}{M}] \leq \ 1 - \alpha \Rightarrow N_2^*[T(x_1, x_2, x_3, ..., x_n), t] \leq \ 1 - \alpha$$

**Theorem 4.4** *Let* $T: X_1 \times X_2 \times ... \times X_n \rightarrow Y_1 \times Y_2 \times ... \times Y_n$ *be a fuzzy-anti-n-linear operator,* $X_1, X_2, ..., X_n$ *and* $Y_1, Y_2, ..., Y_n$ *are subspaces of* $(X, N_1^*)$, $(Y, N_2^*)$ *respectively. If* $T$ *is St-fa-n-bounded, then* $T$ *is Wk-fa-n-bounded but the converse need not be sure.*

*Proof.* Let us assume$T$ is St-fa-n-bounded. Then $\exists M > 0$, such that $\forall (x_1, x_2, x_3, ..., x_n) \in X_1 \times X_2 \times ... \times X_n$ and $\forall t \in R$, $N_2^*[T(x_1, x_2, x_3, ..., x_n), t] \leq N_1^*[(x_1, x_2, x_3, ..., x_n), \frac{t}{M}]$. Thus for any $\alpha \in (0, 1)$, $\exists\, M_\alpha (= M) > 0$, such that

$$N_1^*[(x_1, x_2, x_3, ..., x_n), \frac{t}{M_\alpha}] \leq 1 - \alpha \Rightarrow N_2^*[T(x_1, x_2, x_3, ..., x_n), t] \leq 1 - \alpha.$$

Therefore $T$ is Wk-fa-n-bounded.

The following example tells us that the converse of the theorem is not always true.

*Example 4.5* Let $(X, \|\bullet, \bullet, ..., \bullet\|)$ be a n-normed-linear space over the field $K$, where $K = R$ or $C$. Let $N_1^*, N_2^*$: $X \times X \times ... \times X \times R^+ \rightarrow [0, 1]$ be defined by $N_1^*(x_1, x_2, x_3, ..., x_n, t) = \frac{4\,\|x_1, x_2, x_3, ..., x_n\|^2}{t^2 + 2\,\|x_1, x_2, x_3, ..., x_n\|^2}$ if $t > \|x_1, x_2, x_3, ..., x_n\| = 1$, if $t \leq \|x_1, x_2, x_3, ..., x_n\|$

$$N_2^*(x_1, x_2, x_3, ..., x_n, t) = \frac{\|x_1, x_2, x_3, ..., x_n\|}{t + \|x_1, x_2, x_3, ..., x_n\|}.$$

We know that $(X, N_2^*)$ is a Fa-n-normed linear space.

Now we would prove $(X, N_1^*)$ is a Fa-n-normed linear space.

(i) $\forall t \in R$ with $t \leq 0$ and by definition $N_1^*(x_1, x_2, x_3, ..., x_n, t) = 1$

(ii) $\forall t \in R$ with $t > 0$,

$$N_1^*(x_1, x_2, x_3, ..., x_n, t) = 0 \Leftrightarrow \frac{4\,\|x_1, x_2, x_3, ..., x_n\|^2}{t^2 + 2\,\|x_1, x_2, x_3, ..., x_n\|^2} = 0$$

$$\Leftrightarrow \|x_1, x_2, x_3, ..., x_n\|^2 = 0 \Leftrightarrow x_1, x_2, x_3, ..., x_n \text{ are linearly dependent.}$$

(iii) As $\|x_1, x_2, x_3, ..., x_n\|$ is invariant under any permutation of $x_1, x_2, x_3, ..., x_n$ it follows that $N_1^*(x_1, x_2, x_3, ..., x_n, t)$ is invariant under any permutation of $x_1, x_2, x_3, ..., x_n$.

