Journal of Mathematics Research; Vol. 5, No. 4; 2013
ISSN 1916-9795 E-ISSN 1916-9809
Published by Canadian Center of Science and Education

On Complex Contact Similarity Manifolds

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Received: July 14, 2013 Accepted: August 27, 2013 Online Published: September 29, 2013

doi:10.5539/jmr.v5n4p1 URL: http://dx.doi.org/10.5539/jmr.v5n4p1

Abstract

We shall construct *complex contact similarity manifolds*. Among them there exists a complex contact infranilmanifold $\mathcal{L}/\Gamma$ which is a holomorphic torus fiber space over a quaternionic euclidean orbifold. Specifically taking a connected sum of $\mathcal{L}/\Gamma$ with the complex projective space $\mathbb{CP}^{2n+1}$, we prove that the connected sum admits a complex contact structure. Our examples of complex contact manifolds are different from those known previously as complex Boothby-Wang fibration (Foreman, 2000) or the twistor fibration (Salamon, 1989).

Keywords: complex contact structure, Sasakian 3-structure, Twistor space, complex Boothby-Wang fibration, Infranil-manifold, Quaternionic Kähler manifold

2010 Mathematics Subject Classification: 53D10, 57S25, 32Q55

1. Introduction

There is a construction of three different types of complex contact structure. Given a $4n$-dimensional quaternionic Kähler manifold N of nonzero scalar curvature, the twistor construction produces a complex contact manifold M which is the total space of a fibration: $S^2 \to M \to N$ (cf. Salamon, 1989; Wolf, 1965). Similarly, a quaternionic Kähler manifold N^{4n} of positive (resp. negative) scalar curvature induces a Sasakian 3-structure (resp. pseudo-Sasakian 3-structure) on the total space M^{4n+3} of the principal fibration: $S^3 \to M \to N$. By taking a circle S^1 from S^3, the total space M/S^1 of the quotient bundle $S^2 \to M/S^1 \to N$ admits a complex contact structure. (See Ishihara & Konishi, 1979; Moroianu & Semmelmann, 1996; Tanno, 1996). However, these constructions cannot produce complex contact manifolds for quaternionic Kähler manifolds of vanishing scalar curvature. On the other hand, if N^{4n} is a complex symplectic manifold with a complex symplectic form $\Omega = \Omega_1 + \mathbf{i}\Omega_2$ such that $[\Omega_i] \in H^2(N;\mathbb{Z})$ is an integral class ($i = 1, 2$), then the *complex Boothby-Wang fibration* induces a compact complex contact manifold M which has a connection bundle: $T^2 \to M \to N$ (cf. Foreman, 2000; Blair, 2002). If N^{4n} happens to be a quaternionic Kähler manifold with vanishing scalar curvature, then we have a new example of compact complex manifold. In fact, Foreman (2000) shows that a complex nilmanifold M which is the total space of a principal torus bundle over a complex torus $T^{2n}_{\mathbb{C}}$ admits a complex contact structure. The universal covering $\tilde{M}$ is endowed with a complex nilpotent Lie group structure which is called *generalized complex Heisenberg group* in Foreman, 2000.

In this paper, we study *complex contact transformation groups* by taking into account this specific nilpotent Lie group. We verify this group from the viewpoint of geometric structure in Section 4. In fact the sphere S^{4n+3} admits a canonical quaternionic CR-structure. The sphere S^{4n+3} with one point ∞ removed is isomorphic to the $4n+3$-dimensional quaternionic Heisenberg Lie group $\mathcal{M}$ as a quaternionic CR-structure. $\mathcal{M}$ has a central group extension: $1 \to \mathbb{R}^3 \to \mathcal{M} \xrightarrow{p} \mathbb{H}^n \to 1$ where $\mathbb{R}^3 = \mathrm{Im}\mathbb{H}$ is the imaginary part of the quaternion field $\mathbb{H}$. Taking a quotient of $\mathcal{M}$ by $\mathbb{R}\,(= \mathbb{R}\mathbf{i})$, we obtain a complex nilpotent Lie group $\mathcal{L}\,(= \mathcal{L}_{2n+1})$ which supports a holomorphic principal bundle $\mathbb{C} \to \mathcal{L} \xrightarrow{p} \mathbb{C}^{2n}$. The canonical quaternionic CR-structure on S^{4n+3} restricts a Carnot-Carathéodory structure B to $\mathcal{M}$. Using this bundle B, a left invariant complex contact structure on $\mathcal{L}$ is obtained (cf. Alekseevsky & Kamishima, 2008; Kamishima, 1999).

We are mainly interested in constructing examples of *compact* complex contact manifolds which are not known previously. Let $\mathrm{Sim}(\mathcal{L})$ be the group of complex contact similarity transformations. It is defined to be the semidirect product $\mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot \mathbb{C}^*)$, $(\mathbb{C}^* = S^1 \times \mathbb{R}^+)$. The pair $(\mathrm{Sim}(\mathcal{L}), \mathcal{L})$ is said to be *complex contact similarity geometry*. A manifold M locally modelled on this geometry is called a complex contact similarity manifold. Denote by $\mathrm{Aut}_{cc}(M)$ the group of complex contact transformations of M. We prove the following characterization of compact complex contact similarity manifolds in Section 2 (Compare Fried, 1980; Miner, 1991 for the related results in this direction).

Theorem A *Let M be a compact complex contact similarity manifold of complex dimension $2n + 1$. If $S^1 \leq \mathrm{Aut}_{cc}(M)$ acts on M without fixed points, then M is holomorphically diffeomorphic to a complex contact infranil-manifold $\mathcal{L}/\Gamma$ or a complex contact Hopf manifold $\mathcal{L} - \{0\}/\mathbb{Z}^+$ diffeomorphic to $S^1 \times S^{4n+1}$. Here Γ is a discrete cocompact subgroup in $\mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot S^1)$ or $\mathbb{Z}^+$ is an infinite cyclic subgroup of $\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$.*

In Section 3, we can perform a connected sum of our complex contact infranil-manifolds $\mathcal{L}/\Gamma$ $(\Gamma \leq \mathrm{E}(\mathcal{L}))$.

Theorem B *The connected sum $\mathbb{CP}^{2n+1}\#\mathcal{L}/\Gamma$ admits a complex contact structure.*

By iteration of this procedure there exists a complex contact structure on the connected sum of a finite number of complex contact similarity manifolds and $\mathbb{CP}^{2n+1}$'s. These examples are different from those admitting S^2 (resp. T^2)-fibrations.

2. Complex Contact Structure on the Nilpotent Group

2.1 Definition of Complex Contact Structure

Recall that a complex contact structure on a complex manifold M in complex dimension $2n + 1$ is a collection of local forms $\{U_\alpha, \omega_\alpha\}_{\alpha\in\Lambda}$ which satisfies that (1) $\bigcup_{\alpha\in\Lambda} U_\alpha = M$. (2) Each ω_α is a holomorphic 1-form defined on U_α. Then $\omega_\alpha \wedge (d\omega_\alpha)^n \neq 0$ on U_α. (3) If $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a nonzero holomorphic function $f_{\alpha\beta}$ on $U_\alpha \cap U_\beta$ such that $f_{\alpha\beta} \cdot \omega_\alpha = \omega_\beta$. Unlike contact structures on orientable smooth manifolds, it does not always exist a holomorphic 1-form globally defined on M. Note that if the first Chern class $c_1(M)$ vanishes, then there is a global existence of a complex contact form ω on M. (See Kobayashi, 1959; Lebrun, 1995).

Let h: $M \to M$ be a biholomorphism. Suppose that $h(U_\alpha) \cap U_\beta \neq \emptyset$ for some $\alpha, \beta \in \Lambda$. If there exists a holomorphic function $f_{\alpha\beta}$ on an open subset in U_α such that $h^*\omega_\beta = f_{\alpha\beta}\omega_\alpha$, then we call h a *complex contact transformation* of M. Denote $\mathrm{Aut}_{cc}(M)$ the group of complex contact transformations. It is not necessarily a finite dimensional complex Lie group.