(iv) For all $t \in R$ with $t > 0$ and $c \neq 0$, $c \in K$, we get

$$N_1^*(x_1, x_2, x_3, ..., cx_n, t) = \frac{4\,\|x_1, x_2, x_3, ..., cx_n\|^2}{t^2 + 2\,\|x_1, x_2, x_3, ..., cx_n\|^2} = \frac{|c|^2\, 4\,\|x_1, x_2, x_3, ..., x_n\|^2}{t^2 + |c|^2\, 2\,\|x_1, x_2, x_3, ..., x_n\|^2}$$

$$= \frac{4\,\|x_1, x_2, x_3, ..., x_n\|^2}{\frac{t^2}{|c|^2} + 2\,\|x_1, x_2, x_3, ..., x_n\|^2} = N_1^*[(x_1, x_2, x_3, ..., x_n, \frac{t}{|c|})].$$

(v) For all $s, t \in R$ and $x_1, x_2, x_3, ..., x_n, x_n' \in X$, we have to show that

$$N_1^*(x_1, x_2, ..., x_{n-1}, x_n + x_n', s + t) \leq \max\{N_1^*(x_1, x_2, ..., x_{n-1}, x_n, s), N_1^*(x_1, x_2, ..., x_{n-1}, x_n', t)\}.$$

If (a) $s + t < 0$ (b) $s = t = 0$ (c) $s + t > 0$, $s > 0$, $t < 0$; $s < 0$, $t > 0$, then in the three cases the relation will be trivial.

If (d) $s > 0$, $t > 0$, $s + t > 0$ and

$$\|x_1, x_2, x_3, ...., x_{n-1}, x_n\| + \left\|x_1, x_2, x_3, ...., x_{n-1} x_n'\right\| \geq \left\|x_1, x_2, x_3, ...., x_{n-1}, x_n + x_n'\right\|.$$

Therefore

$$N_1^*(x_1, x_2, x_3, ..., x_{n-1}, x_n + x_n', s + t) = \frac{4\,\left\|x_1, x_2, x_3, ..., x_{n-1}, (x_n + x_n')\right\|^2}{(s + t)^2 + 2\,\left\|x_1, x_2, x_3, ..., x_{n-1}, (x_n + x_n')\right\|^2}$$

$$\leq \frac{4\,\left(\|x_1, x_2, x_3, ..., x_n\| + \left\|x_1, x_2, x_3, ..., x_{n-1}, x_n'\right\|\right)^2}{(s + t)^2 + 2\,\left(\|x_1, x_2, x_3, ..., x_{n-1}, x_n\| + \left\|x_1, x_2, x_3, ..., x_{n-1}, x_n'\right\|\right)^2}$$

$$\leq \frac{4\,\left\|x_1, x_2, x_3, ..., x_{n-1}, x_n'\right\|^2}{t^2 + 2\left\|x_1, x_2, x_3, ..., x_{n-1}, x_n'\right\|^2} = N_1^*(x_1, x_2, x_3, ..., x_{n-1},\ x_n',\ t).$$

Therefore $N_1^*(x_1, x_2, x_3, ..., x_{n-1}, x_n + x_n', s + t) \leq N_1^*(x_1, x_2, x_3, ..., x_{n-1}, x_n', t)$, when $N_1^*(x_1, x_2, x_3, ..., x_{n-1}, x_n, s) \leq N_1^*(x_1, x_2, x_3, ...., x_{n-1}, x_n, t)$. Similarly, $N_1^*(x_1, x_2, x_3, ..., x_{n-1}, x_n + x_n', s + t) \leq N_1^*(x_1, x_2, x_3, ..., x_{n-1}, x_n, s)$, when $N_1^*(x_1, x_2, x_3, ..., x_{n-1}, x_n', t) \leq N_1^*(x_1, x_2, x_3, ..., x_{n-1}, x_n, s)$. Thus $N_1^*(x_1, x_2, ..., x_{n-1}, x_n + x_n', s + t) \leq \max\{N_1^*(x_1, x_2, ..., x_{n-1}, x_n, s), N_1^*(x_1, x_2, ..., x_{n-1}, x_n', t)\}$.

If $t_1 < t_2 \leq 0$, which implies

$$N_1^*(x_1, x_2, ..., x_{n-1}, x_n, t_1) = N_1^*(x_1, x_2, ..., x_{n-1}, x_n, t_2) = 1.$$

If $0 < t_1 < t_2$, then

$$\begin{aligned} & N_1^*(x_1, x_2, ....., x_n, t_1) - N_1^*(x_1, x_2, ...., x_n, t_2) \\ &= \frac{4\ \|x_1, x_2, ...., x_n\|^2}{t_1^2 + 2\ \|x_1, x_2, ...., x_n\|^2} - \frac{4\ \|x_1, x_2, ...., x_n\|^2}{t_2^2 + 2\ \|x_1, x_2, ...., x_n\|^2} \\ &= \frac{4\ \|x_1, x_2, ...., x_n\|^2\,(t_2^2 - t_1^2)}{(t_1^2 + 2\ \|x_1, x_2, ...., x_n\|^2)(t_2^2 + 2\ \|x_1, x_2, ...., x_n\|^2)} > 0 \\ &\Rightarrow N_1^*(x_1, x_2, x_3, ..., x_n, t_1) \geq N_1^*(x_1, x_2, x_3, ...., x_n, t_2). \end{aligned}$$

Thus $N_1^*(x_1, x_2, x_3, ..., x_n, t)$ is a non- increasing function of $t \in R$

$$\lim_{t\to\infty} N_1^*(x_1, x_2, x_3, ..., x_n, t) = \lim_{t\to\infty} \frac{4\ \|x_1, x_2, x_3, ..., x_n\|^2}{t^2 + 2\ \|x_1, x_2, x_3, ..., x_n\|^2} = 0, \forall (x_1, x_2, x_3, ..., x_n) \in X_1 \times X_2 \times ... \times X_n.$$

Therefore $(X, N_1^*)$ is a fuzzy-anti-n-normed linear space.

Now let us consider the mapping $T$: $X_1 \times X_2 \times ... \times X_n \to Y_1 \times Y_2 \times ... \times Y_n$ defined by

$$T(x_1, x_2, x_3, ..., x_n) = (x_1, x_2, x_3, ..., x_n) \forall (x_1, x_2, x_3, ..., x_n) \in X_1 \times X_2 \times X_3 \times ..... \times X_n$$

Let $\alpha \in (0, 1)$ and $t \in R^+$ and choose $M_\alpha = \frac{1}{1-\alpha}$.

We now prove that

$$N_1^*[(x_1, x_2, x_3, ..., x_n), \frac{t}{M_\alpha}] \leq 1 - \alpha \Rightarrow N_2^*[T(x_1, x_2, x_3, ..., x_n), t] \leq 1 - \alpha$$

$$\begin{aligned} & N_1^*[(x_1, x_2, x_3, ..., x_n), \frac{t}{M_\alpha}] \leq 1 - \alpha \Rightarrow \frac{4\ \|x_1, x_2, x_3, ..., x_n\|^2}{t^2(1-\alpha)^2 + 2\ \|x_1, x_2, x_3, ..., x_n\|^2} \leq 1 - \alpha \\ &\Rightarrow 1 - \frac{4\ \|x_1, x_2, x_3, ..., x_n\|^2}{t^2(1-\alpha)^2 + 2\ \|x_1, x_2, x_3, ..., x_n\|^2} \geq 1 - (1 - \alpha) = \alpha \\ &\Rightarrow \frac{t^2(1-\alpha)^2 - 2\ \|x_1, x_2, x_3, ..., x_n\|^2}{t^2(1-\alpha)^2 + 2\ \|x_1, x_2, x_3, ..., x_n\|^2} \geq \alpha \\ &\Rightarrow t^2(1-\alpha)^2 - 2\ \|x_1, x_2, x_3, ..., x_n\|^2 \geq t^2\,\alpha(1-\alpha)^2 + 2\alpha\ \|x_1, x_2, x_3, ..., x_n\|^2 \\ &\Rightarrow t^2(1-\alpha)^3 \geq 2(1+\alpha)\ \|x_1, x_2, x_3, ..., x_n\|^2 \\ &\Rightarrow\ \|x_1, x_2, x_3, ..., x_n\|^2 \leq \frac{t^2(1-\alpha)^3}{2(1+\alpha)} \\ &\Rightarrow\ \|x_1, x_2, x_3, ..., x_n\| \leq \frac{t(1-\alpha)\sqrt{(1-\alpha)}}{\sqrt{2}\sqrt{(1+\alpha)}} \\ &\Rightarrow\ t + \|x_1, x_2, x_3, ..., x_n\| \leq \frac{t\sqrt{2}\ \sqrt{(1+\alpha)} + t(1-\alpha)\sqrt{(1-\alpha)}}{\sqrt{2}\sqrt{(1+\alpha)}} \\ &\Rightarrow\ \frac{t}{t + \|x_1, x_2, x_3, ..., x_n\|} \geq \frac{\sqrt{2}\ \sqrt{(1+\alpha)}}{(1-\alpha)\sqrt{(1-\alpha)} + \sqrt{2}\ \sqrt{(1+\alpha)}} \end{aligned}$$