2.2 The Iwasawa Nilpotent Lie Group $\mathcal{L}_{2n+1}$

Let $\mathcal{L}_{2n+1}$ be the product $\mathbb{C}^{2n+1} = \mathbb{C} \times \mathbb{C}^{2n}$ with group law $(n \geq 1)$:

$$(x, z) \cdot (y, w) = (x + y + \sum_{i=1}^{n} z_{2i-1}w_{2i} - z_{2i}w_{2i-1}, z + w) \tag{2.1}$$

where $z = (z_1, \ldots, z_{2n})$, $w = (w_1, \ldots, w_{2n})$.

Put $\mathcal{L} = \mathcal{L}_{2n+1}$. It is easy to see that $[(x, z), (y, w)] = (2\sum_{i=1}^{n} z_{2i-1}w_{2i} - z_{2i}w_{2i-1}, 0)$ so $[\mathcal{L}, \mathcal{L}] = (\mathbb{C}, (0, \ldots, 0)) = \mathbb{C}$ is the center of $\mathcal{L}$. Thus there is a central group extension: $1 \to \mathbb{C} \to \mathcal{L}_{2n+1} \longrightarrow \mathbb{C}^{2n} \to 1$. It is easy to check that $\mathcal{L}_3$ is isomorphic to the Iwasawa group consisting of 3×3-upper triangular unipotent complex matrices.

Definition 2.1 A complex $2n + 1$-dimensional complex nilpotent Lie group $\mathcal{L}_{2n+1}$ is said to be the Iwasawa Lie group.

See (Foreman, 2000, pp.193-195) for more general construction of this kind of Lie group.

2.3 Construction of Complex Contact Structure on $\mathcal{L}_{2n+1}$

Choose a coordinate $(z_0, z_1, \ldots, z_{2n}) \in \mathcal{L}_{2n+1}$, we define a complex 1-form η:

$$\eta = dz_0 - (\sum_{i=1}^{n} z_{2i-1} \cdot dz_{2i} - z_{2i} \cdot dz_{2i-1}) = dz_0 - (z_1, \ldots, z_{2n})\,\mathsf{J}_n \begin{pmatrix} dz_1 \\ \vdots \\ dz_{2n} \end{pmatrix} \tag{2.2}$$

where $\mathsf{J}_n = \begin{pmatrix} J & & \\ & \ddots & \\ & & J \end{pmatrix}$ with $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Since $\eta \wedge (d\eta)^n$ is a non-vanishing form $2n(-2)^n dz_0 \wedge \cdots \wedge dz_{2n}$ on $\mathcal{L}_{2n+1}$, η is a complex contact structure on $\mathcal{L}_{2n+1}$ by Definition 2.1.

2.4 Complex Contact Transformations

Let $\mathrm{hol}(\mathcal{L}_{2n+1})$ be the group of biholomorphic transformations of $\mathcal{L} = \mathcal{L}_{2n+1}$. The group of complex contact transformations on $\mathcal{L}$ with respect to η is denoted by

$$\mathrm{hol}(\mathcal{L}, \eta) = \{f \in \mathrm{hol}(\mathcal{L}) \mid f^*\eta = \tau \cdot \eta\} \tag{2.3}$$

where τ is a holomorphic function on $\mathcal{L}$.

Let $\mathrm{Sp}(n, \mathbb{C}) = \{A \in M(2n, \mathbb{C}) \mid {}^tA\mathsf{J}_nA = \mathsf{J}_n\}$ be the complex symplectic group. As $\mathrm{Sp}(n, \mathbb{C}) \cap \mathbb{C}^* = \{\pm 1\}$, denote $\mathrm{Sp}(n, \mathbb{C}) \cdot \mathbb{C}^* = \mathrm{Sp}(n, \mathbb{C}) \times \mathbb{C}^*/\{\pm 1\}$. Put

$$\mathrm{A}(\mathcal{L}) = \mathcal{L} \rtimes (\mathrm{Sp}(n, \mathbb{C}) \cdot \mathbb{C}^*) \tag{2.4}$$

which forms a group as follows; write elements $\lambda \cdot A$, $\mu \cdot B \in \mathrm{Sp}(n, \mathbb{C}) \cdot \mathbb{C}^*$ for $A, B \in \mathrm{Sp}(n, \mathbb{C}), \lambda, \mu \in \mathbb{C}^*$. Let $(a, w), (b, z) \in \mathcal{L}$. Define

$$((a, w), \lambda \cdot A) \cdot ((b, z), \mu \cdot B) = ((a + \lambda^2 b + {}^tw\, \mathsf{J}_n(\lambda Az), w + \lambda Az), \lambda\mu \cdot AB).$$

Here ${}^tw\, \mathsf{J}_n(\lambda Az) = \sum_{i=1}^{n} w_{2i-1} \cdot (\lambda Az)_{2i} - w_{2i} \cdot (\lambda Az)_{2i-1}$ as before.

Let $((a, w), \lambda \cdot A) \in \mathrm{A}(\mathcal{L}), (z_0, z) \in \mathcal{L}$. $\mathrm{A}(\mathcal{L})$ acts on $\mathcal{L}$ as

$$((a, w), \lambda \cdot A) \cdot (z_0, z) = (a, w) \cdot (\lambda^2 z_0, \lambda Az) = (a + \lambda^2 z_0 + {}^tw\, \mathsf{J}_n(\lambda Az),\ w + \lambda Az). \tag{2.5}$$

If $h = ((b, w), \mu \cdot B) \in \mathrm{A}(\mathcal{L})$ is an element, then it is easy to see that

$$h^*\eta = \mu^2 \cdot \eta. \tag{2.6}$$

Thus $\mathrm{A}(\mathcal{L})$ preserves the complex contact structure on $\mathcal{L}$ defined by η.

Let $\mathrm{Aff}(\mathbb{C}^{2n+1}) = \mathbb{C}^{2n+1} \rtimes \mathrm{GL}(2n + 1, \mathbb{C})$ be the complex affine group which is a subgroup of $\mathrm{hol}(\mathcal{L})$ since $\mathcal{L}_{2n+1} = \mathbb{C}^{n+1}$ (biholomorphically). We assign to each $((a, w), \lambda \cdot A) \in \mathrm{A}(\mathcal{L})$ an element

$$\left(\begin{bmatrix} a \\ w \end{bmatrix}, \left(\begin{array}{c|c} \lambda^2 & \lambda^t w\, \mathsf{J}_n A \\ \hline 0 & \lambda A \end{array}\right)\right) \in \mathrm{Aff}(\mathbb{C}^{2n+1}). \tag{2.7}$$

Then the action (2.5) of $((a, w), \lambda \cdot A)$ on $\mathcal{L}$ coincides with the above affine transformation of $\mathbb{C}^{2n+1}$. Moreover, it is easy to check that this correspondence is an injective homomorphism:

$$\mathrm{A}(\mathcal{L}) \leq \mathrm{Aff}(\mathbb{C}^{2n+1}). \tag{2.8}$$

As a consequence it follows

$$\mathrm{A}(\mathcal{L}) \leq \mathrm{hol}(\mathcal{L}, \eta). \tag{2.9}$$

Let M be a smooth manifold. Suppose that there exists a maximal collection of charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in\Lambda}$ whose coordinate changes belong to $\mathrm{A}(\mathcal{L})$. More precisely, $M = \bigcup_{\alpha\in\Lambda} U_\alpha$, φ_α: $U_\alpha \to \mathcal{L}$ is a diffeomorphism onto its image. If $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a unique element $g_{\alpha\beta} \in \mathrm{A}(\mathcal{L})$ such that $g_{\alpha\beta} = \varphi_\beta \cdot \varphi_\alpha^{-1}$ on $\varphi_\alpha(U_\alpha \cap U_\beta)$. We say that M is locally modelled on $(\mathrm{A}(\mathcal{L}), \mathcal{L})$ (Compare Kulkarni, 1978).