$$\Rightarrow 1-\frac{t}{t+\|x_1,x_2,x_3,...,x_n\|}\le 1-\frac{\sqrt{2}\sqrt{(1+\alpha)}}{(1-\alpha)\sqrt{(1-\alpha)}+\sqrt{2}\sqrt{(1+\alpha)}}$$

$$\Rightarrow \frac{\|x_1,x_2,x_3,...,x_n\|}{t+\|x_1,x_2,x_3,...,x_n\|}\le \frac{(1-\alpha)\sqrt{(1-\alpha)}}{(1-\alpha)\sqrt{(1-\alpha)}+\sqrt{2}\sqrt{(1+\alpha)}}$$

Now consider

$$\frac{(1-\alpha)\sqrt{(1-\alpha)}}{(1-\alpha)\sqrt{(1-\alpha)}+\sqrt{2}\sqrt{(1+\alpha)}}\le(1-\alpha)$$

$$\Leftrightarrow \sqrt{(1-\alpha)}\le \sqrt{2}\sqrt{(1+\alpha)}+\sqrt{(1-\alpha)}-\alpha\sqrt{(1-\alpha)}$$

$$\Leftrightarrow 0\le \sqrt{2}\sqrt{(1+\alpha)}-\alpha\sqrt{(1-\alpha)} \Leftrightarrow \alpha\sqrt{(1-\alpha)}\le \sqrt{2}\sqrt{(1+\alpha)}$$

$$\Leftrightarrow \alpha^2(1-\alpha)\le 2+2\alpha \Leftrightarrow \alpha^2\le\alpha^3+2\alpha+2,$$

which is true for all $\alpha\in(0,1)$.

Hence

$$N_1^*[(x_1,x_2,x_3,...,x_n),\ \frac{t}{M_\alpha}]\le 1-\alpha \Rightarrow N_2^*[T(x_1,x_2,x_3,...,x_n),\ t]\le 1-\alpha.$$

Therefore $T$ is weakly-fuzzy-anti-n-bounded.

Now conversely, let $T$ be St-fa-n-bounded.

$$N_2^*[T(x_1,x_2,x_3,...,x_n),\ t]\le N_1^*[(x_1,x_2,x_3,...,x_n),\ \frac{t}{M_\alpha}]$$

$$\frac{\|x_1,x_2,x_3,...,x_n\|}{t+\|x_1,x_2,x_3,...,x_n\|}\le\frac{4\ \|x_1,x_2,x_3,...,x_n\|^2}{\frac{t^2}{M^2}+2\ \|x_1,x_2,x_3,...,x_n\|^2}\{M_\alpha=M\}$$

$$\frac{\|x_1,x_2,x_3,...,x_n\|}{t+\|x_1,x_2,x_3,...,x_n\|}\le\frac{4\,M^2\,\|x_1,x_2,x_3,...,x_n\|^2}{t^2+2M^2\,\|x_1,x_2,x_3,...,x_n\|^2}$$

$$\Leftrightarrow t^2\,\|x_1,x_2,x_3,...,x_n\|+2M^2\,\|x_1,x_2,x_3,...,x_n\|^3\le 4tM^2\,\|x_1,x_2,x_3,...,x_n\|^2+4M^2\,\|x_1,x_2,x_3,...,x_n\|^3$$

$$\Leftrightarrow t^2\ \|x_1,x_2,x_3,...,x_n\|\le 4tM^2\,\|x_1,x_2,x_3,...,x_n\|^2+2M^2\,\|x_1,x_2,x_3,...,x_n\|^3$$

$$\Leftrightarrow t^2\le 4tM^2\,\|x_1,x_2,x_3,...,x_n\|+2M^2\,\|x_1,x_2,x_3,...,x_n\|^2$$

$$\Leftrightarrow \frac{t^2}{4t\,\|x_1,x_2,x_3,...,x_n\|+2\,\|x_1,x_2,x_3,...,x_n\|^2}\le M^2,$$

(i.e.)

$$M^2\ge\frac{t^2}{4t\,\|x_1,x_2,x_3,...,x_n\|+2\,\|x_1,x_2,x_3,...,x_n\|^2},\ \text{for } t\in(0,1)$$

$$\Leftrightarrow M\ge\frac{t}{\left(4t\,\|x_1,x_2,x_3,...,x_n\|+2\,\|x_1,x_2,x_3,...,x_n\|^2\right)^{\frac{1}{2}}}.$$

$M=\infty$ as $t\to\infty$. This would be contradiction to above assumption. Therefore $T$ is not St-fa-n-bounded.

**Theorem 4.6** *Let $T\colon X_1\times X_2\times...\times X_n\to Y_1\times Y_2\times...\times Y_n$ be a fuzzy-anti-n-linear mapping, $X_1,X_2,...,X_n$ and $Y_1,Y_2,...,Y_n$ are subspaces of $(X,N_1^*)$, $(Y,N_2^*)$ respectively. Then*

(i) *$T$ is St-fa-n-continuous on $X_1\times X_2\times...\times X_n$, if $T$ is St-fa-n-continuous at a point $(x_0^{(1)},x_0^{(2)},....,x_0^{(n)})\in X_1\times X_2\times ...\times X_n$;*

(ii) *$T$ is St-fa-n-continuous iff $T$ is St-fa-n-bounded.*

*Proof.* (i) Since $T$ is St-fa-n-continuous at $(x_0^{(1)},x_0^{(2)},.....,x_0^{(n)})\in X_1\times X_2\times...\times X_n$, if for each $\varepsilon>0$, there exists $\delta>0$, such that

$$N_2^*[T(x^{(1)},x^{(2)},...,x^{(n)})-T(x_0^{(1)},x_0^{(2)},...,x_0^{(n)}),\varepsilon]\le N_1^*[(x^{(1)},x^{(2)},...,x^{(n)})-(x_0^{(1)},x_0^{(2)},...,x_0^{(n)}),\delta],$$

taking $(y^{(1)}, y^{(2)}, ....., y^{(n)}) \in X_1 \times X_2 \times ... \times X_n$ and replacing $(x^{(1)}, x^{(2)}, ....., x^{(n)})$ by $(x^{(1)}, x^{(2)}, ..., x^{(n)}) + (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}) - (y^{(1)}, y^{(2)}, ..., y^{(n)})$, we get

$$N_2^*[T[(x^{(1)}, x^{(2)}, ..., x^{(n)}) + (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}) - (y^{(1)}, y^{(2)}, ..., y^{(n)})] - T(x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), \varepsilon]$$

$$\leq N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}) + (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}) - (y^{(1)}, y^{(2)}, ..., y^{(n)}) - (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), \delta]$$

$$\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}) - T(y^{(1)}, y^{(2)}, ..., y^{(n)}), \varepsilon] \leq N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}) - (y^{(1)}, y^{(2)}, ..., y^{(n)}), \delta].$$

Since $(y^{(1)}, y^{(2)}, ....., y^{(n)}) \in X_1 \times X_2 \times ... \times X_n$ is arbitrary. Therefore $T$ is St-fa-n-continuous on $X_1 \times X_2 \times ... \times X_n$.

(ii) Now we assume $T$ is St-fa-n-bounded. Thus there exists a positive real number $M$, such that for all $(x^{(1)}, x^{(2)}, ..., x^{(n)}) \in X_1 \times X_2 \times ... \times X_n$ and $\forall \varepsilon \in R^+$,

$$N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}), \varepsilon] \leq N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}), \frac{\varepsilon}{M}]$$

$$N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), \varepsilon] \leq N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}) - (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), \frac{\varepsilon}{M}]$$

$$\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), \varepsilon] \leq N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}) - (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), \delta]$$

where $\delta = \frac{\varepsilon}{M}$. Therefore $T$ is St-fa-n-continuous at $(x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)})$. This implies $T$ is St-fa-n-continuous on $X_1 \times X_2 \times ... \times X_n$.