Here is a sufficient condition for the existence on complex contact structure.

Proposition 2.2 *If a* $(4n+2)$*-dimensional smooth manifold M is locally modelled on* $(\mathrm{A}(\mathcal{L}), \mathcal{L})$*, then M is a complex contact manifold. Moreover, M is also a complex affinely flat manifold.*

Proof. First of all, we define a complex structure on M. Let J_0 be the standard complex structure on $\mathcal{L} = \mathbb{C}^{2n+1}$. Define a complex structure J_α on U_α by setting $\varphi_{\alpha *}J_\alpha = J_0\varphi_{\alpha *}$ on U_α for each $\alpha \in \Lambda$. When $g_{\alpha\beta} \in \mathrm{A}(\mathcal{L})$, note that $g_{\alpha\beta *}J_0 = J_0 g_{\alpha\beta *}$ from (2.8). On $U_\alpha \cap U_\beta$, a calculation shows that $\varphi_{\beta *}J_\alpha = g_{\alpha\beta *}\varphi_{\alpha *}J_\alpha = J_0\varphi_{\beta *}$. Since $\varphi_{\beta *}J_\beta = J_0\varphi_{\beta *}$ by the definition, it follows $J_\alpha = J_\beta$ on $U_\alpha \cap U_\beta$. This defines a complex structure J on M. In particular, each φ_α: $(U_\alpha, J) \to (\mathcal{L}, J_0)\ (= \mathbb{C}^{2n+1})$ is a holomorphic embedding. Let η be the holomorphic 1-form on $\mathcal{L}$ as before. Define a family of local holomorphic 1-forms $\{\omega_\alpha, U_\alpha\}_{\alpha\in\Lambda}$ by

$$\omega_\alpha = \varphi_\alpha^*\eta \text{ on } U_\alpha. \tag{2.10}$$

If $U_\alpha \cap U_\beta \neq \emptyset$, then there exists a unique element $g_{\alpha\beta} \in \mathrm{A}(\mathcal{L})$ such that $g_{\alpha\beta} = \varphi_\beta \cdot \varphi_\alpha^{-1}$. From (2.6), $g_{\alpha\beta}^*\eta = \mu_{\alpha\beta}^2 \cdot \eta$ for some $\mu_{\alpha\beta} \in \mathbb{C}^*$. It follows $\omega_\beta = \mu_{\alpha\beta}^2 \cdot \omega_\alpha$. Thus the family $\{\omega_\alpha, U_\alpha\}_{\alpha\in\Lambda}$ is a complex contact structure on (M, J).

Apart from the complex contact structure, since $\mathrm{A}(\mathcal{L}) \leq \mathrm{Aff}(\mathbb{C}^{2n+1})$ from (2.8), M is also modelled on $(\mathrm{Aff}(\mathbb{C}^{2n+1}), \mathbb{C}^{2n+1})$ where $\mathcal{L} = \mathbb{C}^{2n+1}$. M is a complex affinely flat manifold. □

Remark 2.3 (1) When a subgroup $\Gamma \leq \mathrm{A}(\mathcal{L})$ acts properly discontinuously and freely on a domain Ω of $\mathcal{L}$ with compact quotient, we obtain a compact complex contact manifold Ω/Γ by this proposition. In fact let p: $\Omega \to \Omega/\Gamma$ be a covering holomorphic projection. Take a set of evenly covered neighborhoods $\{U_\alpha\}_{\alpha\in\Lambda}$ of Ω/Γ. Choose a family of open subsets $\tilde{U}_\alpha$ such that $p_\alpha = p_{|\tilde{U}_\alpha}$: $\tilde{U}_\alpha \to U_\alpha$ is a biholomorphism. Put $\omega_\alpha = (p_\alpha^{-1})^*\eta$. Then the family $\{U_\alpha, \omega_\alpha\}_{\alpha\in\Lambda}$ is a complex contact structure on Ω/Γ.

(2) When $\Omega = \mathcal{L}$, $\mathcal{L}/\Gamma$ is said to be a compact complete affinely flat manifold. Concerning the Auslandr-Milnor conjecture, we do not know whether the fundamental group Γ is virtually polycyclic.

(3) By the monodromy argument, there exists a developing immersion: dev: $\tilde{M} \to \mathcal{L}$ from the universal covering $\tilde{M}$ of M. Then note that $\mathrm{dev}_*J = J_0\mathrm{dev}_*$, i.e. dev is a holomorphic map. Here J is the lift of complex structure on $\tilde{M}$ (We wrote the same J on $\tilde{M}$).

When M is a complex manifold, we assume that the complex structure on M coincides with the one constructed in Proposition 2.2.

2.5 Complex Contact Similarity Geometry

It is in general difficult to find such a properly discontinuous group Γ as in Remark 2.3. $\mathrm{Sp}(n,\mathbb{C})$ contains a maximal compact symplectic subgroup $\mathrm{Sp}(n) = \{A \in \mathrm{U}(2n) \,|\, {}^tA\mathsf{J}_nA = \mathsf{J}_n\}$ where $\mathrm{Sp}(n,\mathbb{C}) \cong \mathrm{Sp}(n) \times \mathbb{R}^{n(2n+1)}$.

Definition 2.4 Put $\mathrm{Sim}(\mathcal{L}) = \mathcal{L} \rtimes (\mathrm{Sp}(n)\cdot\mathbb{C}^*) \leq \mathrm{A}(\mathcal{L})$. The pair $(\mathrm{Sim}(\mathcal{L}), \mathcal{L})$ is called complex contact similarity geometry. If a manifold M is locally modelled on this geometry, M is said to be a complex contact similarity manifold. The euclidean subgroup of $\mathrm{Sim}(\mathcal{L})$ is defined to be $\mathrm{E}(\mathcal{L}) = \mathcal{L} \rtimes (\mathrm{Sp}(n)\cdot S^1)$.

For example, choose $c \in \mathbb{C}^*$ with $|c| \neq 1$ and $A \in \mathrm{Sp}(n)$. Put $r = ((0,0), c\cdot A) \in \mathrm{Sim}(\mathcal{L})$. Let $\mathbb{Z}^+$ be an infinite cyclic group generated by r. Then it is easy to see that $\mathbb{Z}^+$ acts freely and properly discontinuously on the complement $\mathcal{L} - \{0\}$. Here $0 = (0,0) \in \mathcal{L}_{2n+1} = \mathcal{L}$. The quotient $\mathcal{L} - \{0\}/\mathbb{Z}^+$ is diffeomorphic to $S^1 \times S^{4n+1}$. By Proposition 2.2 ((1) of Remark 2.3), $S^1 \times S^{4n+1}$ is a complex contact similarity manifold.

Let $\mathbb{H}^n$ be the $4n$-dimensional quaternionic vector space. The quaternionic similarity group $\mathrm{Sim}(\mathbb{H}^n) = \mathbb{H}^n \rtimes ((\mathrm{Sp}(n)\cdot\mathrm{Sp}(1)) \times \mathbb{R}^+)$ (resp. quaternionic euclidean group $\mathrm{E}(\mathbb{H}^n) = \mathbb{H}^n \rtimes (\mathrm{Sp}(n)\cdot\mathrm{Sp}(1))$) has a special subgroup $\widehat{\mathrm{Sim}}(\mathbb{H}^n) = \mathbb{H}^n \rtimes ((\mathrm{Sp}(n)\cdot S^1) \times \mathbb{R}^+)$ (resp. $\widehat{\mathrm{E}}(\mathbb{H}^n) = \mathbb{H}^n \rtimes (\mathrm{Sp}(n)\cdot S^1)$). When we identify $\mathbb{H}^n$ with the complex vector space $\mathbb{C}^{2n}$ by the correspondence $(a + b\mathbf{j}) \mapsto (\bar{a}, b)$, $\widehat{\mathrm{Sim}}(\mathbb{H}^n)$ is canonically isomorphic to the complex similarity subgroup $\mathbb{C}^{2n} \rtimes (\mathrm{Sp}(n)\cdot\mathbb{C}^*)$ where $\mathbb{C}^* = S^1 \times \mathbb{R}^+$. Then there are commutative exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{Sim}(\mathcal{L}) & \xrightarrow{\mathsf{p}} & \widehat{\mathrm{Sim}}(\mathbb{H}^n) & \longrightarrow & 1 \\ & & \| & & \cup & & \cup & & \\ 1 & \longrightarrow & \mathbb{C} & \longrightarrow & \mathrm{E}(\mathcal{L}) & \xrightarrow{\mathsf{p}} & \widehat{\mathrm{E}}(\mathbb{H}^n) & \longrightarrow & 1. \end{array} \tag{2.11}$$

Choosing a torsionfree discrete cocompact subgroup Γ from $\mathrm{E}(\mathcal{L})$, we obtain an infranilmanifold $\mathcal{L}/\Gamma$ of complex dimension $2n+1$. In particular, $\Gamma \cap \mathcal{L}$ is discrete uniform in $\mathcal{L}$ by the Auslander-Bieberbach theorem. As $\mathbb{C}$ is the central subgroup of $\mathcal{L}$, $\Gamma \cap \mathbb{C}$ is discrete uniform in $\mathbb{C}$ and so $\Delta = \mathsf{p}(\Gamma)$ is a discrete uniform subgroup in $\widehat{\mathrm{E}}(\mathbb{H}^n)$. We obtain a Seifert singular fibration over a quaternionic euclidean orbifold $\mathbb{H}^n/\Delta$: $T^1_{\mathbb{C}} \to \mathcal{L}/\Gamma \longrightarrow \mathbb{H}^n/\Delta$. By (1) of Remark 2.3, $\mathcal{L}/\Gamma$ is a complex contact manifold.

Remark 2.5 When we take a finite index nilpotent subgroup Γ' of Γ admitting a central extension: $1 \to \mathbb{Z}^2 \to \Gamma' \longrightarrow \mathbb{Z}^{4n} \to 1$, a nilmanifold $\mathcal{L}/\Gamma'$ admits a holomorphic principal $T^1_{\mathbb{C}}$-bundle over a complex torus $T^{2n}_{\mathbb{C}} = \mathbb{H}^n/\mathbb{Z}^{4n}$. This holomorphic example is a special case of Foreman's T^2- connection bundle over $T^{2n}_{\mathbb{C}}$ (Foreman, 2000).

We give rise to a classification of compact complex contact similarity manifolds under the existence of S^1-actions (Compare Fried, 1980; Miner, 1991 for the related results of similarity manifolds). Recall that $\mathrm{Aut}_{cc}(M)$ is the group of complex contact transformations from Definition 2.1.

Theorem 2.6 *Let M be a $4n+2$-dimensional compact complex contact similarity manifold. If $S^1 \leq \mathrm{Aut}_{cc}(M)$ acts on M without fixed points, then M is holomorphically diffeomorphic to a complex contact infranilmanifold $\mathcal{L}/\Gamma$ or a complex contact Hopf manifold $S^1 \times S^{4n+1}$.*

Proof. Let J be a complex structure on M. Given a collection of charts $\{U_\alpha, \varphi_\alpha, J_\alpha\}$ on M with $J_\alpha = J_{|U_\alpha}$ such that $\varphi_\alpha \colon (U_\alpha, J_\alpha) \to (\mathcal{L}, J_0)$ is a holomorphic diffeomorphism onto its image, the monodromy argument shows that there is a developing pair:

$$(\rho, \mathrm{dev}) : (\mathrm{Aut}_{cc}(\tilde{M}), \tilde{M}) \to (\mathrm{Sim}(\mathcal{L}), \mathcal{L}) \tag{2.12}$$

where $\tilde{M}$ is the universal covering and $\tilde{J}$ is a lift of J to $\tilde{M}$, and $\pi = \pi_1(M) \leq \mathrm{Aut}_{cc}(\tilde{M})$. Then dev is a holomorphic immersion $\mathrm{dev}_* J = J_0 \mathrm{dev}_*$ and $\rho \colon \mathrm{Aut}_{cc}(\tilde{M}) \to \mathrm{Sim}(\mathcal{L})$ is a holonomy homomorphism. Put $\Gamma = \rho(\pi)$. Let $\tilde{S}^1$ be a lift of S^1 to $\tilde{M}$ so that $\rho(\tilde{S}^1) \leq \mathrm{Sim}(\mathcal{L})$.

Case 1) If $\Gamma \leq \mathrm{E}(\mathcal{L})$, then there is a $\mathrm{E}(\mathcal{L})$-invariant Riemannian metric on $\mathcal{L}$. As M is compact, the pullback metric on $\tilde{M}$ by dev is (geodesically) complete, dev: $\tilde{M} \to \mathcal{L}$ is an isometry. As dev becomes a complex contact diffemorphism, M is holomorphically isomorphic to a complex contact infranilmanifold $\mathcal{L}/\Gamma$.

Case 2) Suppose that some $\rho(\gamma)$ has a nontrivial summand in $\mathbb{R}^+ \leq \mathcal{L} \rtimes (\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+) = \mathrm{Sim}(\mathcal{L})$. In view of the affine representation $\rho(\gamma) = (p, P)$ where $P = \left(\begin{array}{c|c} \lambda^2 & \lambda^t w\, \mathsf{J}_n A \\ \hline 0 & \lambda A \end{array}\right)$ from (2.7), we note $|\lambda| \neq 1$, i.e. P has no eigenvalue 1. Then there exists an element $z_0 \in \mathcal{L}$ such that the conjugate $(z_0, I)\rho(\gamma)(-z_0, I) = (0, P)$. We may assume that $\rho(\gamma) = (0, P) \in \mathrm{Aff}(\mathcal{L})$ from the beginning. As $\rho(\tilde{S}^1)$ centralizes Γ, if $\rho(t) = (q, Q) \in \rho(\tilde{S}^1)$, then the equation $\rho(t)\rho(\gamma) = \rho(\gamma)\rho(t)$ implies that $Pq = q$ and so $q = 0$. Thus $\rho(t) = (0, Q) = ((0, 0), \mu_t \cdot B_t) \in \mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+ \leq \mathrm{Sim}(\mathcal{L})$. It follows $\rho(\tilde{S}^1) \leq \mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$. In particular, $\rho(\tilde{S}^1)$ has a non-empty fixed point set $\mathcal{S}$ in $\mathcal{L}$. If $\mathrm{dev}(x) \in \mathcal{S}$, then $\mathrm{dev}(\tilde{S}^1 x) = \rho(\tilde{S}^1)\,\mathrm{dev}(x) = x$. Since dev is an immersion, $\tilde{S}^1 x = x$. As S^1 has no fixed points on M, it is noted that $\mathrm{dev}(\tilde{M}) \subset \mathcal{L} - \mathcal{S}$. Let $\mathrm{Sim}(\mathcal{L} - \mathcal{S})$ be the subgroup of $\mathrm{Sim}(\mathcal{L})$ whose elements leave $\mathcal{S}$ invariant. Note that $\Gamma \leq \mathrm{Sim}(\mathcal{L} - \mathcal{S})$.