Coming to converse let us assume $T$ is St-fa-n-continuous on $X_1 \times X_2 \times ... \times X_n$, applying fuzzy-anti-n-continuity at $(x^{(1)}, x^{(2)}, ..., x^{(n)}) = (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)})$ for $\varepsilon = 1$, there exists $\delta > 0$, such that $\forall (x^{(1)}, x^{(2)}, ..., x^{(n)}) \in X_1 \times X_2 \times ... \times X_n$,

$$N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), 1] \leq N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}) - (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), \delta].$$

If $(x^{(1)}, x^{(2)}, ..., x^{(n)}) \neq (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)})$ and t > 0, putting $(x^{(1)}, x^{(2)}, ..., x^{(n)}) = (u^{(1)}, u^{(2)}, ..., u^{(n)})\, t$

$$N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}),\ t] = N_2^*[T((u^{(1)}, u^{(2)}, ..., u^{(n)})\, t),\ t] = N_2^*[tT(u^{(1)}, u^{(2)}, ..., u^{(n)}),\ t]$$

$$= N_2^*[T(u^{(1)}, u^{(2)}, ..., u^{(n)}),\ 1] \leq N_1^*[(u^{(1)}, u^{(2)}, ..., u^{(n)}),\ \delta]$$

$$= N_1^*\left[\frac{(x^{(1)}, x^{(2)}, ..., x^{(n)})}{t},\ \delta\right] = N_1^*\left[(x^{(1)}, x^{(2)}, ..., x^{(n)}),\ t\delta\right]$$

$$= N_1^*\left[(x^{(1)}, x^{(2)}, ..., x^{(n)}),\ \frac{t}{1/\delta}\right] = N_1^*\left[(x^{(1)}, x^{(2)}, ..., x^{(n)}),\ \frac{t}{M}\right],$$

where $M = \frac{1}{\delta}$, so, $N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}),\ t] \leq N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}),\ \frac{t}{M}]$.

If $(x^{(1)}, x^{(2)}, ..., x^{(n)}) \neq (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)})$ and $t \leq 0$, then

$$N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}),\ t] = N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}),\ \frac{t}{M}] = 1$$

If $(x^{(1)}, x^{(2)}, ..., x^{(n)}) = (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)})$ and $t \in R$, then

$$T(x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}) = (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)})$$

and

$$N_2^*[T(x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), t] = N_1^*[(x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), \frac{t}{M}] = 0$$

if $t > 0$;

$$N_2^*[T(x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}),\ t] = N_1^*[(x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), \frac{t}{M}] = 1$$

if $t \leq 0$. Therefore$T$ is St-fa-n-bounded.

**Theorem 4.7** *Let* $T: X_1 \times X_2 \times ... \times X_n \to Y_1 \times Y_2 \times ... \times Y_n$ *be a fuzzy-anti-n-linear mapping,* $X_1, X_2, ..., X_n$ *and* $Y_1, Y_2, ..., Y_n$ *are subspaces of* $(X, N_1^*)$,$(Y, N_2^*)$ *respectively. Then*

(i) *T is Wk-fa-n-continuous on* $X_1 \times X_2 \times ... \times X_n$ *if T is Wk-fa-n-continuous at a point* $(x_0^{(1)}, x_0^{(2)}, ....., x_0^{(n)}) \in X_1 \times X_2 \times ... \times X_n$.

(ii) *T is Wk-fa-n-continuous if and only if T is Wk-fa-n-bounded.*

*Proof.* (i) Since $T$ is Wk-fa-n-continuous at $(x_0^{(1)}, x_0^{(2)}, ....., x_0^{(n)}) \in X_1 \times X_2 \times ... \times X_n$ for $\varepsilon > 0$, $\alpha \in (0, 1)$, $\exists$ $\delta = \delta(\alpha, \varepsilon) > 0$, such that $\forall (x^{(1)}, x^{(2)}, ..., x^{(n)}) \in X_1 \times X_2 \times ..... \times X_n$

$$N_1^*[(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) - (x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \delta] \leq 1 - \alpha$$