We determine $\mathcal{S}$ and $\mathrm{Sim}(\mathcal{L} - \mathcal{S})$. Since $\rho(\tilde{S}^1)$ belongs to the maximal abelian group $T^{2n} \cdot S^1 \times \mathbb{R}^+$ up to conjugate in $\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$, we can put $\langle \lambda_t \rangle \leq S^1 \times \mathbb{R}^+$, $\langle s_t \rangle = S^1$ and

$$\rho(\tilde{S}^1) = \{((0,0), \mu_t \cdot B_t)) = \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \left(\begin{array}{c|c} \mu_t^2 & 0 \\ \hline 0 & \mu_t B_t \end{array}\right)\right)\}$$

$$B_t = \begin{pmatrix} s_t & & & & & \\ & \ddots & & & & \\ & & s_t & & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix} \in T^{2n} \leq \mathrm{Sp}(n) \tag{2.13}$$

where $\mathrm{Sp}(n) \leq \mathrm{U}(2n)$ is canonically embedded so that $2k$-numbers of s_t's and 2ℓ-numbers of 1's. Recall that $\rho(\tilde{S}^1)$ acts on $\mathcal{L}$ by $\rho(t)(z_0, z) = (\mu_t^2 z_0, \mu_t B_t z)$.

Case I. $\mu_t \neq 1$. Suppose that $\mu_t \lambda_t = 1$. Then $\mathcal{S} = \mathrm{Fix}(\rho(\tilde{S}^1), \mathcal{L}) = \{(0, (z, 0)) \in \mathcal{L} \,|\, z \in \mathbb{C}^{2k}\}$ $(0 \leq k \leq n)$. As the element $((a, w), \lambda \cdot A) \in \mathrm{Sim}(\mathcal{L})$ acts by $((a, w), \lambda \cdot A)(0, (z, 0)) = (a + \lambda^t w\, \mathsf{J}_n A z,\ w + \lambda A z) \in \mathcal{S}$ (cf. (2.5)), we can check that $a = 0$, $w \in \mathbb{C}^{2k}$ and so $\lambda A z \in \mathbb{C}^{2k}$. In particular, $A \in \mathrm{Sp}(k)$. From $w\, \mathsf{J}_n A z = 0$, it follows $w = 0$.

$$\mathrm{Sim}(\mathcal{L} - \mathcal{S}) = \{((0, 0), \lambda \cdot A) \,|\, A \in \mathrm{Sp}(k)\} = \mathrm{Sp}(k) \cdot S^1 \times \mathbb{R}^+. \tag{2.14}$$

Case II. $\mu_t = 1$. Then $\mathcal{S} = \{(z_0, (0, z)) \in \mathcal{L} \,|\, z \in \mathbb{C}^{2\ell}\} = \mathcal{L}_{2\ell+1}$ $(0 \leq \ell \leq n-1)$. It follows as above

$$\begin{aligned}\mathrm{Sim}(\mathcal{L} - \mathcal{S}) &= \{((a, w), \lambda \cdot A) \,|\, w \in \mathbb{C}^{2\ell}, A \in \mathrm{Sp}(\ell)\} \\ &= \mathcal{L}_{2\ell+1} \rtimes (\mathrm{Sp}(\ell) \cdot S^1 \times \mathbb{R}^+) = \mathrm{Sim}(\mathcal{L}_{2\ell+1}).\end{aligned} \tag{2.15}$$

We need the following lemma.

Lemma 2.7 $\mathrm{Sim}(\mathcal{L} - \mathcal{S})$ *acts properly on* $\mathcal{L} - \mathcal{S}$.

Proof. **Case I.** There is an equivariant inclusion

$$(\mathrm{Sp}(k) \cdot S^1 \times \mathbb{R}^+, \mathcal{L} - \mathcal{S}) \subset (\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+, \mathcal{L} - \{0\}).$$

As there is an $\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+$-invariant Riemannian metric on $\mathcal{L} - \{0\}$ and $\mathrm{Sp}(k) \cdot S^1 \times \mathbb{R}^+$ is a closed subgroup, it acts properly on $\mathcal{L} - \mathcal{S}$.

Case II. Let $G = \mathbb{C}^{2\ell} \rtimes (\mathrm{Sp}(\ell) \cdot S^1 \times \mathbb{R}^+)$ be the semidirect group which preserves the complement $\mathbb{C}^{2n} - \mathbb{C}^{2\ell}$. Then there is an equivariant principal bundle:

$$(\mathbb{C}, \mathbb{C}) \to (\mathrm{Sim}(\mathcal{L}_{2\ell+1}), \mathcal{L} - \mathcal{L}_{2\ell+1}) \longrightarrow (G, \mathbb{C}^{2n} - \mathbb{C}^{2\ell}). \tag{2.16}$$

We note that G acts properly on $\mathbb{C}^{2n} - \mathbb{C}^{2\ell}$. For this, we observe that

$$\mathbb{C}^{2n} - \mathbb{C}^{2\ell} = S^{4n} - S^{4\ell} = \mathbb{H}_{\mathbb{R}}^{4\ell+1} \times S^{4n-4\ell-1} \tag{2.17}$$

in which

$$G \leq \mathbb{R}^{4\ell} \rtimes (\mathrm{O}(4\ell) \times \mathbb{R}^+) = \mathrm{Sim}(\mathbb{R}^{4\ell}) \leq \mathrm{PO}(4\ell + 1, 1). \tag{2.18}$$

As $\mathrm{PO}(4\ell+1, 1) \times \mathrm{O}(4n-4\ell) = \mathrm{Isom}(\mathbb{H}_{\mathbb{R}}^{4\ell+1} \times S^{4n-4\ell-1})$ and G is a closed subgroup of $\mathrm{PO}(4\ell+1, 1)$, G acts properly on $\mathbb{C}^{2n} - \mathbb{C}^{2\ell}$.

Since $\mathbb{C}$ acts properly on $\mathcal{L} - \mathcal{L}_{2\ell+1}$, the above principal bundle (2.16) implies that $\mathrm{Sim}(\mathcal{L}_{2\ell+1})$ acts properly on $\mathcal{L} - \mathcal{L}_{2\ell+1}$. □

We continue the proof of Theorem 2.6. For **Case I**, there is an $\mathrm{Sp}(k) \cdot S^1 \times \mathbb{R}^+$-invariant Riemannian metric on $\mathcal{L} - \mathcal{S}$. Put $H = \mathrm{Sp}(k) \cdot S^1 \times \mathbb{R}^+$. As $\mathcal{L} - \mathcal{S} = \mathbb{C}^{2n+1} - \mathbb{C}^{2k} = \mathbb{H}_{\mathbb{R}}^{4k+1} \times S^{4n-4k+1}$ where $H \leq \mathrm{Sim}(\mathbb{R}^{4k}) \leq \mathrm{PO}(4k+1, 1)$, note that the quotient $\mathcal{L} - \mathcal{S}/H$ is a Hausdorff space. On the other hand,

$$\begin{aligned}\mathcal{L} - \mathcal{S}/H &= \mathbb{H}_{\mathbb{R}}^{4k+1}/H \times S^{4n-4k+1} \\ &= \mathbb{R}^{4k} \rtimes \mathbb{R}^+/H \times S^{4n-4k+1} \\ &= \mathbb{R}^{4k}/(\mathrm{Sp}(k) \cdot S^1) \times S^{4n-4k+1}.\end{aligned} \tag{2.19}$$

$\mathcal{L} - \mathcal{S}/H$ cannot be compact unless $k = 0$.

On the other hand, as M is compact and $\Gamma \leq \mathrm{Sim}(\mathcal{L} - \mathcal{S})$, using Lemma 2.7, $\mathrm{dev} : \tilde{M} \to \mathcal{L} - \mathcal{S}$ is a covering map. $\mathcal{L} - \mathcal{S}$ is simply connected unless $k = n$. Then $M \cong \mathcal{L} - \mathcal{S}/\Gamma$ is compact $(k \neq n)$. If we consider the fiber space $\mathcal{L} - \mathcal{S}/\Gamma \to \mathcal{L} - \mathcal{S}/H$, $\mathcal{L} - \mathcal{S}/H$ must be compact, which cannot occur except for $k = 0$.

If $\mathcal{L} - \mathcal{S} = \mathbb{H}_{\mathbb{R}}^{4n+1} \times S^1$ $(k = n)$, then there is a lift of dev, $\widetilde{\mathrm{dev}}$: $\tilde{M} \to \mathbb{H}_{\mathbb{R}}^{4n+1} \times \mathbb{R}$ which is a diffeomorphism. The group $\tilde{\Gamma} = \widetilde{\mathrm{dev}} \circ \pi \circ \widetilde{\mathrm{dev}}^{-1}$ acts properly discontinuously and freely on $\mathbb{H}_{\mathbb{R}}^{4n+1} \times \mathbb{R}$ such that $\mathbb{H}_{\mathbb{R}}^{4n+1} \times \mathbb{R}/\tilde{\Gamma}$ is compact. As there is the canonical projection:

$$\mathbb{H}_{\mathbb{R}}^{4n+1} \times \mathbb{R}/\tilde{\Gamma} \to \mathbb{H}_{\mathbb{R}}^{4n+1} \times S^1/H, \tag{2.20}$$

$\mathbb{H}_{\mathbb{R}}^{4n+1} \times S^1/H$ is compact. This case is also impossible.

For $k = 0$, $\mathcal{S} = \{0\}$, dev: $\tilde{M} \to \mathcal{L} - \{0\}$ is a diffeomorphism. As $\Gamma \leq S^1 \times \mathbb{R}^+$ acting freely on $\mathcal{L} - \{0\} = \mathbb{H}_{\mathbb{R}}^1 \times S^{4n+1}$, M is biholomorphic to $\mathcal{L} - \{0\}/\Gamma$ which is diffeomorphic with $S^1 \times S^{4n+1}$.

For **Case II**, $\mathcal{L} - \mathcal{L}_{2\ell+1}$ is always simply connected (cf. (2.16)). Then M is diffeomorphic to $\mathcal{L} - \mathcal{L}_{2\ell+1}/\Gamma$ so that $\Gamma \leq \mathrm{Sim}(\mathcal{L}_{2\ell+1}) = \mathcal{L}_{2\ell+1} \rtimes (\mathrm{Sp}(\ell) \cdot S^1 \times \mathbb{R}^+)$ is a discrete subgroup. As there is a fiber space

$$\mathrm{Sim}(\mathcal{L}_{2\ell+1})/\Gamma \to \mathcal{L} - \mathcal{L}_{2\ell+1}/\Gamma \longrightarrow \mathcal{L} - \mathcal{L}_{2\ell+1}/\mathrm{Sim}(\mathcal{L}_{2\ell+1}), \tag{2.21}$$

it follows that $\mathrm{Sim}(\mathcal{L}_{2\ell+1})/\Gamma$ is compact. Since $\mathcal{L}_{2\ell+1}$ is a maximal nilpotent subgroup of $\mathrm{Sim}(\mathcal{L}_{2\ell+1})$, $\mathcal{L}_{2\ell+1} \cap \Gamma$ is discrete uniform in $\mathcal{L}_{2\ell+1}$. As $\mathbb{R}^+$ acts on $\mathcal{L}$ as multiplication, Γ cannot have a nontrivial summand in $\mathbb{R}^+$. This contradicts the hypothesis of **Case 2**. So **Case II** does not occur. This proves the theorem. □

3. Connected Sum

In Kobayashi (1959), there is a complex contact structure on the complex projective space $\mathbb{CP}^{2n+1}$; let $\omega = \sum_{i=1}^{n+1}(z_{2i-1} \cdot dz_{2i} - z_{2i} \cdot dz_{2i-1})$ be a holomorphic 1-form on $\mathbb{C}^{2n+2}$. Put $U_i = \{[w_0, \ldots, w_{2n+1}] \,|\, w_i \neq 0\}$ which forms a cover $\{U_i\}$ of $\mathbb{CP}^{2n+1}$. If s_i is a holomorphic cross-section of the principal bundle $\mathbb{C}^* \to \mathbb{C}^{2n+2} - \{0\} \longrightarrow \mathbb{CP}^{2n+1}$ restricted to U_i, setting $\omega_i = s_i^*\omega$, $\{\omega_i\}$ defines a complex contact structure on $\mathbb{CP}^{2n+1}$. For example, let $\iota\colon U_0 \to \mathbb{C}^{2n+1}$ be the local coordinate system defined by $\iota([w_0, \ldots, w_{2n+1}]) = (z_0, \ldots, z_{2n})$ such that $w_{i+1}/w_0 = z_i$. A holomorphic map $s_0\colon U_0 \to \mathbb{C}^{2n+2} - \{0\}$ may be defined as

$$s_0 \circ \iota^{-1}(z_0, \ldots, z_{2n}) = (1, z_0, -z_1, z_2, -z_3, z_4, \ldots, -z_{2n-1}, z_{2n}).$$

Then the holomorphic 1-form $(s_0 \circ \iota^{-1})^*\omega$ on $\iota(U_0)$ is described as

$$(s_0 \circ \iota^{-1})^*\omega = dz_0 - \sum_{i=1}^{n}(z_{2i-1} \cdot dz_{2i} - z_{2i} \cdot dz_{2i-1}). \tag{3.1}$$

For this,

$$\begin{aligned}(s_0 \circ \iota^{-1})^*\omega &= (s_0 \circ \iota^{-1})^*((z_1dz_2 - z_2dz_1) + (z_3dz_4 - z_4dz_3) + \cdots + (z_{2n+1}dz_{2n+2} - z_{2n+2}dz_{2n+1}))\\ &= dz_0 - (z_1dz_2 - z_2dz_1) - \cdots - (z_{2n-1} \cdot dz_{2n} - z_{2n}dz_{2n-1}).\end{aligned}$$

So $(s_0 \circ \iota^{-1})^*\omega$ is equivalent with $\eta_{|\iota(U_0)}$ of (2.2).

Let $p\colon \mathcal{L} \to \mathcal{L}/\Gamma$ be the holomorphic covering map. Put $V_0 = p(\iota(U_0))$ and $p(0) = x$. Then the map $p \circ \iota\colon U_0 \to V_0$ is a holomorphic map with $p \circ \iota([1, 0, \ldots, 0]) = x$. Choose a neighborhood $U'_0 \subset U_0$ such that $\iota(U'_0)$ is a closed ball B at the origin in $\mathbb{C}^{2n+1}$. Put $p(B) = V'_0 \subset V_0$ so that $p \circ \iota\colon U'_0 \to V'_0$ is a biholomorphism. Then a connected sum $\mathbb{CP}^{2n+1}\#\mathcal{L}/\Gamma$ is obtained by glueing $\mathbb{CP}^{2n+1} - \mathrm{int}U'_0$ and $\mathcal{L}/\Gamma - \mathrm{int}V'_0$ along the boundaries $\partial U'_0$ and $\partial V'_0$ by $p \circ \iota$.