$$\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, x^{(3)}, ..., x^{(n)}) - T(x_0^{(1)}, x_0^{(2)}, x_0^{(3)}, ..., x_0^{(n)}), \varepsilon] \leq 1 - \alpha$$

taking $(y^{(1)}, y^{(2)}, ....., y^{(n)}) \in X_1 \times X_2 \times ... \times X_n$ and replacing $(x^{(1)}, x^{(2)}, ....., x^{(n)})$ by $(x^{(1)}, x^{(2)}, ..., x^{(n)}) + (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}) - (y^{(1)}, y^{(2)}, ..., y^{(n)})$, we get

$$N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}) + (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}) - (y^{(1)}, y^{(2)}, ..., y^{(n)}) - (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), \delta] \leq 1 - \alpha$$

$$N_2^*[T[(x^{(1)}, x^{(2)}, ..., x^{(n)}) + (x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}) - (y^{(1)}, y^{(2)}, ..., y^{(n)})] - T(x_0^{(1)}, x_0^{(2)}, ..., x_0^{(n)}), \varepsilon] \leq 1 - \alpha$$

(i.e.)

$$N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}) - (y^{(1)}, y^{(2)}, ..., y^{(n)}), \delta] \leq 1 - \alpha$$
$$N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}) - T(y^{(1)}, y^{(2)}, ..., y^{(n)}), \varepsilon] \leq 1 - \alpha.$$

Since $(y^{(1)}, y^{(2)}, ....., y^{(n)}) \in X_1 \times X_2 \times ... \times X_n$ is arbitrary, $T$ is Wk-fa-n-continuous on $X_1 \times X_2 \times ... \times X_n$.

(ii) Now we assume $T$ is Wk-fa-n-bounded. Thus for any $\alpha \in (0, 1)$ there exists $M_\alpha > 0$, such that $\forall\ t \in R$ and for all $(x^{(1)}, x^{(2)}, ..., x^{(n)}) \in X_1 \times X_2 \times ... \times X_n$, we have

$$N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}), \frac{t}{M}] \leq 1 - \alpha \Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}), t] \leq 1 - \alpha.$$

Therefore

$$N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}) - (\theta^{(1)}, \theta^{(2)}, ..., \theta^{(n)}), \frac{t}{M}] \leq 1 - \alpha$$

$$\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}) - T(\theta^{(1)}, \theta^{(2)}, ..., \theta^{(n)}), t] \leq 1 - \alpha$$

(i.e.)

$$N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}) - (\theta^{(1)}, \theta^{(2)}, ..., \theta^{(n)}), \frac{\varepsilon}{M_\alpha}] \leq 1 - \alpha$$

$$\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}) - T(\theta^{(1)}, \theta^{(2)}, ..., \theta^{(n)}),\ \varepsilon] \leq 1 - \alpha$$

(i.e.)

$$N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}) - (\theta^{(1)}, \theta^{(2)}, ..., \theta^{(n)}),\ \delta] \leq 1 - \alpha$$

$$\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}) - T(\theta^{(1)}, \theta^{(2)}, ..., \theta^{(n)}),\ \varepsilon] \leq 1 - \alpha$$

where $\frac{\varepsilon}{M_\alpha} = \delta$. Therefore $T$ is Wk-fa-n-continuous at $(x_0^{(1)}, x_0^{(2)}, ....., x_0^{(n)})$, which implies $T$ is Wk-fa-n-continuous on $X_1 \times X_2 \times ... \times X_n$.

Coming to converse let us assume $T$ is Wk-fa-n-continuous on $X_1 \times X_2 \times ... \times X_n$, applying continuity of T at $(\theta^{(1)}, \theta^{(2)}, ..., \theta^{(n)})$ and take $\varepsilon = 1$, we have $\forall \alpha \in (0, 1)\ \exists \delta(\alpha, 1) > 0$, such that $\forall (x^{(1)}, x^{(2)}, ..., x^{(n)}) \in X_1 \times X_2 \times ... \times X_n$, (i.e.)

$$N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}) - (\theta^{(1)}, \theta^{(2)}, ..., \theta^{(n)}),\ \delta] \leq 1 - \alpha$$

$$\Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}) - T(\theta^{(1)}, \theta^{(2)}, ..., \theta^{(n)}),\ 1] \leq 1 - \alpha$$