Proposition 3.1 *The connected sum $\mathbb{CP}^{2n+1}\#\mathcal{L}/\Gamma$ admits a complex contact structure.*

Proof. As above $(s_0 \circ \iota^{-1})^*\omega = \eta$ on $\iota(U_0)$. Note that $\omega_0 = s_0^*\omega = \iota^*\eta$ on U_0. On the other hand, the complex contact structure $\{\eta_i\}$ on $\mathcal{L}/\Gamma$ satisfies that $p^*\eta_0 = \eta$ on $\iota(U_0)$. The holomorphic map $p \circ \iota\colon U_0 \to V_0$ satisfies that $(p \circ \iota)^*\eta_0 = \omega_0$. Since $J(p \circ \iota)_* = (p \circ \iota)_*J$ on U_0, the complex structure J is naturally extended to a complex structure on $\mathbb{CP}^{2n+1}\#\mathcal{L}/\Gamma$ along the boundary $\partial U'_0$. □

Since any complex contact similarity manifold M is locally modelled on $(\mathrm{Sim}(\mathcal{L}), \mathcal{L})$ by the definition, every point of M has a neighborhood U on which the complex contact structure is equivalent to a restriction of $(\eta, \mathcal{L})$. Similarly to the above proof, we have

Theorem 3.2 *Any connected sum $M_1\# \cdots \#M_k\#\ell\mathbb{CP}^{2n+1}$ admits a complex contact structure for a finite number of complex contact similarity manifolds $M_1, \ldots, M_k$ and ℓ-copies of $\mathbb{CP}^{2n+1}$.*

4. Contact Complex Structure from Quaternionic Heisenberg Lie Group

4.1 Quaternionic Heisenberg Geometry

Denote $\mathbb{R}^3 = \mathrm{Im}\,\mathbb{H}$ which is the imaginary part of the quaternion field $\mathbb{H}$. $\mathcal{M}$ is the product $\mathbb{R}^3 \times \mathbb{H}^n$ with group law:

$$(\alpha, u) \cdot (\beta, v) = (\alpha + \beta + \mathrm{Im}\langle u, v\rangle, u + v).$$

Here $\langle u, v\rangle = {}^t\bar{u} \cdot v = \sum_{i=1}^{n} \bar{u}_i v_i$ is the Hermitian inner product where $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_n)$ is the quaternion conjugate. $\mathcal{M}$ is nilpotent because $[\mathcal{M}, \mathcal{M}] = \mathbb{R}^3$ which is the center consisting of the form $((a, b, c), 0)$ $(a, b, c \in \mathbb{R})$. $\mathcal{M}$ is called *quaternionic Heisenberg Lie group*. The similarity subgroup $\mathrm{Sim}(\mathcal{M})$ is defined to be the semidirect product $\mathcal{M} \rtimes (\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+)$. The action of $\mathrm{Sim}(\mathcal{M})$ on $\mathcal{M}$ is given as follows; for $h = ((\alpha, u), (A \cdot g, t)) \in \mathcal{M} \rtimes (\mathrm{Sp}(n) \cdot \mathrm{Sp}(1) \times \mathbb{R}^+)$, $(\beta, v) \in \mathcal{M}$,

$$h \circ (\beta, v) = (\alpha + t^2 g\beta g^{-1} + \mathrm{Im}\langle u, tAvg^{-1}\rangle,\ u + t \cdot Avg^{-1}).$$

The pair $(\mathrm{Sim}(\mathcal{M}), \mathcal{M})$ is called *quaternionic Heisenberg geometry.*

Let $u_i = z_i + w_i\mathbf{j} \in \mathbb{H}$ $(z_i, w_i \in \mathbb{C})$. It is easy to check that the correspondence $\mathsf{p}\colon \mathcal{M} \to \mathcal{L}$ defined by

$$(a\mathbf{i} + b\mathbf{j} + c\mathbf{k}, (u_1, \ldots, u_n)) \mapsto (b + c\mathbf{i}, (\bar{z}_1, w_1, \bar{z}_2, w_2, \ldots, \bar{z}_n, w_n)) \tag{4.1}$$

is a Lie group homomorphism. Let $\widehat{\mathrm{Sim}}(\mathcal{M}) = \mathcal{M} \rtimes (\mathrm{Sp}(n) \cdot S^1 \times \mathbb{R}^+)$ be the subgroup of $\mathrm{Sim}(\mathcal{M})$. Then $\mathsf{p}\colon \mathcal{M} \to \mathcal{L}$ induces a homomorphism $\mathsf{q}\colon \widehat{\mathrm{Sim}}(\mathcal{M}) \to \mathrm{Sim}(\mathcal{L})$ for which $(\mathsf{q}, \mathsf{p})\colon (\widehat{\mathrm{Sim}}(\mathcal{M}), \mathcal{M}) \to (\mathrm{Sim}(\mathcal{L}), \mathcal{L})$ is equivariant.

Take the coordinates $(a, b, c) \in \mathbb{R}^3$, $u = (u_1, \ldots, u_n) \in \mathbb{H}^n$. Define a $\mathrm{Im}\,\mathbb{H}$-valued 1-form on $\mathcal{M}$ to be

$$\omega = da\mathbf{i} + db\mathbf{j} + dc\mathbf{k} - \mathrm{Im}\langle u, du\rangle. \tag{4.2}$$

We may put

$$\omega = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k} \tag{4.3}$$

for some real 1-forms $\omega_1, \omega_2, \omega_3$ on $\mathcal{M}$. Noting (4.1), $p^*\eta \cdot \mathsf{j}$ is a $\mathbb{C}\mathsf{j}\,(\leq \mathbb{H})$-valued 1-form on $\mathcal{M}$. A calculation shows that

$$\omega - \mathsf{p}^*\eta \cdot \mathsf{j} = da\mathsf{i} + \sum_{i=1}^{n} (\bar{z}_i dz_i - z_i d\bar{z}_i + w_i d\bar{w}_i - \bar{w}_i dw_i) \tag{4.4}$$

which is an $\mathbb{R}\mathsf{i}$-valued 1-form. Then we have from (4.3) that

$$\omega - \mathsf{p}^*\eta \cdot \mathsf{j} = \omega_1 \cdot \mathsf{i}. \tag{4.5}$$

In particular when $\mathsf{p}_*\colon T\mathcal{M} \to T\mathcal{L}$ is the differential map, this equality shows

$$\mathsf{p}_*(\mathrm{Ker}\,\omega) = \mathrm{Ker}\,\eta. \tag{4.6}$$

4.2 Quaternionic Carnot-Carathéodory Structure on $\mathcal{M}^{4n+3}$

Let $\nu\colon \mathcal{M} \to \mathbb{H}^n$ be the projection defined by $\nu((a, b, c), u) = u$. Then it is easy to check that $\nu_*\colon \mathrm{Ker}\,\omega \to T\mathbb{H}^n$ is an isomorphism at each point. By the pullback of this isomorphism, the standard quaternionic structure $\{J_1, J_2, J_3\}$ on $\mathbb{H}^n$ induces an almost quaternionic structure on $\mathrm{Ker}\,\omega$. (We write it as $\{J_1, J_2, J_3\}$ also.) As $[\mathrm{Ker}\,\omega, \mathrm{Ker}\,\omega] = \mathbb{R}^3$, $(\mathrm{Ker}\,\omega, \{J_\alpha\}_{\alpha=1,2,3})$ is said to be *quaternionic Carnot-Carathéodory* structure on $\mathcal{M}^{4n+3}$ (cf. Alekseevsky & Kamishima, 2008).

Set $u_i = z_i + w_i\mathbf{j} = x_i + y_i\mathbf{i} + (p_i + q_i\mathbf{i})\mathbf{j}$, so that

$$g = |du|^2 = \sum_{i=1}^{n} (dx_i^2 + dy_i^2 + dp_i^2 + dq_i^2)$$

is the standard positive definite symmetric bilinear form on $\mathrm{Ker}\,\omega$. Since $d\omega = -d\bar{u} \wedge du = d\omega_1\mathbf{i} + d\omega_2\mathbf{j} + d\omega_3\mathbf{k}$ from (4.2), (4.3), a reciprocity of the quaternionic structure shows that

$$d\omega_1(J_1X, Y) = d\omega_2(J_2X, Y) = d\omega_3(J_3X, Y) = -g(X, Y). \quad (\forall\, X, Y \in \mathrm{Ker}\,\omega). \tag{4.7}$$

Let J_0 be the complex structure on $\mathcal{L}$ and $\mu\colon \mathcal{L} \to \mathbb{C}^{2n}$ the canonical projection. Since η is a holomorphic 1-form, $\mu_*\colon (\mathrm{Ker}\,\eta, J_0) \to (T\mathbb{C}^{2n}, J_0)$ is an equivariant isomorphism. If $\mathsf{q}\colon \mathbb{H}^n \to \mathbb{C}^{2n}$ is an isomorphism defined by $\mathsf{q}(u_1, \ldots, u_n) = (\bar{z}_1, w_1, \ldots, \bar{z}_n, w_n)$, then there is the commutative diagram:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\ \nu\ } & \mathbb{H}^n \\ {\scriptstyle \mathsf{p}}\big\downarrow & & {\scriptstyle \mathsf{q}}\big\downarrow \\ \mathcal{L} & \xrightarrow{\ \mu\ } & \mathbb{C}^{2n}, \end{array} \tag{4.8}$$

By the definition of J_1, $\mathsf{q}_* \circ J_1 = J_0 \circ \mathsf{q}_*$ on $T\mathbb{H}^n$.