(i.e.)

$$N_1^*[(x^{(1)}, x^{(2)}, ..., x^{(n)}),\ \delta] \leq 1 - \alpha \Rightarrow N_2^*[T(x^{(1)}, x^{(2)}, ..., x^{(n)}), 1] \leq 1 - \alpha.$$

If $(x^{(1)}, x^{(2)}, ..., x^{(n)}) \neq (\theta^{(1)}, \theta^{(2)}, ..., \theta^{(n)})$ and $t > 0$, putting $(x^{(1)}, x^{(2)}, ...., x^{(n)}) = \frac{(u^{(1)}, u^{(2)}, ...., u^{(n)})}{t}$

$$N_1^*\left(\frac{(u^{(1)}, u^{(2)}, ...., u^{(n)})}{t},\ \delta\right) \leq 1 - \alpha \Rightarrow N_2^*\left(T\left(\frac{(u^{(1)}, u^{(2)}, ...., u^{(n)})}{t}\right),\ 1\right) \leq 1 - \alpha$$

(i.e.)

$$N_1^*\left((u^{(1)},u^{(2)},....,u^{(n)}),\ t\,\delta\right)\le 1-\alpha\Rightarrow N_2^*\left(T\left(\frac{(u^{(1)},u^{(2)},....,u^{(n)})}{t}\right),\ 1\right)\le 1-\alpha$$

(i.e.)

$$N_1^*\left((u^{(1)},u^{(2)},....,u^{(n)}),\ \frac{t}{M_\alpha}\right)\le 1-\alpha\Rightarrow N_2^*\left(T\left(\frac{(u^{(1)},u^{(2)},....,u^{(n)})}{t}\right),\ 1\right)\le 1-\alpha$$

where $M_\alpha=\frac{1}{\delta(\alpha,1)}$. So

$$N_1^*\left[t(x^{(1)},x^{(2)},....,x^{(n)}),\ \frac{t}{M_\alpha}\right]\le 1-\alpha\Rightarrow N_2^*\left[T(x^{(1)},x^{(2)},....,x^{(n)}),1\right]\le 1-\alpha$$

$$N_1^*\left[(x^{(1)},x^{(2)},....,x^{(n)}),\ \frac{t}{M_\alpha}\right]\le 1-\alpha\Rightarrow N_2^*\left[T\left(\frac{(x^{(1)},x^{(2)},....,x^{(n)})}{t}\right),\ 1\right]\le 1-\alpha$$

$$N_1^*\left[(x^{(1)},x^{(2)},....,x^{(n)}),\ \frac{t}{M_\alpha}\right]\le 1-\alpha\Rightarrow N_2^*\left[T\left((x^{(1)},x^{(2)},....,x^{(n)})\right),\ t\right]\le 1-\alpha,$$

where $M_\alpha=\frac{1}{\delta(\alpha,1)}$. If $(x^{(1)},x^{(2)},...,x^{(n)})\ne(\theta^{(1)},\theta^{(2)},...,\theta^{(n)})$ and $t\le 0$,

$$N_1^*\left[(x^{(1)},x^{(2)},....,x^{(n)}),\ \frac{t}{M_\alpha}\right]=N_2^*\left[T\left((x^{(1)},x^{(2)},....,x^{(n)})\right),\ t\right]=1\text{ for any }M_\alpha>0.$$

If $(x^{(1)},x^{(2)},...,x^{(n)})=(\theta^{(1)},\theta^{(2)},...,\theta^{(n)})$, then for $M_\alpha>0$,

$$N_1^*\left[(x^{(1)},x^{(2)},....,x^{(n)}),\ \frac{t}{M_\alpha}\right]=N_2^*\left[T\left((x^{(1)},x^{(2)},....,x^{(n)})\right),\ t\right]=0,\ \text{ if }t>0,$$

$$N_1^*\left[(x^{(1)},x^{(2)},....,x^{(n)}),\ \frac{t}{M_\alpha}\right]=N_2^*\left[T\left((x^{(1)},x^{(2)},....,x^{(n)})\right),\ t\right]=1,\ \text{ if }t\le 0.$$

Therefore $T$ is Wk-fa-n-bounded.

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