Note that $\mathrm{Ker}\,\omega_1 = \mathrm{Ker}\,\omega \oplus \langle \frac{d}{db}, \frac{d}{dc}\rangle$ with $\omega_1(\frac{d}{da}) = 1$ and $T\mathcal{L} = \mathrm{Ker}\,\eta \oplus \langle \frac{d}{db}, \frac{d}{dc}\rangle$. Since $\mathsf{p}_*\langle \frac{d}{db}, \frac{d}{dc}\rangle = \langle \frac{d}{db}, \frac{d}{dc}\rangle$ (cf. (4.1)) and by (4.6), $\mathsf{p}_*\colon \mathrm{Ker}\,\omega_1 \to T\mathcal{L}$ is an isomorphism.

4.3 Complex Contact Bundle on $\mathcal{L}$

As $\mathbb{R}^3$ acts as translations on $\mathcal{M}$, $\mathbb{R}^3$ leaves ω (resp. ω_i $(i = 1, 2, 3)$) invariant. $\mathbb{R}^3$ induces the distribution of vector fields $\langle \frac{d}{da}, \frac{d}{db}, \frac{d}{dc}\rangle$ on $\mathcal{M}$. Define an almost complex structure $\bar{J}_1$ on $\mathrm{Ker}\,\omega_1$ as

$$\bar{J}_1|\mathrm{Ker}\,\omega_1 = J_1,\ \bar{J}_1\frac{d}{db} = \frac{d}{dc},\ \bar{J}_1\frac{d}{dc} = -\frac{d}{db}. \tag{4.9}$$

Lemma 4.1 $\mathsf{p}_*\circ\bar{J}_1 = J_0\circ\mathsf{p}_*$ *on* $\mathrm{Ker}\,\omega_1$.

Proof. Let $X \in \mathrm{Ker}\,\omega$. By the commutativity of (4.8)

$$\mu_*(\mathsf{p}_*(J_1X)) = \mathsf{q}_*\nu_*(J_1X) = J_0\mathsf{q}_*\nu_*(X) = \mu_*(J_0\mathsf{p}_*(X)), \tag{4.10}$$

so $\mathsf{p}_*(J_1X) = J_0\mathsf{p}_*(X)$.

Obviously, $\mathsf{p}_*(\bar{J}_1(\frac{d}{db}, \frac{d}{dc})) = J_0\mathsf{p}_*(\frac{d}{db}, \frac{d}{dc})$. □

Lemma 4.2 $\bar{J}_1$ *is integrable on* $\mathrm{Ker}\,\omega_1$.

Proof. Let $\mathrm{Ker}\,\omega_1\otimes\mathbb{C} = T^{1,0}_{\omega_1}\oplus T^{0,1}_{\omega_1}$ be the eigenspace decomposition. Then $T^{1,0}_{\omega_1} = T^{1,0}_{\omega}\oplus\langle\frac{d}{db} - \frac{d}{dc}\mathbf{i}\rangle$. If we note that $d\omega_1(\bar{J}_1X, \bar{J}_1Y) = d\omega_1(X, Y)$ $(X, Y \in \mathrm{Ker}\,\omega_1)$ from (4.7), then $[T^{1,0}_{\omega_1}, T^{1,0}_{\omega_1}] \subset \mathrm{Ker}\,\omega_1\otimes\mathbb{C}$. Then $\mathsf{p}_*([T^{1,0}_{\omega_1}, T^{1,0}_{\omega_1}]) = [T^{1,0}(\mathcal{L}), T^{1,0}(\mathcal{L})]$. Since J_0 is the complex structure on $\mathcal{L}$, $[T^{1,0}(\mathcal{L}), T^{1,0}(\mathcal{L})] \subset T^{1,0}(\mathcal{L})$. It follows

$$[T^{1,0}_{\omega_1}, T^{1,0}_{\omega_1}] \subset T^{1,0}_{\omega_1}. \tag{4.11}$$

□

Remark 4.3 The pair $(\mathrm{Ker}\,\omega_1, \bar{J}_1)$ is not a strictly pseudoconvex *CR*-structure on $\mathcal{M}$ unlike Sasakian 3-structures. For this, $[\frac{d}{db}, \frac{d}{dc}] = 0$ in $\mathrm{Ker}\,\omega_1 = \mathrm{Ker}\,\omega\oplus\langle\frac{d}{db}, \frac{d}{dc}\rangle$, so $d\omega_1(\frac{d}{db}, \frac{d}{dc}) = 0$. However, $d\omega_1\colon \mathrm{Ker}\,\omega\times\mathrm{Ker}\,\omega \to \mathbb{R}$ is nondegenerate from (4.7).

We put $\mathrm{Ker}\,\eta\otimes\mathbb{C} = T^{1,0}_{\eta}\oplus T^{0,1}_{\eta}$. Let $\mathsf{p}_*\colon \mathrm{Ker}\,\omega_1\otimes\mathbb{C} \to T\mathcal{L}\otimes\mathbb{C}$ be an isomorphism so that $\mathsf{p}_*(\frac{d}{db} - \frac{d}{dc}\mathbf{i}) = \frac{d}{db} - \frac{d}{dc}\mathbf{i}$. By Lemma 4.1, we have $\mathsf{p}_*(T^{1,0}_{\omega}) = T^{1,0}_{\eta}$.

Theorem 4.4 *The complex* $2n$*-dimensional holomorphic subbundle* $T^{1,0}_{\eta}$ *is a complex contact subbundle on* $\mathcal{L}$.

Proof. Let $T^{1,0}_{\omega_1}\otimes\mathbb{C} = T^{1,0}_{\omega}\oplus\langle\frac{d}{db} - \frac{d}{dc}\mathbf{i}\rangle$ and $T^{1,0}(\mathcal{L}) = T^{1,0}_{\eta}\oplus\langle\frac{d}{db} - \frac{d}{dc}\mathbf{i}\rangle$ as above. From Remark 4.3, $d\omega_1\colon T^{1,0}_{\omega}\times\bar{T}^{1,0}_{\omega} \to \mathbb{C}$ is nondegenerate. Since $J_1(J_3X) = -\mathbf{i}(J_3X)$, $J_3X \in \bar{T}^{1,0}_{\omega}$. Then $d\omega_1(J_3X, Y) = -d\omega_2(X, Y) = \omega_2([X, Y])$ from (4.7). Thus $\omega_2([T^{1,0}_{\omega}, T^{1,0}_{\omega}]) = \mathbb{C}$. In particular, $[T^{1,0}_{\omega}, T^{1,0}_{\omega}] \neq \{0\}$. As $[T^{1,0}_{\omega}, T^{1,0}_{\omega}] \subset T^{1,0}_{\omega_1} = T^{1,0}_{\omega}\oplus\langle\frac{d}{db} - \frac{d}{dc}\mathbf{i}\rangle$ by Lemma 4.2, it follows

$$[T^{1,0}_{\eta}, T^{1,0}_{\eta}] \equiv \langle\frac{d}{db} - \frac{d}{dc}\mathbf{i}\rangle \bmod T^{1,0}_{\eta}. \tag{4.12}$$

Hence $T^{1,0}_{\eta}$ is a complex contact subbundle on $\mathcal{L}$. □

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